PRESSURE-TIME CURVE IN INTERNAL BALLISTICS OF SOLID-FUEL ROCKETS AS DEDUCED FROM THE THEORY OF INTERNAL BALLISTICS OF RECOIL-LESS GUNS

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1. Introduction

The theories of Internal Ballistics of solid-fuel rockets (Kershner, 1944; Wimpress, 1950) and of the Internal Ballistics of recoil-less guns (Corner, 1947, 1950) have been developed independently; but it is obvious that there should be some intrinsic relation between them [Corner (1950), 16], as in both cases the escape of gases is regulated by a convergent-divergent nozzle. The main difference, however, in the two cases is that while in a R.C.L. gun the change of volume available to the gases occurs on two accounts, (i) motion of the shot, (ii) burning of the propellant, in the case of the rocket the change occurs only on account of the latter.

Two of the important problems in the theory of Internal Ballistics of solid-fuel rockets are the investigations of the pressure-time curve and of the equilibrium pressure. Kershner (1944) obtained these on the following assumptions:

(i) Constant-burning surface,
(ii) constant pressure in the rocket motor at any instant,
(iii) constant temperature in the rocket motor (isothermal assumption),
(iv) co-volume η per unit mass is taken as zero,
(v) neglect of change of volume available to the gases in the rocket motor, due to the burning of the propellant.

Gupta and Mehta (1954) obtained the equation of the pressure-time curve when the change in volume is taken into account; their other assumptions were the same as those of Kershner.

In the present paper, we are able to deduce the equations of Internal Ballistics of rockets from those of the R.C.L. gun without making four of the five assumptions the exception being (ii) which requires a separate investigation. Of course, in the general case, analytic integration of the equations is not possible.

We first obtain here the pressure-time curve for the case discussed by Gupta and Mehta except that we do not assume η to be necessarily zero. Even when η is put equal to zero, our equation does not reduce to theirs. It appears that the difference is due to their taking the gas-density ρ_g proportional to pressure in one part of their discussion and as constant in the other part. On taking ρ_g proportional to pressure throughout, we get, from their approach, the same equation as ours.

The isothermal assumption with necessary adjustments in the charge parameters, while fairly satisfactory when the charge is burning, is very poor after all-burnt, since the gas considerably cools in this period by expansion. We obtain here the equation of the Transient Exhaust Curve taking into consideration the effect of change of temperature. As a matter of fact we obtain here an explicit expression for the variation of temperature with time after all-burnt.

While for most commonly used shapes, the tube and the multitube, the burning surface area remains more or less constant; we also use other shapes (cruciform, triform, hexaform, octaform, slab, etc.) for which the burning is either progressive or degressive. Even for a single-perforation tubular grain, the burning is slightly regressive. We have obtained here the pressure-time curve when the propellant has a general quadratic form-function. This enables us to discuss the pressure-time curve not only for a single charge but also for composite and moderated charges.

2. Equations of Internal Ballistics of Rockets deduced from those of Recoil-less Guns

Corner (1950, page 260) gives the following equations of Internal Ballistics of R.C.L. guns

$$p\left[U+Ax-\frac{C(1-z)}{\delta}-CN\eta\right]=CNRT\left(1+\frac{kCN}{6W}\right) \quad \ldots \quad \ldots \quad (1)$$

$$W_1\frac{d^2x}{dt^2}=Ap \quad \ldots \quad (2)$$

$$W_1=W+\tfrac{1}{2}kCN \quad \ldots \quad (3)$$

$$D\frac{df}{dt}=-B\cdot p^{\alpha} \quad \ldots \quad (4)$$

$$z=(1-f)(1+\theta f) \quad \ldots \quad (5)$$

$$\frac{dN}{dt}=\frac{dz}{dt}-\frac{\psi S_0 p}{C(RT)^{\frac{1}{2}}} \quad \ldots \quad (6)$$

$$\frac{d}{dt}(NT)=-(\bar{\gamma}-1)\frac{Ap}{CR}\frac{dx}{dt}+T_0\frac{dz}{dt}-\frac{\gamma\psi S_0 p(RT)^{\frac{1}{2}}}{CR} \quad \ldots \quad (7)$$

Now for a rocket, x does not change; accordingly we can put $x=0$ in (1), $\frac{dx}{dt}=0$ in (7) and omit equations (2) and (3) altogether. Thus the equations reduce to

$$p\left[U-\frac{C(1-z)}{\delta}-CN\eta\right]=CNRT\left(1+\frac{kCN}{6W}\right) \quad \ldots \quad (8)$$

$$D\frac{df}{dt}=-(a+bp) \quad \ldots \quad (9)$$

$$z=(1-f)(1+\theta f) \quad \ldots \quad (10)$$

$$\frac{dN}{dt}=\frac{dz}{dt}-\frac{\psi S_0 p}{C(RT)^{\frac{1}{2}}} \quad \ldots \quad (11)$$

$$\frac{d}{dt}(NT)=T_0\frac{dz}{dt}-\frac{\gamma\psi S_0 p(RT)^{\frac{1}{2}}}{CR}, \quad \ldots \quad (12)$$

where we have taken the Linear Law for the rate of burning.

We may also point out here that equation (1) is deduced from the following two equations

$$\rho_g\left[U+Ax-\frac{C(1-z)}{\delta}\right]=NC \quad \ldots \quad (13)$$

and

$$p\left[\frac{1}{\rho_g}-\eta\right]=RT\left(1+\frac{kCN}{6W}\right) \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

where in the first equation, the assumption is made that the gas density ρ_g is uniform in the motor at any instant.

