ON THE CHANGE IN SHAPE OF A GRAVITATING SPHERE SUBJECT TO THE INFLUENCE OF A MAGNETIC FIELD

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1. In a recent fundamental paper Chandrasekhar and Fermi (1953) (later referred to as C.F.) have, amongst other problems, considered the problem of the flattening of a gravitating sphere subject to the influence of a magnetic field, the material of the sphere is supposed to be inviscid, homogeneous and infinitely conducting. The magnetic field is assumed to be uniform inside the sphere and a dipole-field outside it, the moment of the dipole being $\mu = \frac{1}{2}\mathbf{H}_0 R^3$, where $\mathbf{H}_0$ is the inside field and R the radius of the sphere. This field is generated by *surface currents* and using spherical polar co-ordinates, with the Z-axis along $\mathbf{H}_0$, the surface current-density (current per unit arc) is given by

$$\mathbf{j} = \left(0, \quad 0, \quad \frac{3H_0}{8\pi}\sin\theta\right). \tag{1}$$

C.F. have shown that under the influence of the magnetic field the sphere undergoes a flattening and the equilibrium configuration is an *oblate* spheroid of ellipticity which, when small, is given by

$$\frac{\epsilon}{R} = -\frac{35}{24}\frac{H_0^2 R^4}{GM^2}, \tag{2}$$

where M is the mass of the sphere and G the gravitational constant. The ratio of the axes of the spheroid is $\left(1+\frac{3\epsilon}{R}\right)^{\frac{1}{2}}$.

In the C.F. problem the sphere is subject to its own magnetic field, there being no field of external origin. Miss Gjellestad (1954), following the C.F. method, has considered the problem of a sphere subject to an external field. If in the C.F. problem we superimpose (throughout space) a uniform field equal and opposite to the field inside the sphere, then the resulting field vanishes inside it. Outside the sphere the field is the resultant of the superimposed field and the dipole field, the axis of the dipole being opposite to the direction of the superimposed field. For this case Miss Gjellestad has given for the ellipticity the expression

$$\frac{\epsilon}{R} = \frac{25}{24}\frac{H_0^2 R^4}{GM^2}. \tag{3}$$

The numerical factor in the above expression should be, as shown below, $\frac{5}{8}$ instead of $\frac{25}{24}$.

The purpose of the present paper is two-fold:

(i) We give what we believe to be a correct derivation of the ellipticity for the Gjellestad problem. Her derivation, apart from an error in the sign of the integral in equation (16) of her paper, is incorrect on account of the energy of the magnetic field becoming infinite*. (In fact, her treatment when corrected for the sign of the integral referred to above leads to an *oblate* spheroid instead of a prolate one, which is obviously incorrect.)

(ii) We consider a straightforward generalisation of the C.F. problem. Instead of superimposing in the C.F. problem a field equal and opposite to $\mathbf{H}_0$, as has been done by Miss Gjellestad, we superimpose a field $\mathbf{H}_1$, $\mathbf{H}_1$ being parallel to $\mathbf{H}_0$. Thus the field inside the sphere is $\mathbf{H}_0+\mathbf{H}_1$ and outside it is the resultant of $\mathbf{H}_1$ and the dipole-field of the sphere. For $\mathbf{H}_1 = 0$ we have the C.F. case and for $\mathbf{H}_1 = -\mathbf{H}_0$ we have the Gjellestad case.

The magnetic field on the surface of the sphere at (R, θ, ϕ) is given by

$$[(H_0+H_1)\cos\theta,\ -(H_0+H_1)\sin\theta,\ 0] \text{ inside,} \tag{4}$$

and

$$\left[(H_0+H_1)\cos\theta,\ \left(\frac{H_0}{2}-H_1\right)\sin\theta,\ 0\right] \text{ outside.}$$

The interaction between the magnetic field and the surface current (1) gives rise to a mechanical force $\mathbf{F}$. Its components per unit area, as is readily seen, are given by

$$\mathbf{F} = \mathbf{j}\times\mathbf{H} = \left[\frac{3H_0}{8\pi}\left(\frac{H_0}{4}+H_1\right)\sin^2\theta,\ \frac{3H_0}{8\pi}(H_0+H_1)\sin\theta\cos\theta,\ 0\right]. \tag{5}†$$

The presence of the surface forces F_θ is an unsatisfactory feature of the C.F. model and the generalisation we are considering‡. This difficulty disappears, of course, for the special cases when $\mathbf{H}_0+\mathbf{H}_1 = 0$, i.e. the Gjellestad problem. In spite of the foregoing difficulty it may be worth while, following C.F., to derive expressions for $\frac{\epsilon}{R}$ for the equilibrium configuration using the energy method. We do this in the next section and revert again in the last section to the question of surface forces.

