ON THE (RELATIVISTIC) STATISTICAL THERMODYNAMICS OF AN ASSEMBLY IN MASS-MOTION

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(Communicated by D. S. Kothari, F.N.I.)

(*Received December 23, 1956; read January 12, 1957*)

The present paper deals with a relativistic study of the Statistical Thermodynamics of an ideal gaseous assembly in mass-motion. In order to determine the microcanonical distribution appropriate to the laboratory system K, we introduce the conservation of the net linear momentum $\mathbf{P}$ of the assembly as an additional constraint besides the usual constraints of the total number N of the particles and the total energy E.* The distribution function thus obtained enables us to set up expressions for various thermodynamical quantities pertaining to the assembly. The transformation equations connecting these expressions with the corresponding ones appropriate to the rest system K^0 are then derived and it is established that the state of degeneracy of the assembly is identical for all observers.

In the last section of the paper we discuss certain dynamical aspects of our results. The Lagrangian and the Hamiltonian are introduced, the latter being identical with the enthalpy $(E+PV)$.† Since the assembly under consideration consists of a thermodynamic fluid contained in a vessel under the influence of the 'external' pressure from the walls of the vessel, it is actually the enthalpy $(E+PV)$ which represents the energy transferred by the moving fluid. Consequently the assembly behaves as if it possessed an inertial mass given by $(E+PV)/c^2$.

1. The restrictive conditions controlling the microcanonical distribution with respect to the laboratory system K are

$$\left.\begin{aligned} \sum_j n_j &= N, \\ \sum_j n_j \epsilon_j &= E, \\ \sum_j n_j \mathbf{p}_j &= \mathbf{P}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The probability $W(n_1, n_2, \ldots, n_j, \ldots)$ that there are n_j particles in the g_j-fold degenerate energy level is given by

$$lnW = \sum_j [n_j \,.\, ln(g_j/n_j - a) - g_j/a \,.\, ln(1 - a \,.\, n_j/g_j)], \quad \ldots \quad \ldots \quad (2)$$

where $a = +1$ or -1, according as the particles constituting the assembly are fermions or bosons, respectively. The limiting case $a = 0$ gives the Boltzmannian results.

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* A similar treatment has also been given by W. Band (cf. Introduction to Quantum Statistics, D. Van Nostrand Company, 1955). He has, however, confined himself to the non-relativistic case.

† The symbols $\mathbf{P}$ and P representing the momentum and the pressure, respectively, should not be confused with each other.

Maximizing the probability of distribution under the conservation-constraints (1), we have for any variation δn_j of the n_j

$$\left.\begin{aligned}\delta(ln W) &= \sum_j ln(g/n_j - a)\delta n_j = 0\\ \delta N &= \sum_j \delta n_j = 0, \quad \delta E = \sum_j \epsilon_j \delta n_j = 0, \quad \delta \mathbf{P} = \sum_j \mathbf{p}_j \delta n_j = 0\end{aligned}\right\} \quad .. \quad (3)$$

Let α, β and $\boldsymbol{\gamma}$ be the Lagrange's undetermined multipliers that take care of the conservation of the total number of particles, the total energy (including the rest energy) and the total momentum of the assembly, respectively. We then have for any arbitrary choice of the δn_j

$$\delta(ln W) + \alpha\delta N + \beta\delta E + \boldsymbol{\gamma} \cdot \delta\mathbf{P} = 0, \quad .. \quad .. \quad .. \quad (4)$$

Hence it follows that

$$ln(g_j/n_j - a) + \alpha + \beta\epsilon_j + \boldsymbol{\gamma} \cdot \mathbf{p}_j = 0$$

$$\therefore \quad n_j = \frac{g_j}{\exp(-\alpha - \beta\epsilon_j - \boldsymbol{\gamma} \cdot \mathbf{p}_j) + a} \quad .. \quad .. \quad .. \quad (5)$$

The entropy of the assembly in its equilibrium state is given by

$$S = kln W_{\text{max.}}, \quad .. \quad .. \quad .. \quad .. \quad (6)$$

and is consequently obtained by substituting (5) into (2), so that

$$\frac{S}{k} = -\alpha N - \beta E - \boldsymbol{\gamma} \cdot \mathbf{P} + \sum_j g_j/a \,.\, ln[1 + a \exp(\alpha + \beta\epsilon_j + \boldsymbol{\gamma} \cdot \mathbf{p}_j)] \quad .. \quad (7)$$