We have written here the most general equations. We next examine the effect of each of the assumptions of Kershner on these equations.

Assumption (i) implies $\theta=0$

Assumption (ii) implies that $1+\frac{kCN}{6W}$ is taken as unity, since space mean-pressure $=\frac{p}{1+\frac{kCN}{6W}}$. In rockets, the pressure distribution is approximately parabolic and is important when discussing 'regression' and 'erosive' burning. Here, however, pressure-drop is neglected.

Assumption (iii) implies that T is taken as constant. The formal effect is to make (11) and (12) identical.

Assumption (iv) implies that η is taken as zero.

Assumption (v) implies that $\frac{Cz}{\delta}-CN\eta$ is neglected.

From assumptions (ii), (iii), (iv) and (14) we get

$$p=\rho_g RT=\lambda\rho_g \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

or

$$\rho_g=Bp,$$

where

$$B=\frac{1}{\lambda} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

and we have written λ for RT.

Making all the above assumptions except (iv) and (v), the equations reduce to

$$p\left[U-\frac{C(1-z)}{\delta}-CN\eta\right]=CN\lambda \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$D\frac{df}{dt}=-(a+bp) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$z=1-f \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

$$\frac{dN}{dt}=\frac{dz}{dt}-\frac{\psi S_0 p}{C\sqrt{\lambda}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

3. The Pressure-Time Curve

From (18), (19) and (20)

$$\frac{dN}{dt}=-\frac{df}{dt}+\frac{\psi S_0}{bC\sqrt{\lambda}}\left[D\frac{df}{dt}+a\right]$$

$$=-\frac{df}{dt}+\Psi\left[\frac{df}{dt}+\frac{a}{D}\right], \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where

$$\Psi = \frac{\psi S_0 D}{bC\sqrt{\lambda}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

is the dimensionless leakage parameter.

From (17), (18), (19) and (21)

$$\frac{dp}{dt}\left[U - \frac{Cf}{\delta} - CN\eta\right] + p\left[\left(\frac{C}{\delta} - C\eta\right)\frac{a+bp}{D} + \frac{\eta\Psi S_0}{\sqrt{\lambda}}p\right]$$

$$= C\lambda\left[\frac{a+bp}{D}(1-\Psi) + \frac{a}{D}\Psi\right]$$

or

$$U - \frac{Cf}{\delta} - CN\eta$$

$$= \frac{\dfrac{C\lambda}{D}\left[(a+bp)(1-\Psi)+a\Psi\right] + \dfrac{pC}{D}\left[\eta - \dfrac{1}{\delta}\right][a+bp] - p\dfrac{\eta\Psi bC}{D}}{\dfrac{dp}{dt}}$$

Differentiating with respect to t, making use of (18) and (20) and simplifying, we get

$$\frac{d^2p}{dt^2}\left\{bBp^2(\eta\rho-1) + p[aB(\eta\rho-1) - K\eta B + b\rho - K] + a\rho\right\}$$

$$= \left(\frac{dp}{dt}\right)^2\left\{p[3bB(\eta\rho-1) - K\eta B] + b\rho - K + 2aB(\eta\rho-1) - \eta BK\right\}, \quad \ldots \quad (23)$$

where

$$\lambda = \frac{1}{B}, \; \delta = \rho, \; \rho b\Psi = K \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

When $\eta = 0$, (23) becomes,

$$\frac{d^2p}{dt^2}\left[a\rho + p(b\rho - aB - K) - bBp^2\right] = \left(\frac{dp}{dt}\right)^2\left[b\rho - K - 2aB - 3bBp\right] \quad \ldots \quad (25)$$

A better approximation is, however, obtained by putting $\eta\rho \sim 1$. In this case (23) gives

$$\frac{d^2p}{dt^2}\left[p\left(b\rho - K - \frac{BK}{\rho}\right) + a\rho\right] = \left(\frac{dp}{dt}\right)^2\left[-\frac{KBp}{\rho} + b\rho - K - \frac{BK}{\rho}\right] \quad \ldots \quad (26)$$

Each of the three equations (23), (25) and (26) can be integrated to give the equation of the pressure-time curve.

Case (i): η is neglected.

By putting $\dfrac{dp}{dt} = q$, (25) becomes

$$\frac{dq}{q} = dp\left[-\frac{E}{\alpha - p} + \frac{F}{p+\beta}\right],$$

where

$$E+F = 3 \qquad (27a)$$

$$F\alpha - E\beta = \frac{b\rho - K - 2aB}{bB} \qquad (27b)$$

$$\alpha - \beta = \frac{b\rho - aB - K}{bB} \qquad (27c)$$

and

$$\alpha\beta = \frac{a\rho}{bB} \qquad (27d)$$

Integrating

$$\frac{dp}{dt} = q = \frac{C\lambda a}{D\left[U - \frac{C}{\delta}\right]}\left[1 - \frac{p}{\alpha}\right]^E \left[1 + \frac{p}{\beta}\right]^F$$

$$= K_0[\alpha - p]^E\,[\beta + p]^F, \qquad (28)$$

since when $p = 0, f = 1,$ $\qquad q = \dfrac{\frac{C\lambda a}{D}}{U - \frac{C}{\delta}}$