2. We shall follow closely the 'energy method' of C.F. and Miss Gjellestad. However, in order to overcome the difficulty of the infinite magnetic energy referred to earlier we shall assume the external field H_1 to be due to a magnetic pole on the Z-axis of strength $H_1 d^2$, where d is the distance of the pole from the centre of the sphere. Further we take $R/d\to 0$. This device makes the external field uniform and equal to H_1 in the neighbourhood of the sphere but makes it vanish at infinity and thus overcomes the divergence difficulty.

Let us deform the sphere in such a way that the deformed boundary is (we follow the notation of C.F.):

$$r(\mu) = R+\epsilon P_l(\mu), \tag{6}$$

where $\epsilon << R$, $\mu=\cos\theta$ and $P_l(\mu)$ is the Legendre polynomial of order l. We call such a deformation of the sphere a P_l-deformation. Now an arbitrary deformation of an incompressible body can be realised by applying at each point of the body a displacement $\boldsymbol{\xi}$ which is the gradient of a scalar function ψ satisfying Laplace's equation

$$\nabla^2\Psi = 0. \tag{7}$$

* This divergence difficulty does not arise in the C.F. case. It arises in the Gjellestad case on account of the field not vanishing at infinity.

† We notice that F_r is the difference of the magnetic pressure on the two sides of the surface.

‡ As surface forces F_θ cannot be balanced by hydrostatic forces, it means that *strictly* the model has no steady-state configurations.

A solution of this equation corresponding to the displacement (6) can be written

$$\Psi = \frac{\epsilon}{l}\frac{r^l}{R^{l-1}} P_l(\mu). \tag{8}$$

It satisfies the relation

$$\xi_r = \frac{\partial \Psi}{\partial r} = \epsilon P_l(\mu) \text{ at } r = R.$$

Thus at any point we have

$$\xi_r = \epsilon \left(\frac{r}{R}\right)^{l-1} P_l(\mu)$$

and

$$\xi_\theta = -\frac{\epsilon}{l}\left(\frac{r}{R}\right)^{l-1} P_l'(\mu) \sin\theta, \tag{9}$$

where dash denotes differentiation with respect to μ. For infinite electrical conductivity, the change in the internal magnetic field is given by

$$\delta H^{(i)} = \text{curl } (\boldsymbol{\xi} \times \mathbf{H}) = (\mathbf{H}.\text{ grad})\, \boldsymbol{\xi}$$

since

$$\text{div } \boldsymbol{\xi} = 0. \tag{10}$$

Initially the radial and the transverse components of the magnetic field are:

$$H_r^{(i)} = (H_0+H_1)\cos\theta,\ H_\theta^{(i)} = -(H_0+H_1)\sin\theta \quad (r < R) \tag{11}$$

Substituting these values in (10) we get

$$\delta H_r^{(i)} = \epsilon\,(l-1)\,(H_0+H_1)\frac{r^{l-2}}{R^{l-1}} P_{l-1}(\mu).$$

and

$$\delta H_\theta^{(i)} = -\epsilon\,(H_0+H_1)\frac{r^{l-2}}{R^{l-1}} P_{l-1}'(\mu)\sin\theta, \tag{12}$$

The change in the internal magnetic energy to a first order is, therefore, given by

$$\Delta W^{(i)} = \frac{1}{4\pi}\int H.^{(i)}\delta H^{(i)}\, d\tau$$

$$= \epsilon(l-1)\frac{(H_0+H_1)^2}{4\pi}\int \frac{r^{l-1}}{R^{l-1}} P_{l-2}(\mu)\, d\tau, \tag{13}$$

the integration being carried over the volume of the sphere. Thus

$$\Delta W^{(i)} = \frac{1}{3}\epsilon(H_0+H_1)^2 R^2 \quad \text{for } l = 2 \tag{14}$$

and $\qquad = 0 \qquad \text{for } l \neq 2$

We shall now calculate $\Delta W^{(e)}$, the change in the magnetic energy outside the sphere. We assume that initially the external field consists of a field of dipole moment $\frac{1}{2} H_0 R^3$ plus a field due to a magnetic pole of pole-strength $H_1 d^2$. Let

O in figure 1 be the magnetic pole placed at distance d from the centre C of the sphere ($d \gg R$). The field inside the sphere is parallel to OC. The magnetic field at any point outside the deformed sphere is given by