The first law of thermodynamics for reversible processes, including adiabatic changes in the imposed momentum, then reads

$$dE = TdS - PdV + \mu dN + \mathbf{v} \cdot d\mathbf{P}, \quad .. \quad .. \quad .. \quad (8)$$

where the various symbols have their usual meanings and

$$\mathbf{v} = \left(\frac{\partial E}{\partial \mathbf{P}}\right)_{S,\, V,\, N}$$

is velocity of mass-motion. It further follows that

$$\left(\frac{\partial S}{\partial N}\right)_{V,\, E,\, \mathbf{P}} = -\frac{\mu}{T}, \quad \left(\frac{\partial S}{\partial E}\right)_{V,\, N,\, \mathbf{P}} = \frac{1}{T}, \quad \left(\frac{\partial S}{\partial \mathbf{P}}\right)_{V,\, N,\, E} = -\frac{\mathbf{v}}{T} \quad .. \quad (9)$$

Comparing these results with the corresponding ones deduced from (40) and (6), one readily obtains

$$\alpha = \frac{\mu}{kT}, \; \beta = -\frac{1}{kT}, \; \boldsymbol{\gamma} = \frac{\mathbf{v}}{kT}. \quad .. \quad .. \quad .. \quad (10)$$

We take the z-axis of our co-ordinate system in the direction of $\mathbf{v}$, so that the distribution function (5) may be written as

$$n(\mathbf{p}) = \frac{g(\mathbf{p})}{\exp[-\alpha + (\epsilon - vp_z)/kT] + a} \quad .. \quad .. \quad .. \quad (5')$$

For an observer in the rest system, $v = 0$ and hence the distribution law in K^0 would be

$$n^0(\mathbf{p}^0) = \frac{g^0(\mathbf{p}^0)}{\exp\left[-\alpha_0+\frac{\epsilon^0}{kT_0}\right]+a} \qquad \text{(11)}$$

According to the Lorentz transformation of the four-vector $\left(\mathbf{p}, \frac{i}{c}\epsilon\right)$, we have

$$\epsilon - vp_z = \epsilon^0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}.$$

It follows that the functional dependence of the occupation number in one system of reference is related to that in the other system through a Lorentz transformation if

$$T = T_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}, \qquad \text{(12)}$$

which is the transformation equation for temperature.

2. The expression for the total number of particles of the assembly, as observed in K^0, follows from the distribution function (11):

$$N = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0}{\exp\left(-\alpha_0+\frac{\epsilon^0}{kT_0}\right)+a} \qquad \text{(13)}$$

An observer in K, however, gets from his distribution function (5′) and the restrictive conditions (1)

$$N = \frac{V}{h^3}\int_{-\infty}^{\infty} \frac{d^3p}{\exp\left(-\alpha+\frac{\epsilon - vp_z}{kT}\right)+a} \qquad \text{(14)}$$

We make the substitutions

$$\left.\begin{aligned} p_x = p_x^0,\ p_y = p_y^0,\ p_z &= \frac{p_z^0+\frac{v}{c^2}\epsilon^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \\ \epsilon &= \frac{\epsilon^0+vp_z^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \end{aligned}\right\} \qquad \text{(15)}$$

so that

$$c^2(p_x^2+p_y^2+p_z^2)-\epsilon^2 = c^2(p_x^{02}+p_y^{02}+p_z^{02})-\epsilon^{02} = -m^2c^4$$

and

$$d^3p = d^3p^0\frac{\partial(p_x, p_y, p_z)}{\partial(p_x^0, p_y^0, p_z^0)} = d^3p^0\frac{1+vp_z^0/\epsilon^0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad \text{(16)}$$

Also as a consequence of the Lorentz contraction

$$V = V_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} \qquad \text{(17)}$$

Equation (14) then becomes

$$N = \frac{V_0}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0(1+vp_z^0/\epsilon^0)}{\exp\left(-\alpha+\dfrac{\epsilon^0}{kT_0}\right)+a} \quad \ldots \quad (18)$$

The second part in this integral vanishes and the remaining integral, when compared with its equal (13), yields the result

$$\alpha = \alpha_0 \quad \ldots \quad (19)$$

We thus establish the invariance of the first Lagrange multiplier. It is to be noted that this result and many others that follow owe for their simplicity to the fact that we are including in ϵ_j's the rest energies of the particles.