From (28), when $p = \alpha$, $\dfrac{dp}{dt} = 0$. After this $p = \alpha$ continues to satisfy (25) and thus the pressure remains constant till all-burnt position. Thus the equilibrium pressure is the positive root of

$$a\rho + [b\rho - aB - K]\,p - bBp^2 = 0 \qquad (29)$$

Time from ignition to the position of equilibrium is given by

$$t_e = \frac{1}{K_0}\int_0^\alpha \frac{dp}{(\alpha - p)^E(\beta + p)^F}$$

$$= \frac{(\alpha+\beta)^{1-E-F}}{K_0}\int_0^{\frac{\alpha}{\alpha+\beta}} u^{-E}(1-u)^{-F}\,du$$

$$= \frac{1}{k_0\,(\alpha+\beta)^2}\,\beta_{\frac{\alpha}{\alpha+\beta}}\,(1-E,\, E-2) \qquad (30)$$

Thus the time to position of equilibrium and as a matter of fact to any given value of p less than the equilibrium pressure can be expressed in terms of Incomplete Beta Functions provided the corresponding integral is convergent.

Case (ii) $\eta p \sim 1$ is neglected.

In this case equilibrium pressure is given by

$$p_{\text{eq.}} = \frac{a\rho}{K + \frac{BK}{\rho} - b\rho} \qquad (31)$$

(26) gives

$$\frac{dq}{q} = \frac{\frac{KB}{\rho} p_{eq.} \quad +a\rho}{p-p_{eq.}} dp$$

Integrating

$$q = \frac{C\lambda a}{U-\frac{c}{\delta}} e^{Mp}\left(1-\frac{p}{p_{eq.}}\right)^{L}, \qquad (32)$$

where

$$L = 1+\frac{KB}{a\rho^2} p_{eq.}{}^2 \qquad (33a)$$

$$M = \frac{kB}{\rho} p_{eq.} = \frac{bB}{a\rho}\Psi p_{eq.} \qquad (33b)$$

Integrating further and remembering that

$$\lambda = \frac{1}{B}, C = \rho SD, U-\frac{c}{\delta} = V_0,$$

we get

$$\frac{aS\rho}{BV_0} t = \int_0^p e^{-Mp}\left(1-\frac{p}{p_{eq.}}\right)^{-L} dp \qquad (34)$$

Equation (34) is similar to, though not exactly the same as the corresponding equation of Gupta and Mehta. Mp is still very small of the order 0·001 and we can integrate (34) in finite terms to a very good approximation.

Case (iii) $\eta\neq 0$ $n\rho\neq 1$

From (23) equilibrium pressure would be a root of

$$bBp^2[\eta\rho-1]+p[aB(\eta\rho-1)+b\rho-K-\eta BK]+a\rho = 0 \qquad (35)$$

Since for most of the propellants in use $\eta\rho > 1$, the roots of (35) are either both positive or both negative. In the second case the equilibrium pressure does not exist. In the first case, the smaller of the two positive roots determines this pressure.

On integrating (23) twice, we get formulae of the type

$$\frac{dp}{dt} = q = K_1(p+\alpha_1)^{A_1}(p+\beta_1)^{B_1} \qquad (36)$$

and

$$t = \frac{1}{K_1}\int_0^p \frac{dp}{(p+\alpha_1)^{A_1}(p+\beta_1)^{B_1}} \qquad (37)$$

4. Transient Exhaust Curve

After all-burnt $z = 1$, and (17) and (20) give

$$p[U-CN\eta] = CN\lambda \qquad (38)$$

$$\frac{dN}{dt} = -\frac{\psi S_0 p}{C\sqrt{\lambda}} = -\frac{b}{D}\Psi p \qquad (39)$$

From (38) and (39)

$$\frac{1}{p[\lambda+p\eta]^2}\,dp = -\frac{bC\Psi}{D\lambda U}\,dt \qquad \ldots \qquad (40)$$

Integrating

$$\frac{1}{\lambda}\left[\log\frac{p}{p_B} - \log\frac{\lambda+p\eta}{\lambda+p_B\eta}\right] + \left[\frac{1}{\lambda+\eta p} - \frac{1}{\lambda+\eta p_B}\right] = -\frac{bC\Psi}{DU}[t-t_B] \qquad \ldots \qquad (41)$$

If $\eta = 0$, this gives

$$p = p_B\, e^{-\frac{C\lambda b}{D\Psi}U(t-t_B)}$$

or

$$p = p_B\, e^{-\frac{KS}{BU}(t-t_B)} \qquad \ldots \qquad (42)$$

Here U is the volume of the empty motor and S is the constant surface area of the charge. (42) is the same as the corresponding equation of Kershner, as it should be since after all-burnt there is no change in volume available to the gases and we have put $\eta = 0$. (41) is an extension of his result when we take the contribution of η into account.

As noted in the Introduction, the isothermal assumption is very poor for the exhaust curve. We shall, therefore, take the variation of temperature into account. For the sake of simplicity, however, we shall neglect η in the rest of the section.