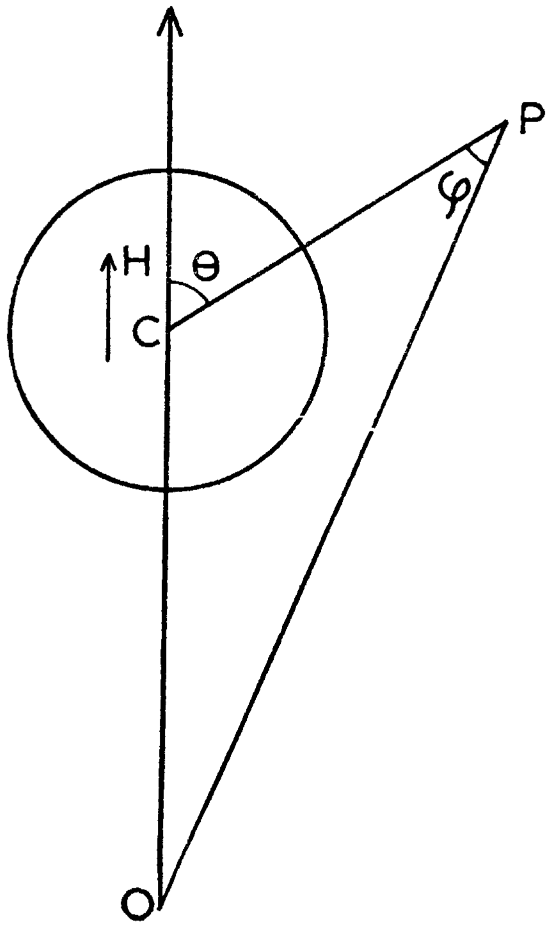

FIG. 1.

$$H_r^{(e)} = \frac{H_1 d^2}{s^2} \cos \phi + \frac{H_0 R^3}{r^3} \cos \theta$$

$$+ (H_0 + H_1) \frac{\epsilon}{R} \sum_{n=1}^{\infty} n(n+1) a_n \left(\frac{R}{r}\right)^{n+2} P_n(\mu), \tag{15}$$

$$H_\theta^{(e)} = -\frac{H_1 d^2}{s^2} \sin \phi + \frac{H_0 R^3}{2r^3} \sin \theta$$

$$+ (H_0 + H_1) \frac{\epsilon}{R} \sum_{n=1}^{\infty} n a_n \left(\frac{R}{r}\right)^{n+2} P_n'(\mu) \sin \theta, \tag{16}$$

where s is the distance of P from the pole O, ϕ is the angle that OP makes with CP. We have

$$\sin\phi = \frac{d}{s}\sin\theta, \quad \cos\phi = \frac{r+d\cos\theta}{s}. \tag{17}$$

The coefficients a_n can be determined from the continuity of the normal component of the field on the deformed surface.

$$[H_r^i]_{R+\epsilon P_l} + [H_\theta^{(i)}]_R \frac{\epsilon}{R} P_l'(\mu)\sin\theta$$

$$= [H_r^e]_{R+\epsilon Pl} + [H_\theta^e]_R \frac{\epsilon}{R} P_e'(\mu)\sin\theta \tag{18}$$

Since $\frac{R}{d} \to 0$, we have on the surface of the sphere $s \sim d$ and $\phi \sim \theta$. Substituting the values from the equations (11), (12), (15), (16) and (17) and making some simplifications we obtain

$$(H_0+H_1)\mu + (H_0+H_1)\frac{\epsilon}{R}[(l-1)P_{l-1} - (1-\mu^2)P_l']$$

$$= (H_0+H_1)\mu - \frac{3\epsilon}{R} H_0\mu P_l + \frac{\epsilon}{R}\left[-H_1 + \frac{1}{2}H_0\right](1-\mu^2)P_e'$$

$$+ (H_0+H_1)\frac{\epsilon}{R}\sum_{n=1}^{\infty} n(n+1)\, a_n P_n(\mu),$$

and hence we have

$$\sum_{n=1}^{\infty} n(n+1)\, a_n P_n(\mu)$$

$$= \frac{(l-1)}{2(2l+1)}\left(l+2+\frac{3H_1 l}{H_0+H_1}\right)P_{l-1}$$

$$+\frac{3(l+1)(l+2)}{2(2l+1)}\,\frac{H_0}{H_0+H_1}\,P_{l+1}. \tag{19}$$

From the orthogonality of the Legendre's polynomials it follows that

$$a_n = 0 \quad \text{for} \quad n \neq l \pm 1 \tag{20}$$

$$a_{l-1} = \frac{1}{2l(2l+1)}\left[l+2+\frac{3H_1 l}{H_0+H_1}\right] \tag{21}$$

and

$$a_{l+1} = \frac{3}{2(2l+1)}\,\frac{H_0}{H_0+H_1}. \tag{22}$$

Thus the field components outside the sphere are

$$H_r^{(e)} = \left(H_1 + \frac{H_0R^3}{r^3}\right)\mu + \frac{\epsilon}{R}\left[\frac{l-1}{2(2l+1)}\{(H_0+H_1)(l+2)+3H_1 l\}\times\right.$$

