SECONDARY FLOW IN A ROTATING ELLIPTIC PIPE

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1. INTRODUCTION

Secondary flow in stationary curved pipes has been studied in the past from both experimental and theoretical standpoints. Dean (1927) obtained analytically the secondary flow which results from the curvature in a pipe when a viscous fluid passes through it at low Reynolds' number. White (1929) and Adler (1934) made extensive experimental investigations which confirm Dean's calculations. Secondary flow of a viscous liquid in a rotating straight pipe of circular cross-section has been studied in detail by Barua (1954). It has been shown that there is a motion in which the flow at every circular cross-section is the same and that the velocity components at any point depend only on the x, y co-ordinates of the point, ox, xy being rectangular axes through the centre of the circular cross-section in its plane. In the present paper it has been shown that exactly similar secondary flow occurs when a viscous fluid is flowing under a pressure gradient through a straight pipe with elliptic cross-section, when the pipe is rotating about an axis perpendicular to its length.

2. FORMULATION OF THE PROBLEM

We consider a Cartesian system of co-ordinates (x, y, z) with z measured from the axis of rotation which is the y-axis and is along the minor axis of the elliptic cross-section of the pipe. The x-axis is taken along the major axis of the ellipse. If we denote the angular velocity about the axis of rotation by Ω, and the velocities relative to the x, y and z axes by u, v and w respectively, the equations of motion for steady state may be written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+2\Omega w=-\frac{\partial}{\partial x}(p/\rho)+\Omega^2 x+\nu\nabla^2 u. \quad \ldots \quad \ldots \quad (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{\partial}{\partial y}(p/\rho)+\nu\nabla^2 v. \quad \ldots \quad \ldots \quad (2)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-2\Omega u=-\frac{\partial}{\partial z}(p/\rho)+\Omega^2 z+\nu\nabla^2 w. \quad \ldots \quad \ldots \quad (3)$$

The equation of continuity may be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

If we now introduce the assumption that the motion is the same in each cross-section of the pipe, the equations of motion simplify to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+2\Omega w = -\frac{\partial \chi}{\partial x}+\nu\nabla^2 u, \quad \ldots \quad (5)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial \chi}{\partial y}+\nu\nabla^2 v, \quad \ldots \quad (6)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}-2\Omega u = -\frac{\partial \chi}{\partial z}+\nu\nabla^2 w, \quad \ldots \quad (7)$$

where

$$\chi = \frac{p}{\rho}-\frac{1}{2}\Omega^2(x^2+z^2)$$

The equation of continuity becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \quad \ldots \quad (8)$$

As a consequence of the foregoing assumption, the form of χ is restricted to

$$\chi = c'z+\phi(x, y) \quad \ldots \quad (9)$$

where c' is a constant, and may be termed the gradient of χ along the axis of the pipe. From (8) one can write

$$u = -\frac{\partial \psi}{\partial y}, \; v = \frac{\partial \psi}{\partial x} \quad \ldots \quad (10)$$

where ψ is a function of x and y only. Making use of these relations and eliminating χ from (5) and (6) we get

$$-2\Omega\frac{\partial w}{\partial y}+\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = \nu\nabla^2\nabla^2\psi. \quad \ldots \quad (11)$$

From (7), (9) and (10)

$$2\Omega\frac{\partial \psi}{\partial y}+\frac{\partial(\psi, w)}{\partial(x, y)} = -c'+\nu\nabla^2 w, \quad \ldots \quad (12)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Let us now make a transformation by putting $x+iy = c\cosh(\xi+i\eta)$, and let $\xi = \xi_0$ correspond to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,$$

where $a = c \cosh \xi_0$ and $b = c \sinh \xi_0$. In this case (11) and (12) become

$$2\Omega c\left[\cosh \xi \sin \eta \frac{\partial}{\partial \xi}+\sinh \xi \cos \eta \frac{\partial}{\partial \eta}\right]w = \frac{\partial(\psi, Q\nabla'^2\psi)}{\partial(\xi, \eta)} - \nu\nabla'^2(Q\nabla'^2\psi) \quad (13)$$

$$2\Omega c\left[\cosh \xi \sin \eta \frac{\partial}{\partial \xi}+\sinh \xi \cos \eta \frac{\partial}{\partial \eta}\right]\psi = -\frac{c'}{Q}+\nu\nabla'^2 w-\frac{\partial(\psi, w)}{\partial(\xi, \eta)}, \quad .. \quad (14)$$

where

$$\nabla'^2 = \frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}; \; Q = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \frac{1}{c^2\,(\sinh^2 \xi+\sin^2 \eta)} = \frac{2}{c^2\,(\cosh 2\xi-\cos 2\eta)}.$$