We now define, after Lorentz, the pressure P of the moving assembly. Consider for this purpose a surface element of area $d\mathbf{s}$ somewhere in the gas. The pressure will then be given by the total 'normal' momentum transported per second per unit area through this element in the direction of the positive normal $\hat{\mathbf{n}}$. Since the element under consideration is itself moving with a velocity $\mathbf{v}$, a particle whose momentum and velocity are $\mathbf{p}$ and $\mathbf{u}$ respectively, will be able to traverse it at time t only if at time zero the position vector of the element with respect to it (the particle) was $(\mathbf{u}-\mathbf{v})t$. Thus all such particles as lying within the cylindrical region of length $(\mathbf{u}-\mathbf{v})$ and area of cross-section $d\mathbf{s}$ will be able to traverse the element in one second. Their number will be given by

$$n(\mathbf{p})\,.\,\frac{1}{V}\,[(\mathbf{u}-\mathbf{v})\,.\,d\mathbf{s}] \quad \ldots \quad (20)$$

Each of these particles carries a normal momentum so that the contribution of such particles to the net normal momentum transferred per second per unit area in the direction of the positive normal will be

$$n(\mathbf{p})\,.\,\frac{1}{V}\,[(\mathbf{u}-\mathbf{v})\,.\,d\mathbf{s}]\,.\,(\mathbf{p}\,.\,\hat{\mathbf{n}}) \quad \ldots \quad (21)$$

The pressure P will be obtained by summing up (21) over all $\mathbf{p}$. We thus have

$$P = \frac{1}{h^3}\int_{-\infty}^{\infty} \frac{d\mathbf{p}(\mathbf{p}\,.\,\hat{\mathbf{n}})[(\mathbf{u}-\mathbf{v})\,.\,\hat{\mathbf{n}}]}{\exp\left[-\alpha+\dfrac{\epsilon-vp_z}{kT}\right]+a} \quad \ldots \quad (22)$$

Without loss of generality, we take $\hat{\mathbf{n}}$ in the +ve direction of the z-axis, i.e., in the direction of $\mathbf{v}$, the velocity of mass-motion.

$$\therefore\quad P = \frac{1}{h^3}\int_{-\infty}^{\infty} \frac{d^3p\,.\,p_z(u_z-v)}{\exp\left[-\alpha+\dfrac{\epsilon-vp_z}{kT}\right]+a} \quad \ldots \quad (23)$$

The corresponding expression in the rest system would be

$$P_0 = \frac{1}{h^3}\int_{-\infty}^{\infty} \frac{d^3p^0.p_z^0u_z^0}{\exp\left[-\alpha_0+\dfrac{\epsilon^0}{kT_0}\right]+a} \quad \ldots \quad (24)$$

When we simplify the integral in (23) with the help of our substitutions (15) and (16) and our relations (12) and (19) it reduces to that in (24) so that pressure is an invariant under a Lorentz transformation:

$$P = P_0 \quad \ldots \quad (25)$$

Setting up similarly the integrals for the total energy E and the total momentum $\mathbf{P}$, the following results readily follow (Pathria, 1955):

$$E = \frac{E_0 + P_0 V_0 \frac{v^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \quad \ldots \quad (26)$$

and

$$\mathbf{P} = \frac{E_0 + P_0 V_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \frac{\mathbf{v}}{c^2}. \quad \ldots \quad (27)$$

Further, the equation (26) coupled with (17) and (25) gives

$$E + PV = \frac{E_0 + P_0 V_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \quad \ldots \quad (28)$$

From (26) and (27) one gets

$$\frac{E - \mathbf{v} \cdot \mathbf{P}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = E_0, \quad \ldots \quad (29)$$

while (27) and (28) give

$$\frac{(E + PV) - \mathbf{v} \cdot \mathbf{P}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = E_0 + P_0 V_0 \quad \ldots \quad (30)$$