$$\left.\left(\frac{R}{r}\right)^{l+1} P_{l-1}(\mu) + \frac{3(l+1)(l+2)}{2(2l+1)} H_0 \left(\frac{R}{r}\right)^{l+3} P_{l+1}(\mu)\right] \tag{23}$$

and

$$H_0^{(e)} = -\left(H_1 - \frac{1}{2}H_0\frac{R^3}{r^3}\right)\sin\theta$$

$$+\frac{\epsilon}{R}\left[\frac{l-1}{2l(2l+1)}\{(H_0+H_1)\ (l+2)+3H_1 l\}\left(\frac{R}{r}\right)^{l+1}P'_{l-1}(\mu)\right.$$

$$\left.+\frac{3(l+1)}{2(2l+1)}H_0\left(\frac{R}{r}\right)^{l+3}P'_{l+1}(\mu)\right]\sin\theta. \tag{24}$$

To the first order in ϵ, the change ΔW (e) in the energy of the outside field is given by

$$\Delta W^{(e)} = -\frac{H_0^2R^6}{8}\int_{-1}^{+1}\int_{R}^{R+\epsilon P_l}\frac{1+P_2}{r^4}\,drd\mu -$$

$$-\frac{H_0H_1d^2R^3}{2}\int_{-1}^{1}\int_{R}^{R+\epsilon P_l}\frac{rP_1+dP_2}{s^3r}\,drd\mu$$

$$+\frac{\epsilon}{R}\int_{-1}^{1}\int_{R}^{\infty}\left[\frac{H_1\{(H_0+H_1)(l+2)+3H_1l\}(l-1)d^2}{4(2l+1)s^3}(rP_{l-1}+dP_l)\left(\frac{R}{r}\right)^{l+1}\right.$$

$$+\frac{3H_0H_1d^2}{4s^3}\cdot\frac{(l+1)(l+2)}{2l+1}\left(\frac{R}{r}\right)^{l+3}(rP_{l+1}+dP_{l+2})$$

$$+\frac{(l-1)H_0\{(H_0+H_1)(l+2)+3H_1l\}}{8(2l-1)(2l+1)}\left(\frac{R}{r}\right)^{l+4}\{3(l-1)P_{l-2}+(l+1)P_l\}$$

$$\left.+\frac{3(l+1)(l+2)H_0^2}{8(2l+1)(2l+3)}\left(\frac{R}{r}\right)^{l+6}\{3(l+1)P_l+(l+3)P_l+2\}\right]r^2drd\mu \tag{25}$$

and after evaluating the integrals we obtain

$$\Delta W^{(e)} = \frac{1}{60}(H_0^2-10H_0H_1-20H_1^2)R^2\epsilon \quad \text{for } l=2 \tag{26}$$

$$\text{and} = 0 \qquad\qquad \text{for } l \neq 2$$

This gives, for the total change in the magnetic energy, for the P_2-deformation

$$\Delta W_2 = \Delta W^{(i)} + \Delta W^{(e)}$$

$$= \frac{1}{2_0}H_0(7H_0+10\,H_1)R^2\epsilon. \tag{27}$$

The change in the gravitational energy $\Delta\Omega$ due to a P_l-deformation is given by

$$\Delta\Omega_2 = \frac{3(l-1)}{(2l+1)^2}\left(\frac{\epsilon}{R}\right)^2\frac{GM^2}{R}. \tag{28}$$

Therefore the change for P_2-deformation in the magnetic energy is of the order of ϵ and the change in the gravitational energy is of the order of ϵ^2. The total energy change for a P_2-deformation is

$$\Delta E = \Delta\Omega_2 + \Delta W_2$$

$$= \frac{3}{25}\frac{GM^2}{R^3}\epsilon^2 + \frac{1}{20}H_0(7H_0+10H_1)R^2\epsilon. \tag{29}$$