3. Solution of the problem

When there is no rotation, the problem reduces to that of the flow through an elliptic pipe under a pressure gradient, and it is known that equations (13) and (14) can then be satisfied by

$$\left.\begin{aligned} &\psi = 0 \\ \text{and} \quad & \\ &w = w_0 = \frac{c^2c'}{8\nu \cosh 2\xi_0}(\cosh 2\xi-\cosh 2\xi_0)(\cosh 2\xi_0-\cos 2\eta) \end{aligned}\right\} \quad .. \quad (15)$$

In the general case, when the pipe is rotating, let us assume that the solution can be expressed in a series of the powers of 2Ω (supposed small) as

$$\psi = 2\Omega\psi_1+(2\Omega)^2\Psi_2+\ldots \quad .. \quad .. \quad .. \quad .. \quad (16)$$

$$w = w_0+2\Omega w_1+(2\Omega)^2 w_2+\ldots \quad .. \quad .. \quad .. \quad (17)$$

Substituting for Ψ and w from (16) and (17), and equating the coefficients of powers of 2Ω we get from (13) and (14) a set of equation for the Ψ's and w's. Firstly from (14)

$$\nabla'^2 w_0 = \frac{c'}{\nu Q} = \frac{c^2c'}{\nu}(\sinh^2 \xi+\sin^2 \eta)$$

which is exactly satisfied because of our choice of w_0. From (13) equating coefficient of 2Ω we obtain

$$\nu\nabla'^2(Q\nabla'^2\psi_1) = -c\left[\cosh \xi \sin \eta \frac{\partial}{\partial \xi}+\sinh \xi \cos \eta \frac{\partial}{\partial \eta}\right]w_0$$

or *in extenso*

$$\left[(\cosh 2\xi-\cos 2\eta)\nabla'^2+4(\cosh 2\xi+\cos 2\eta)-4\left(\sinh 2\xi \frac{\partial}{\partial \xi}+\sin 2\eta \frac{\partial}{\partial \eta}\right)\right]\nabla'^2\psi_1$$
$$= A(\sin 3\eta \sinh \xi-\sin \eta \sinh 3\xi)(\cosh 2\xi-\cos 2\eta)^2, \quad .. \quad (18)$$

where

$$A = \frac{c'c^5}{16\nu^2}\left(1+\frac{1}{\cosh 2\xi_0}\right). \quad .. \quad .. \quad .. \quad (19)$$

The solution of this equation appropriate to the boundary conditions

$$\frac{\partial\psi_1}{\partial\xi} = \frac{\partial\psi_1}{\partial\eta} = 0,$$

when $\xi = \xi_0$, for all η, is given by *

$$\begin{aligned}\psi_1 = \sin\eta(A_1 \sinh\xi+(B_1-B-2B_2)\sinh 3\xi-B'\sinh 5\xi)\\ +\sin 3\eta((B_1-B-2B_2)\sinh\xi+A_2\sinh 3\xi+B_2\sinh 5\xi)\\ +\sin 5\eta(-B'\sinh\xi+B_2\sinh 3\xi+A_3\sinh 5\xi),\end{aligned} \qquad \text{(20)}$$

where

$$\left.\begin{aligned}A_1 &= -2B'(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1);\\ A_2 &= -B'\frac{(1+4\cosh 2\xi_0)^2}{2(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1)}\\ A_3 &= -B'\frac{1}{2(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1)};\\ B_2 &= B'\frac{(1+4\cosh 2\xi_0)}{2(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1)}\\ B_1 &= B+2B'\frac{(1+4\cosh 2\xi_0)(1+\cosh 2\xi_0+\cosh^2 2\xi_0)}{(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1)};\\ B &= \frac{A}{128},\ B' = \frac{B}{3}.\end{aligned}\right\} \qquad \text{(21)}$$