Making the other assumptions of Kershner, but allowing for the variation of T, we get from (8), (11) and (12), after all-burnt

$$pU = CNRT \qquad \ldots \qquad (43)$$

$$\frac{dN}{dt} = -\frac{\psi S_0}{C(RT)^{\frac{1}{2}}}\,p \qquad \ldots \qquad (44)$$

$$\frac{d}{dt}(NT) = -\frac{\gamma\psi S_0 p}{CR}(RT)^{\frac{1}{2}} \qquad \ldots \qquad (45)$$

From (43), (44) and (45)

$$\frac{dp}{dt}U = -\frac{CR\gamma\psi S_0 p}{CR}(RT)^{\frac{1}{2}} = -\gamma\psi S_0 (RT)^{\frac{1}{2}} p$$

$$= -\gamma C\frac{dN}{dt}RT = -\gamma\frac{1}{N}\frac{dN}{dt}pU$$

so that

$$\frac{1}{p}\frac{dp}{dt} = \gamma\frac{1}{N}\frac{dN}{dt}. \qquad \ldots \qquad (46)$$

Integrating

$$\frac{p}{p_B} = \left(\frac{N}{N_B}\right)^{\gamma} \qquad \ldots \qquad (47)$$

Also from (43) and (47)

$$U\frac{dp}{dt} = -\gamma\psi S_0 p(RT)^{\frac{1}{2}} = -\gamma\psi S_0 p\left(\frac{pU}{CN}\right)^{\frac{1}{2}}$$

or
$$\frac{dp}{dt} = -\mu\, p^{\frac{3}{2}-\frac{1}{2\gamma}}, \qquad (48)$$

where
$$\mu = \frac{\gamma\psi S_0 (p_B)^{\frac{1}{2\gamma}}}{U^{\frac{1}{2}} C^{\frac{1}{2}} N_B^{\frac{1}{2}}} \qquad (49)$$

Integrating (48)
$$p^{-\frac{1}{2}+\frac{1}{2\gamma}} - p_B^{-\frac{1}{2}+\frac{1}{2\gamma}} = \mu\left(\frac{1}{2}-\frac{1}{2\gamma}\right)(t-t_B) \qquad (50)$$

From this the time of reaching a pressure of one atmosphere can be calculated.

Now (47) is not a new result, in as much as it is a consequence of the assumption of the adiabatic expansion made in deriving (7). From the same assumption or from (43) and (46) or from (44) and (45), we get

$$\frac{T}{T_B} = \left(\frac{N}{N_B}\right)^{\gamma-1} = \left(\frac{p}{p_B}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_g}{\rho_{g,B}}\right)^{\gamma-1} \qquad (51)$$

and
$$\frac{p}{NT} = \frac{p_B}{N_B T_B} = \frac{CR}{U} = \text{const.} \qquad (52)$$

which, in terms of the dimensionless variables

$$\frac{T}{T_B} = T',\ \frac{N}{N_B} = N',\ \frac{p}{p_B} = p',\ \frac{\rho_g}{\rho_{g,B}} = \rho_g' \qquad (53)$$

become

$$T' = (N')^{\gamma-1} = (p')^{\frac{\gamma-1}{\gamma}} = (\rho_g')^{\gamma-1} \qquad (54)$$

and
$$p' = N'T' \qquad (55)$$

respectively.

From (50), (51) and (52), we get

$$p = p_B\left[1+\frac{t-t_B}{\tau}\right]^{-\frac{2\gamma}{\gamma-1}} \qquad (56)$$

$$N = N_B\left[1+\frac{t-t_B}{\tau}\right]^{-\frac{2}{\gamma-1}} \qquad (57)$$

$$T = T_B\left[1+\frac{t-t_B}{\tau}\right]^{-2}, \qquad (58)$$

where

$$\tau = \frac{2}{\gamma-1}\frac{1}{\gamma^{\frac{1}{2}}}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{1}{S_0}p_B^{-1+\frac{1}{2\gamma}}(CUN_B)^{\frac{1}{2}} \qquad (59)$$

To apply the above formulae, we require the value of p_B, N_B, T_B and t_B. p_B is in general either the equilibrium pressure α or very nearly equal to it (see, however, next section). To find N_B, we have on integrating (21)

$$N = (1-\Psi)(1-f) + \frac{a}{D}\Psi t, \qquad (60)$$

since initially $f = 1$, $t = 0$, $N = 0$

$$\therefore N_B = 1 - \Psi + \frac{a}{D} \Psi t_B \quad \ldots \quad (61)$$

Knowing p_B, N_B (43) at once determines T_B. We still require the value of t_B This we determine in the next section.

5. Time till All-burnt

Integrating (9), we get

$$D = \int_0^{t_B} (a+bp)dt \quad \ldots \quad (62)$$

Integration will depend on the pressure-time relation we use ; which in its turn depends on the assumptions we make. We discuss the following three cases

Case (i) η is neglected.

In this case, from (28) and (62),

$$D = \frac{1}{K_0} \int_0^{p_B} \frac{a+bp}{(\alpha-p)^E (\beta+p)^F} dp$$

or

$$K_0 D = \frac{a+b\alpha}{(\alpha+\beta)^2} \int_{\frac{\alpha-p_B}{\alpha+\beta}}^{\frac{\alpha}{\alpha+\beta}} u^{-E}(1-u)^{-F} du - \frac{b}{\alpha+\beta} \int_{\frac{\alpha-p_B}{\alpha+\beta}}^{\frac{\alpha}{\alpha+\beta}} u^{1-E}(1-u)^{-F} du \quad (63)$$

Thus, in general, in the equation to determine the pressure at all-burnt and then t_B, the time at all-burnt is given by

$$t_B = \frac{1}{K_0} \int_0^{p_B} \frac{dp}{(\alpha-p)^E (\beta+p)^F}$$

$$= \frac{1}{K_0} \frac{1}{(\alpha+\beta)^2} \int_{\frac{\alpha-p_B}{\alpha+\beta}}^{\frac{\alpha}{\alpha+\beta}} u^{-E}(1-u)^{-F} du \quad \ldots \quad (64)$$

Equations (63) and (64) can be solved by means of the tables of Incomplete Beta Functions.