We now take up the transformation of entropy. We have from (7) and (10)

$$\frac{S}{k} = -\alpha N + \frac{E - \mathbf{v} \cdot \mathbf{p}}{kT} + \frac{1}{ah^3} \int_{-\infty}^{\infty} V ln \left[1 + a \cdot \exp\left(\alpha - \frac{\epsilon - v p_x}{kT}\right)\right] d^3p \quad \ldots \quad (31)$$

The corresponding expression in the rest system would be

$$\frac{S_0}{k} = -\alpha_0 N + \frac{E_0}{kT_0} + \frac{1}{ah^3} \int_{-\infty}^{\infty} V_0 ln \left[1 + a \exp\left(\alpha_0 - \frac{\epsilon^0}{kT_0}\right)\right] d^3p^0. \quad \ldots \quad (32)$$

The first terms on the right hand sides of (31) and (32) are equal by virtue of (19), the second terms are so by virtue of (12) and (29) while the integrals involved also turn out to be equal when we simplify the first one with the help of our substitutions. We, therefore, get

$$S = S_0, \quad \ldots \quad (33)$$

i.e., the entropy of the assembly is also an invariant under a Lorentz transformation. This result is in accordance with the statistical mechanical interpretation of entropy in terms of probability, since the probability of finding a system in a given state should evidently be independent of the velocity of the observer relative to it.

Before we go over to other thermodynamical quantities, we introduce the q-potential of Kramers (D.ter Harr, 1954), adapted to the case of an assembly in mass-motion. It is a dimensionless quantity and will be defined by the equation

$$q = ln W_{\text{max.}} + \alpha N + \beta E + \boldsymbol{\gamma} \cdot \mathbf{P} \quad \ldots \quad (34)$$

We have for its variation

$$dq = \frac{dS}{k} + (\alpha\, dN + N\, d\alpha) + (\beta\, dE + E\, d\beta) + (\boldsymbol{\gamma} \,.\, d\mathbf{P} + \mathbf{P} \,.\, d\boldsymbol{\gamma}).$$

With the help of (8) and (10), it reduces to

$$dq = \frac{P}{kT} dV + N\, d\alpha + E\, d\beta + \mathbf{P} \,.\, d\boldsymbol{\gamma}.$$

It follows that

$$N = \left(\frac{\partial q}{\partial \alpha}\right)_{V,\beta,\gamma}, \quad E = \left(\frac{\partial q}{\partial \beta}\right)_{V,\gamma,\alpha}, \quad \mathbf{P} = \left(\frac{\partial q}{\partial \boldsymbol{\gamma}}\right)_{V,\alpha,\beta}$$

and

$$P = kT\left(\frac{\partial q}{\partial V}\right)_{\alpha,\beta,\gamma} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

The last result will give us the equation of state, once q has been computed. Comparing (34) with (6) and (7) one readily obtains

$$q = \sum_j g_j/a \; ln[1 + a \exp(\alpha + \beta\epsilon_j + \boldsymbol{\gamma} \,.\, \mathbf{p}_j)], \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

which means that q is the logarithm of the Grand Partition Function. Expressed in the form of an integral,

$$q = \frac{V}{ah^3} \int\!\!\int\!\!\int_{-\infty}^{\infty} ln\left[1 + a \exp\left(\alpha - \frac{\epsilon - vp_z}{kT}\right)\right] dp_x\, dp_y\, dp_z \qquad \ldots \quad (37)$$

Integrating by parts with respect to the variable p_z, the integrated portion vanishes and we are left with

$$q = \frac{-V}{ah^3} \int\!\!\int\!\!\int_{-\infty}^{\infty} p_z \cdot \frac{a \,.\, \exp\left[\alpha - \frac{\epsilon - vp_z}{kT}\right] . \frac{-1}{kT} \cdot \left(\frac{\partial \epsilon}{\partial p_z} - v\right)}{1 + a \exp\left[\alpha - \frac{\epsilon - vp_z}{kT}\right]} dp_x\, dp_y\, dp_z$$

$$= \frac{V}{kT} \cdot \frac{1}{h^3} \int_{-\infty}^{\infty} \frac{p_z(u_z - v) d^3p}{\exp\left[-\alpha + \frac{\epsilon - vp_z}{kT}\right] + a}.$$