On equating the coefficients of 2Ω, one gets from (14)

$$\begin{aligned}\nabla'^2 w_1 = \frac{1}{\nu}\frac{\partial(\psi_1, w_0)}{\partial(\xi,\eta)} = -\frac{c^2c'}{8\nu^2}[\cos\eta(0+a_1\cosh 3\xi+a_2\cosh 5\xi-B'\cosh 7\xi)\\ +\cos 3\eta(-a_1\cosh\xi+0+a_3\cosh 5\xi+3B_2\cosh 7\xi)\\ +\cos 5\eta(-a_2\cosh\xi-a_3\cosh 3\xi+0+5A_3\cosh 7\xi)\\ +\cos 7\eta(B'\cosh\xi-3B_2\cosh 3\xi-5A_3\cosh 5\xi+0)]\\ +\frac{c^2c'}{16\nu^2\cosh 2\xi_0}[\cos\eta(0+a_4\cosh 3\xi+a_5\cosh 5\xi+a_6\cosh 7\xi)\\ +\cos 3\eta(-a_4\cosh\xi+0+a_7\cosh 5\xi+a_8\cosh 7\xi)\\ +\cos 5\eta(-a_5\cosh\xi-a_7\cosh 3\xi+0-2B_2\cosh 7\xi)\\ +\cos 7\eta(-a_6\cosh\xi-a_8\cosh 3\xi+2B_2\cosh 5\xi+0)],\end{aligned} \qquad \text{(22)}$$

where

$$\left.\begin{aligned}a_1 &= 3A_2+A_1+B'+3(B_1-B-2B_2);\\ a_2 &= 5(B_2-B')+(B_1-B-2B_2);\\ a_3 &= 3A_2+5(A_3+B');\\ a_4 &= 2A_1+2B_2-4B'+4(B_1-B-2B_2);\\ a_5 &= 6A_2+4(B_1-B-2B_2);\\ a_6 &= 8B_2-6B';\\ a_7 &= 8B_2-2(B_1-B-2B_2);\\ a_8 &= 10A_3+4B'.\end{aligned}\right\} \qquad \text{(23)}$$

The solution of this equation appropriate to the boundary condition $w_1 = 0$, when $\xi = \xi_0$, for all η is given by

$$\left.\begin{aligned}w_1 = \cos\eta(c_1\cosh\xi+b_1\cosh 3\xi+b_2\cosh 5\xi+b_3\cosh 7\xi)\\ +\cos 3\eta(b_1\cosh\xi+c_3\cosh 3\xi+b_4\cosh 5\xi+b_5\cosh 7\xi)\\ +\cos 5\eta(b_2\cosh\xi+b_4\cosh 3\xi+c_5\cosh 5\xi+b_6\cosh 7\xi)\\ +\cos 7\eta(b_3\cosh\xi+b_5\cosh 3\xi+b_6\cosh 5\xi+c_7\cosh 7\xi)\end{aligned}\right\} \qquad \text{(24)}$$

* It can be verified that the six equations obtained from these boundary conditions containing only five constants, A_1, A_2, A_3, B_1, B_2 in (20), are consistent and uniquely determine these five constants.