This has a minimum value when

$$\frac{\epsilon}{R} = -\frac{5}{24} H_0 (7H_0+10 H_1) \frac{R^4}{GM^2}, \tag{30}$$

$$\frac{\epsilon}{R} = -\frac{5}{24} (H_i-H_1) (7H_i+3H_1) \frac{R^4}{GM^2} \tag{30a}$$

or

where $H_i = H_0+H_1$ is the resulting field inside the sphere*.
For $H_1 = 0$ which is the C.F. case, we have

$$\frac{\epsilon}{R} = -\frac{35}{24} \frac{H_0^2 R^4}{GM^2}. \tag{31}$$

For $H_1 = H_0$ which is the Gjellestad case, we have

$$\frac{\epsilon}{R} = \frac{5}{8} \frac{H_0^2 R^4}{GM^2}. \tag{32}$$

For $H_1 = -\frac{7}{10} H_0$ the ellipticity vanishes.

3. It is of interest to observe that the expression (27) for the change in magnetic energy can be readily obtained from the expression (5) for the mechanical force. In a displacement defined by (9) the work done on the system is

$$-\int \mathbf{F}.\xi dS = -2\pi \int_0^\pi \mathbf{F}.\xi R^2 \sin\theta \, d\theta, \tag{33}$$

which on integration gives for the change in the magnetic energy, in the case $l = 2$

$$\Delta W_2 = \frac{1}{5} H_0 \left(\frac{H_0}{4}+H_1\right) R^2\epsilon + \frac{3}{10} H_0 (H_0+H_1) R^2\epsilon \tag{34}$$

The first term arises on account of the radial pressure F_r and the second on account of the tangential force F_θ. Because of the presence of the surface force F_θ it is difficult to give a direct physical meaning to the value of $\frac{\epsilon}{R}$ obtained by the energy method of the previous section. It may perhaps be more satisfactory to ignore the surface force altogether and calculate the ellipticity on this assumption. We then take for ΔW only the first term in (34) and using the expression for $\Delta\Omega_2$ from (28) we have for

$$\Delta E = \frac{3}{25} \frac{GM^2}{R^3} \epsilon^2 + \frac{1}{5} H_0 \left(\frac{H_0}{4}+H_1\right) R^2\epsilon \tag{35}$$

Hence, minimising the above we obtain

$$\frac{\epsilon}{R} = -\frac{5}{24} H_0 (H_0+4H_1) \frac{R^4}{GM^2}, \tag{36}$$

$$= -\frac{5}{24} (H_i-H_1) (H_i+3H_1) \frac{R^4}{GM^2} \tag{36a}$$

* Expression (30a) would be applicable to the case of a conducting sphere having an internal field H_i and entering a region of uniform field H_1.

In the case of the Gjellestad problem ($\mathbf{H}_0+\mathbf{H}_1 = 0$) the above equation gives, as would be expected, a result identical with (32). For the C.F. problem $H_1 = 0$ we get, instead of (31), the value

$$\frac{\epsilon}{R} = -\frac{5}{24}\frac{H_0^2 R^4}{GM^2}. \tag{37}$$

An alternative derivation of (36) is worth noting. It is easily seen from the usual gravitational theory that if the surface of a gravitating sphere is subjected to a pressure $p = p_0 \sin^2\theta$ (p_0 being a constant), then it is deformed into a spheroid of ellipticity $\frac{\epsilon}{R}$ which when small is given by

$$\frac{\epsilon}{R} = -\frac{20\pi}{9}\frac{p_0 R^4}{GM^2}. \tag{38}$$

From equation (5) we have

$$F_r = \frac{3H_0}{32\pi}(H_0+4H_1)\sin^2\theta \tag{39}$$

and hence substituting for

$$p_0 = \frac{3}{32\pi}H_0(H_0+4H_1)$$

we get

$$\frac{\epsilon}{R} = -\frac{5}{24}H_0(H_0+4H_1)\frac{R^4}{GM^2},$$

which is the same as (36).

Summary

This paper is largely concerned with the flattening of gravitating, infinitely conducting sphere subject to the influence of a uniform field (H_i) inside it and a uniform field (H_1) outside it. For the case of zero field inside, the ellipticity is given by

$$\frac{\epsilon}{R} = \frac{5}{8}\frac{H_1^2 R^4}{GM^2}.$$

References

Chandrasekhar, S., and Fermi, E. (1953). Problems of gravitational stability in the presence of a magnetic field. *Ap. J.*, **118**, 116.

Gjellestad, G. (1954). On the elongation of a fluid sphere in an external magnetic field. *Ap. J.*, **120**, 172.

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