We may point out here that theoretically two possibilities exist

(i) $E \geqslant 1$ in which case the integral in (30) is divergent and the theoretical time of reaching the equilibrium pressure would be infinite. Of course before this the charge would be all-burnt and thus the equilibrium pressure would not be reached and $p_{\max.} < p_{\text{eq.}}$ In this case $p = p_{\text{eq.}}$ would be asymptotic to the pressure-time curve. In practice, the pressure would rise very rapidly to a value very near the equilibrium pressure and would then slowly tend to the asymptotic value, reaching $p_B = p_{\max.}$ at all-burnt and after then it would fall exponentially according to equation (42).

In this case p_B and t_B have to be determined from (63) and (64).

(ii) $E<1$. In this case the integral in (28) is convergent and determines a finite value for t_e. Let

$$D_e=\frac{a+b\alpha}{K_0(\alpha+\beta)^2}\int_0^{\frac{\alpha}{\alpha+\beta}}u^{-E}(1-u)^{-F}du-\frac{1}{K_0}\frac{b}{\alpha+\beta}\int_0^{\frac{\alpha}{\alpha+\beta}}u^{1-E}(1-u)^{-F}du, \quad (65)$$

then three sub-cases arise:

(*a*) $D_e<D$; (*b*) $D_e=D$; (*c*) $D_e>D$.

(*a*) In the first case, the charge is not completely burnt when the equilibrium pressure is reached and from that time till all-burnt, the pressure will remain constant and the time till all-burnt is given by

$$D_e+\int_{t_e}^{t_b}(a+b\alpha)\,dt=D$$

or

$$t_b=t_e+\frac{D-D_e}{a+b\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

(*b*) When $D_e=D$, $t_b=t_e$ and there are only two stages, the transient rise to equilibrium pressure and then a transient fall to a pressure of one atmosphere.

(*c*) $D_e>D$. Here the *p-t* curve consists of a transient rise until all the powder is burnt at time t_b and a pressure-maximum $p_{\text{max.}}$ (less than $p_{\text{eq.}}$) followed by a transient exhaust. In spite of the disadvantageous shape of the *p-t* curve such a rocket has some advantages—notably a short burning time and a relative insensitivity of both t_b and $p_{\text{max.}}$ to small variations in burning rate. The high-velocity rocket grenade (super-Bazooka) operates on this principle.

Sub-case (*c*) arises when the charge has a thin web, and a similar possibility can also arise when $E\geqslant 1$. Strictly speaking this is the only possibility which can arise when $E\geqslant 1$, since first integral on the L.H.S. of (65) would be infinite, but in general the difference $p_{\text{eq.}}-p_{\text{max.}}$ would be small. We shall talk of the possibility arising in practice if this difference is substantial.

Case (ii): $\eta\rho\sim 1$ *is neglected.*

From (34), since $L>1$, time of reaching the equilibrium pressure would be infinite and consequently $p_B=p_{\text{max.}}<p_{\text{eq.}}$ p_B is given by

$$D=\int_0^{p_B}(a+bp)\frac{D\left(U-\frac{C}{\delta}\right)}{C\lambda a}e^{-Mp}\left(1-\frac{p}{p_{\text{eq.}}}\right)^{-L}dp$$

or

$$\frac{C\lambda a}{U-\frac{C}{\delta}}=\int_0^{p_B}e^{-Mp}(a+bp)\left(1-\frac{p}{p_{\text{eq.}}}\right)^{-L}dp \quad \ldots \quad \ldots \quad (67)$$

The integral on the R.H.S. can be easily evaluated to any degree of approximation, since Mp is small and to sufficient accuracy we can write

$$e^{-Mp}=1-Mp+\tfrac{1}{2}(Mp)^2$$

Then the time to all-burnt is determined from

$$t_B=\frac{D\left(U-\frac{C}{\delta}\right)}{C\lambda a}\int_0^{p_B}e^{-Mp}\left(1-\frac{p}{p_{\text{eq.}}}\right)^{-L}dp \quad \ldots \quad \ldots \quad (68)$$

Case (iii): When neither η nor $\eta p \sim 1$ is neglected.

In this case

$$D = \frac{1}{K_1}\int_0^{p_B} (a+bp)(p+\alpha_1)^{-A_1}(p+\beta_1)^{-B_1}\,dp \quad \ldots \quad (69)$$

and

$$t_B = \frac{1}{K_1}\int_0^{p_B} \frac{dp}{(p+\alpha_1)^{A_1}(p+\beta_1)^{B_1}} \quad \ldots \quad (70)$$

The rest of the discussion will be similar to that for Case (*i*).