We thus arrive at the important thermodynamical result

$$q = \frac{PV}{kT} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Combining (35) and (38) we can write

$$q = V\left(\frac{\partial q}{\partial V}\right)_{\alpha,\beta,\gamma}.$$

For a homogeneous assembly $\frac{P}{kT}$ is everywhere the same, with the result that Kramers potential is directly proportional to the size of the assembly. The transformation equations (12), (17) and (25) for T, V and P respectively, now show that q-potential, and consequently the grand partition function, is an invariant:

$$q = q_0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

We finally get

$$TS = -\mu N + E - \mathbf{v}\,.\,\mathbf{P} + PV.$$

Now the Gibbs potential G and the Helmholtz free energy F are given by

$$G = E + PV - TS = \mu N + \mathbf{v}\,.\,\mathbf{P} \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

and

$$F = E - TS = G - PV. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

The corresponding equations in the rest system would be

$$G_0 = \mu_0 N, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

and

$$F_0 = G_0 - P_0 V_0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

For the transformation of the chemical potential $\mu(=\alpha kT)$ we have from (12) and (19)

$$\mu = \mu_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

Equations (40), (42) and (44) give the transformation equation for the Gibbs potential:

$$\frac{G - \mathbf{v}\,.\,\mathbf{P}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = G_0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

The transformation equation for the Helmholtz free energy is obtained by combining (17), (25), (41), (43) and (45) so that

$$\frac{F - \mathbf{v}\,.\,\mathbf{P}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = F_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

At this stage we take up the question of invariance, or otherwise, of the criterion of degeneracy for a Fermi-Dirac or Bose-Einstien gas. For this purpose we have to find out to what extent the statistical distribution among the particles differs from the classical distribution. This is done by comparing the lower limit of the quantity $\exp\left[-\alpha + \frac{\epsilon - vp_z}{kT}\right]$ with unity. Now, from the results obtained above it follows that this quantity, being equal to $\exp\left[-\alpha_0 + \frac{\epsilon^0}{kT_0}\right]$, would have the same magnitude in all inertial systems, and consequently each observer finds the assembly to be in the same degree of degeneracy. We thus establish that the 'degeneracy-criterion' is Lorentz-invariant.

3. We now take up the dynamical aspect of our results, particularly of the equations (27) and (28) whereby

$$\mathbf{P} = \frac{E_0 + P_0 V_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} \frac{\mathbf{v}}{c^2} = \frac{E + PV}{c^2} \mathbf{v}.$$

If we keep the internal state, as measured in the rest system, of the assembly intact and accelerate it in a direction perpendicular to that of its mass-motion so that v^2

remains unchanged, then the effective transverse mass, given by the ratio of the force acting to the acceleration produced, is evidently given by the quotient $\mathbf{P}/\mathbf{v}$:

$$M_T = \frac{(E_0+P_0V_0)/c^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{E+PV}{c^2}.$$

It appears natural to regard $(E_0+P_0V_0)/c^2$ as the effective rest mass of the assembly. So we may denote it by M_0. Then we have

$$M_T = \frac{M_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \quad \ldots \quad (47)$$

If, however, we accelerate the assembly in the direction of the mass-motion (keeping its internal state, of course, intact), then the effective longitudinal mass will be given by the differential coefficient $\left(\frac{d\mathbf{P}}{d\mathbf{v}}\right)_{M_0}$:

$$M_L = \frac{M_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \quad \ldots \quad (48)$$

The expressions (47) and (48) have the same form as those for the velocity dependence of the mass of the Lorentz electorn.