6. Equation of Pressure-Time Curve from First Principles

At any instant, the mass-rate of production of gas in the rocket motor must be equal to the mass rate of discharge plus the rate of accumulation of gas in the chamber. This gives

$$S\rho r = C_D A_t p + \frac{d}{dt}(V\rho_g)$$

where S is the constant burning surface area, r is the linear rate of burning, C_D is the discharge coefficient, A_t is the throat area and V is the volume available to the gas at any instant. Now neglecting the co-volume η

$$\frac{dV}{dt} = Sr, \quad \rho_g = Bp \quad \ldots \quad (71)$$

so that

$$S(\rho-\rho_g)r = C_D A_t p + V\frac{d}{dt}(\rho_g) \quad \ldots \quad (72a)$$

or

$$S(\rho-\rho_g)(a+bp) = C_D A_t p + BV\frac{dp}{dt} \quad \ldots \quad (72b)$$

or

$$S(\rho-Bp)\,(a+bp) = C_D A_t p + BV\frac{dp}{dt} \quad \ldots \quad (72c)$$

or

$$\frac{d}{dt}\left[\frac{(\rho-Bp)(a+bp)-Kp}{\dfrac{dp}{dt}}\right] = B(a+bp), \quad \ldots \quad (73)$$

where

$$K = \frac{C_D A_t}{S} \quad \ldots \quad (74)$$

Simplifying (73), we get

$$\frac{d^2p}{dt^2}[a\rho+p(b\rho-aB-K)-bBp^2]$$

$$= \left(\frac{dp}{dt}\right)^2[b\rho-2aB-K-3bBp] \quad \ldots \quad (75)$$

which is the same as (25).

In (72*b*), Kershner takes both V and ρ_g to be constant. Gupta and Mehta take ρ_g to be constant, but allow for the variation in V. We have, here allowed for the variation of both V and ρ_g. This explains why our pressure-time equation differs from theirs. Our equation (23) is still more general in as much as it takes the contribution of η also into account. While ρ_g is small as compared with p, it is still desirable to use an assumption consistently especially when we see that it is possible to do so.

Comparison of (24) and (74) gives

$$b\rho\Psi = \frac{C_D A_t}{S} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (76)$$

This can be otherwise verified, since using (22) and

$$C = S\rho D,\, S_0 = A_t$$

we get

$$C_D = \frac{\Psi}{\sqrt{\lambda}}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (77)$$

which is true for the Isothermal Model.

For 4·5″ rocket, for which

$$\left.\begin{aligned} S &= 555 \text{ in.}^2 \\ \rho &= 0{\cdot}059 \text{ lb./in.}^3 \\ a &= 0{\cdot}28 \text{ in./sec.} \\ b &= 0{\cdot}00037 \text{ in.}^3\text{/lb. sec.} \\ C_D &= 0{\cdot}007 \frac{\text{lb. (mass)}}{\text{lb.(force)}-\text{sec.}} \\ A_t &= 2{\cdot}64 \text{ in.}^2 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

we find that $\Psi = 1{\cdot}5$ (nearly) (79)

Thus the dimensionless parameter Ψ which has a value of the order of 0·1 for leaking guns, of the order of 0·4–0·6 for R.C.L. and H/L guns, has a value of the order of 1·5 for rockets.

Consequently, the method of expansion in powers of Ψ which is useful for leaking guns (Jain, 1956) and is of limited application for R.C.L. guns (Thiruvenkatachar and Venkatesan, 1953) and H/L guns (Thiruvenkatachar, 1956) cannot at all be used for rockets.

We may also note here that since a single perforated tubular charge burns from both sides, D stands for $\frac{r_1-r_2}{2}$ where r_1 and r_2 are the radii of the outer and inner bounding cylinders respectively.

7. General form-function

We make here the last four assumptions of Kershner, but instead of a constant-burning surface, we consider the more general case, when the charge has the quadratic form-function. The equations for this case are

$$p\left[U - \frac{C}{\delta}\right] = CN\lambda \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (80)$$

$$D\frac{df}{dt} = -(a+bp) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (81)$$

$$z = (1-f)(1+\theta f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

$$\frac{dN}{dt} = \frac{dz}{dt} - \frac{\psi S_0 p}{C\sqrt{\lambda}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83)$$

From (22), (80), (81), (82) and (83)

$$\frac{dp}{dt}\left[U-\frac{C}{\delta}\right] = C\lambda\left[\frac{dz}{df}\frac{df}{dt}-\frac{\psi S_0 p}{C\sqrt{\lambda}}\right]$$

$$= C\lambda\left[(-1+\theta-2\theta f)\left(\frac{a+bp}{-D}\right)-\frac{b\Psi}{D}p\right]$$

or

$$\frac{d}{dt}\left[\frac{BV_0\frac{dp}{dt}+\frac{Cb}{D}\Psi p}{a+bp}\right] = -2\theta\frac{C}{D}\frac{a+bp}{D} \quad \ldots \quad \ldots \quad (84)$$

Integrating

$$\frac{1}{2}\left\{\frac{BV_0\frac{dp}{dt}-\frac{Ca\Psi}{D}}{a+bp}\right\}^2$$

$$= \text{Const.}-\frac{2\theta C}{D^2}\left[BV_0 p-\frac{Ca\Psi}{D}t\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (85)$$

When $t = 0, f = 1, p = 0,$

$$\frac{dp}{dt} = \frac{aC}{BV_0 D}(1+\theta) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (86)$$

and the constant in (85) can be evaluated easily. In general, equation (85) will have to be integrated numerically to get p explicitly as a function of t.

Alternatively equating the mass-rate of production of gas to the mass-rate of discharge plus the mass-rate of accumulation

$$C\frac{dz}{dt} = C_D A_t p+\frac{d}{dt}(V\rho_g)$$

but

$$\rho_g = Bp, \quad \frac{dV}{dt} = \frac{1}{\rho}C\frac{dz}{dt}$$

$$\therefore \quad C\frac{dz}{dt}\left(1-\frac{\rho_g}{\rho}\right) = C_D A_t p+BV\frac{dp}{dt}$$

or

$$C\frac{dz}{df}\left(-\frac{a+bp}{D}\right)\left(1-\frac{\rho_g}{\rho}\right) = C_D A_t p+BV\frac{dp}{dt} \quad \ldots \quad \ldots \quad (87)$$

This is a generalisation of (72).

$$\therefore \quad \frac{d}{dt}\left[\frac{C_D A_t p+BV\frac{dp}{dt}}{\frac{a+bp}{D}C\left(1-\frac{\rho_g}{\rho}\right)}\right] = -2\theta\frac{a+bp}{D} \quad \ldots \quad \ldots \quad (88)$$

We consider three cases.

Case (*i*): (Kershner approximation.) We regard V and ρ_g to be constant. In this case (88) can be regarded as a generalisation of (84) on taking variation of volume into account. The effect of this variation is simulated by reducing the form-factor by about 2%. The first integral is similar to (85) and a first order differential equation has to be numerically integrated.

Case (ii): V is regarded as constant, but ρ_g is taken as B_p. In this case, we have to integrate numerically a second order differential equation.

Case (iii):

(*a*) ρ_g is constant, variation of V is taken into account.

(*b*) Variation of both is taken into account.

Here the problem is reduced to the integration of a third order differential equation:

$$\frac{d}{dt}\left[\frac{X}{Y}\right] = \theta(a+bp), \quad \ldots \quad (89)$$

where

$$X = \frac{d^2p}{dt^2}\left[\frac{C}{D}(1+\theta)(a+bp)\left(1-\frac{Bp}{\rho}\right) - C_D A_t p\right]$$
$$+\left(\frac{dp}{dt}\right)^2\left[(1+\theta)\frac{\beta C}{D\rho}(a+bp) - \frac{C}{D}(1+\theta)\left(b-\frac{Ba}{\rho}-\frac{2bBp}{\rho}\right)+C_D A_t\right]$$
$$+\frac{dp}{dt}\,2\theta\frac{C}{D^2}(a+bp)^2\left(1-\frac{Bp}{\rho}\right)$$

and

$$Y = \frac{d^2p}{dt^2}\left[\frac{2C}{D}(a+bp)\left(1-\frac{Bp}{\rho}\right)\right]$$
$$+\left(\frac{dp}{dt}\right)^2\left[\frac{2BC}{\rho D^2}(a+bp) - \frac{2BC}{D}\left(b-\frac{B}{\rho}-\frac{2bBp}{\rho}\right)\right]$$

It is easily verified that when $\theta = 0$, (89) reduces to (25).

The initial conditions for the integration of (89) are:

$$t = 0,\ p = 0,\ \frac{dp}{dt} = \frac{aC}{BV_0D}(1+\theta)$$

$$\frac{d^2p}{dt^2} = \frac{1}{BV_0}\left[\frac{aC^2}{BV_0D^2}(1+\theta)^2\left(b-\frac{2aB}{\rho}\right)-\frac{2\theta Ca^2}{D^2}\right.$$
$$\left.-\frac{aC}{BV_0D}(1+\theta)\,C_D A_t\right]. \quad \ldots \quad (90)$$

If the charge has its form-function in the cubic form

$$z = (1-f)(1+\theta f+\theta' f^2) \quad \ldots \quad (91)$$

(84) is replaced by the third order differential equation

$$\frac{d}{dt}\left\{\frac{D}{a+bp}\frac{d}{dt}\left[\frac{BV_0\dfrac{dp}{dt}+\dfrac{Cb}{D}\Psi p}{\dfrac{C}{D}(a+bp)}\right]\right\} = 6\theta'\frac{a+bp}{D}. \quad \ldots \quad (92)$$

We may point out here that unlike (25), equations (85) and (88) are not satisfied by $p =$ const. and so an equilibrium pressure in the strict sense may not exist for the general form-function.

8. Composite Charges

Let the composite charge consist of n component charges with parameters $F_i, C_i, D_i, b_i, \theta_i$ and with same value of $\frac{a_i}{b_i}$ and γ_i. This will include the important case when the component charges have the same composition, but different shapes and ballistic sizes. Let F, C, D, b, θ, γ, z, f refer to the equivalent charge, then the form-function for the rth stage for the equivalent charge is (Kapur, 1956)

$$z = A_r + B_r(1-f) - E_r(1-f)^2 \qquad (93)$$

where

$$A_r = \sum_{i=1}^{r-1} \lambda_i \qquad (94a)$$

$$B_r = \sum_{i=r}^{n} \lambda_i k_i (1+\theta_i) \qquad (94b)$$

$$E_r = \sum_{i=r}^{n} \lambda_i k_i^2 \theta_i \qquad (94c)$$

$$\lambda_i = \frac{F_i C_i}{FC} \qquad (94d)$$

$$k_i = \frac{D_i/B_i}{D/B} \qquad (94e)$$

Proceeding as in section 7, the differential equation for the pressure-time curve in the rth stage, we get

$$\frac{1}{2}\left[\frac{BV_0 \frac{dp}{dt} - \frac{Ca}{D}\Psi}{a+bp}\right]^2 = -2E_r\left[BV_0 p - \frac{Ca\Psi}{D}t\right] + K_r, \qquad (95)$$