It is interesting to note that the momentum $\mathbf{P}$ may be derived from the kinetic potential $L(\mathbf{v}, P, S, N)$ which is defined by

$$L = \mathbf{v}\,.\,\mathbf{P}-E-PV \quad \ldots \quad (49)$$

$$= -M_0c^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} \quad \ldots \quad (50)$$

Transformation equation for L follows immediately

$$L = L_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} \quad \ldots \quad (51)$$

For the variation of L, we have from (49) and (8)

$$dL = \mathbf{P}\,.\,d\mathbf{v}-V\,dP-T\,dS-\mu\,dN, \quad \ldots \quad (52)$$

so that

$$\mathbf{P} = \left(\frac{\partial L}{\partial \mathbf{v}}\right)_{P,\,S,\,N} \quad \ldots \quad (53)$$

Since P, S, N are invariants, their constancy during differentiation can be secured by keeping the internal state of the assembly intact, which would imply a constant Mo. Thus we may write

$$\mathbf{P} = \left(\frac{\partial L}{\partial \mathbf{v}}\right)_{M_0}.$$

Equation (50) immediately gives

$$\mathbf{P} = \frac{M_0\mathbf{v}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}.$$

We thus see that L plays the rôle of the 'Lagrangian function'. Finally we may go over from the Lagrangian function $L(\mathbf{v}, P, S, N)$ to the Hamiltonian function $H(\mathbf{P}, P, S, N)$ by the well-known step

$$H = \mathbf{P}\,.\,\mathbf{v} - L \qquad \ldots \qquad (54)$$

$$= E + PV \qquad \ldots \qquad (55)$$

Obviously the transformation equation for H is

$$H = \frac{H_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} \qquad \ldots \qquad (56)$$

or

$$\frac{H - \mathbf{v}\,.\,\mathbf{P}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = H_0, \quad \left(\because \quad \mathbf{v}\,.\,\mathbf{P} = H\,.\,\frac{v^2}{c^2}\right). \qquad \ldots \qquad (57)$$

For the variation of H, we have from (54) and (52)

$$dH = \mathbf{v}\,.\,d\mathbf{P} + V\,dP + T\,dS + \mu\,dN$$

or

$$\mathbf{v} = \left(\frac{\partial H}{\partial \mathbf{P}}\right)_{M_0} \qquad \ldots \qquad (58)$$

which is one of the canonical equation of Hamilton; the other one does not appear because we are dealing with a time-independent momentum. Further, since we can write

$$H = \frac{M_0 c^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

$$= M_0 c^2 \left[1 + \left(\frac{\mathbf{P}}{M_0 c}\right)^2\right]^{\frac{1}{2}},$$

the equation (58) is easily verified.

4. It may be mentioned in passing that the general treatment of the text involves the following particular cases of interest:

(i) When there is no restriction on the number of particles in the assembly, i.e., the conservation-constraint

$$\sum_j n_j = N$$

is absent. This entails the loss of a thermodynamic degree of freedom characterized by the vanishing of the 'chemical potential'. Thus we have in this case

$$\alpha = \alpha_0 = 0,$$

and

$$\mu = \mu_0 = 0.$$

The immediate effect of this would be to modify the equations (40) and (42) to

$$G = \mathbf{v}\,.\,\mathbf{P},$$

and

$$G_0 = 0.$$

The notable examples of this case are an assembly of photons and an assembly of phonons (London, 1954).

(ii) $kT_0 \gg mc^2$, so that the single-particle energy spectrum may be taken to be

$$\epsilon^0 = |\mathbf{p}^0| c$$

This is the extreme relativistic case. The same energy spectrum holds for phonons also, c, in that case, being the velocity of sound.

(iii) $kT_0 \ll mc^2$, so that the energy-momentum relation becomes

$$\epsilon^0 = mc^2 + \frac{1}{2m}\mathbf{p}^{02}.$$

This refers to the case when the particle velocities in the rest system are non-relativistic. Further, the velocity of mass-motion may also be either relativistic or non-relativistic.

5. It is, indeed, a pleasure to acknowledge my gratitude to professor D. S. Kothari and Professor F. C. Auluck for their interest and encouragement during the course of this work. The award of a Fellowship from the Department of Atomic Energy, Government of India is gratefully acknowledged.

Summary

Statistical Thermodynamics of an ideal relativistic gaseous assembly in mass-motion is studied by introducing the constraint of a fixed non-zero momentum. The transformation equations, connecting the observations in the laboratory system K and in the rest system K^0, are obtained for various thermodynamical quantities, and the invariance of the degree of degeneracy is brought out. The dynamical aspect of the results is discussed, showing thereby that the assembly behaves as if it possessed an inertial mass given by $(E+PV)/c^2$.

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Issued September 12, 1957.