where the constant K_1 is determined from the condition

$$t=0,\ f=1,\ p=0,\ \frac{dp}{dt} = \frac{aC}{BV_0 D}B_1 \qquad (96)$$

but to determine K_r from the conditions at the end of the $(r-1)$th stage, we have to remember that $\frac{dp}{dt}$ can be discontinuous in crossing from $(r-1)$th stage to rth. Since

$$-\frac{BV_0}{C}\frac{dp}{dt} = \left\{\frac{b\Psi}{D}p + \frac{a+bp}{D}\frac{dz}{df}\right\} = \left\{\frac{b\Psi}{D}p + \frac{a+bp}{D}[-B_r + 2E_r(1-f)]\right\}$$

and $\frac{dz}{df}$ is discontinuous at the end of the rth stage unless $\theta_r = 1$. Thus at the end of the $(r-1)$th stage, the initial conditions for the determination of K_r are

$$t = t_{r-1},\ p = p_{r-1}$$

$$\frac{dp}{dt} = -\frac{C}{BV_0}\left\{\frac{b\Psi}{D}p_{r-1} + \frac{a+bp_{r-1}}{D}\left[-B_r + \frac{2E_r}{k_{r-1}}\right]\right\} \qquad (97)$$

where suffix $(r-1)$ corresponds to the end of the $(r-1)$th stage.

9. Particular case of tabular component charges

In this case $\theta_i = 0$ ($i = 1, 2, 3, \ldots\ldots n$) and therefore from (94)

$$B_r = \sum_{i=r}^{n} \lambda_i k_i$$

$$E_r = 0$$

∴ from (95)

$$\frac{BV_0 \dfrac{dp}{dt} - \dfrac{Ca\Psi}{D}}{a+bp} = \nu_r \quad \ldots \quad (98)$$

Integrating

$$p = \mu_r e^{\frac{b\nu_r t}{BV_0}} - \frac{\dfrac{Ca\Psi}{D} + a\nu_r}{b\nu_r} \quad \ldots \quad (99)$$

Now

$$\nu_1 = \frac{\dfrac{aC}{D}B_1 - \dfrac{aC\Psi}{D}}{a} = \frac{C}{D}[B_1 - \Psi]$$

$$\mu_1 = \frac{aB_1}{b[B_1 - \Psi]}$$

∴ in the first stage

$$p = \mu_1\left(e^{\frac{b\nu_1 t}{BV_0}} - 1\right) \quad \ldots \quad (100)$$

Since

$$D\frac{df}{dt} = -b(p+p_0)$$

$$\int_1^{1-\frac{1}{k_1}} D\,df = -b\int_0^{t_1} (p+p_0)\,dt$$

or

$$\frac{D}{k_1} = b\left\{(p_0 - \mu_1)t_1 + \mu_1 \frac{BV_0}{b\nu_1}\left(e^{\frac{b\nu_1 t_1}{BV_0}} - 1\right)\right\} \quad \ldots \quad (101)$$

From this we can determine t_1 and then p_1, the pressure at the end of the first stage is given by

$$p_1 = \frac{\nu_1}{BV_0}\left[\frac{D}{k_1} - b(p_0 - \mu_1)t_1\right] \quad \ldots \quad (102)$$

From (97) and (98)

$$\nu_2 = \frac{C}{D}[B_2 - \Psi]$$

and in general

$$\nu_r = \frac{C}{D}[B_r - \Psi] \quad \ldots \quad (103)$$

Similarly from (99), μ_2 is given by

$$\mu_2 = e^{-\frac{bv_2 t_1}{BV_0}}\left[p_1 + \frac{a}{b}\frac{B_2}{B_2 - \Psi}\right],$$

and in general

$$\mu_r = e^{-\frac{bv_r t_{r-1}}{BV_0}}\left[p_{r-1} + \frac{a}{b}\frac{B_r}{B_r - \Psi}\right] \quad \ldots \quad \ldots \quad \ldots (104)$$

From (99), (103) and (104), the pressure-time curve is known in all the stages of burning.

After all-burnt, the pressure-time curve is the same as in section 4.

The pressure-time derivative will, of course, be discontinuous at the ends of all the stages of burning and at all-burnt, for the case of tubular component charges.

If some of the component charges have the same effective ballistic sizes, they burn out simultaneously and for all practical purposes behave as a single charge. No separate discussion is therefore necessary.

We can easily take into account the change in volume. Proceeding as in section 3, the differential equation for the rth stage is

$$\frac{d^2p}{dt^2}[a\rho B_r + p(b\rho B_r - K - aBB_r) - bBB_r p^2]$$

$$= \left(\frac{dp}{dt}\right)^2 [b\rho B_r - K - 2aBB_r - 3bBB_r p] \quad \ldots \quad \ldots \quad \ldots (105)$$

If $B_r = 1$, it reduces to (25). Comparison of (25) and (105) shows that the latter can be obtained formally from the former by replacing ρ by ρB_r and B by BB_r.

The integral of this can be easily expressed in terms of Incomplete Beta Functions. The initial conditions for each stage and the effect of discontinuity in $\frac{dp}{dt}$ on these can be taken into account as indicated earlier.

Summary

In the present paper, we have deduced the equations of Internal Ballistics of Solid-Fuel rockets from those of R.C.L. guns. We have investigated the pressure-time curve taking into account the change in volume available to the gases till all-burnt and variation of temperature into account after all-burnt. The treatment has been extended to single charges with quadratic form-functions and to composite charges.

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