where

$$\left.\begin{aligned}
c_1 \cosh \xi_0 &= \frac{c^2c'}{8\nu^2}\left(\frac{a_1}{8}\cosh 3\xi_0+\frac{a_2}{24}\cosh 5\xi_0-\frac{B'}{48}\cosh 7\xi_0\right)\\
&\quad -\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\left(\frac{a_4}{8}\cosh 3\xi_0+\frac{a_5}{24}\cosh 5\xi_0+\frac{a_6}{48}\cosh 7\xi_0\right)\\
c_3 \cosh 3\xi_0 &= \frac{c^2c'}{8\nu^2}\left(\frac{a_1}{8}\cosh \xi_0+\frac{a_3}{16}\cosh 5\xi_0+\frac{3B_2}{40}\cosh 7\xi_0\right)\\
&\quad -\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\left(\frac{a_4}{8}\cosh \xi_0+\frac{a_7}{16}\cosh 5\xi_0+\frac{a_8}{40}\cosh 7\xi_0\right)\\
c_5 \cosh 5\xi_0 &= \frac{c^2c'}{8\nu^2}\left(\frac{a_2}{24}\cosh \xi_0+\frac{a_3}{16}\cosh 3\xi_0+\frac{5A_3}{24}\cosh 7\xi_0\right)\\
&\quad -\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\left(\frac{a_5}{24}\cosh \xi_0+\frac{a_7}{16}\cosh 3\xi_0-\frac{B_2}{12}\cosh 7\xi_0\right)\\
c_7 \cosh 7\xi_0 &= \frac{c^2c'}{8\nu^2}\left(-\frac{B'}{48}\cosh \xi_0+\frac{3B_2}{40}\cosh 3\xi_0+\frac{5A_3}{24}\cosh 5\xi_0\right)\\
&\quad -\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\left(\frac{a_6}{48}\cosh \xi_0+\frac{a_8}{40}\cosh 3\xi_0-\frac{B_2}{12}\cosh 5\xi_0\right)
\end{aligned}\right\} \cdots \quad (25)$$

$$b_1 = -\frac{c^2c'}{8\nu^2}\cdot\frac{a_1}{8}+\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{a_4}{8}; \qquad b_2 = -\frac{c^2c'}{8\nu^2}\frac{a_2}{24}+\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{a_5}{24}$$

$$b_3 = \frac{c^2c'}{8\nu^2}\cdot\frac{B'}{48}+\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{a_6}{48}; \qquad b_4 = -\frac{c^2c'}{8\nu^2}\frac{a_3}{16}+\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{a_7}{16}$$

$$b_5 = -\frac{c^2c'}{8\nu^2}\cdot\frac{3B_2}{40}+\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{a_8}{40}; \quad b_6 = -\frac{c^2c'}{8\nu^2}\cdot\frac{5A_3}{24}-\frac{c^2c'}{16\nu^2\cosh 2\xi_0}\cdot\frac{B_2}{12}$$

If we restrict ourselves to the first order solution of the problem and neglect terms of the higher order in (2Ω) in (16) and (17) equations (15), (20) and (24) are sufficient to give the first order solution. We shall here examine the nature of the stream lines for solution of this order. Let us consider a typical stream line. We shall look for the projections of the motion along this stream line on the planes $y = 0$ and $z = 0$. The central plane $y = 0$ has two parallel generating lines of the cylinder cut by the major axis of the cross-section on which the portion between the foci is given by $\xi = 0$, and on two sides of this segment, $\eta = 0$, or π.

It can be seen from (20) that in either case

$$u_1 = -\sqrt{Q}\frac{\partial\psi}{\partial\eta} \neq 0, \quad v_1 = \sqrt{Q}\frac{\partial\psi}{\partial\xi} = 0.$$

So a particle of fluid once in this central plane $y = 0$, does not leave it in the subsequent motion. The differential equations of the stream lines are

$$\frac{d\xi}{-2\Omega Q\dfrac{\partial\psi_1}{\partial\eta}} = \frac{d\eta}{2\Omega Q\dfrac{\partial\psi_1}{\partial\xi}} = \frac{dz}{w_0}$$

To find the projection on the central plane we substitute the values of w_0 from (15) and $\dfrac{\partial\psi_1}{\partial\eta}$ from (20), and get

$$dz = -\frac{48\nu}{2\Omega c}\frac{(2\cosh^2 2\xi_0+2\cosh 2\xi_0+1)\sinh\xi d\xi}{\sinh^2 2\xi_0(\cosh^2\xi_0-\cosh^2\xi)} \text{ for the portion } \eta = 0$$

and

$$dz = \frac{48\nu}{2\Omega c} \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)\sinh \xi d\xi}{\sinh^2 2\xi_0(\cosh^2 \xi_0 - \cosh^2 \xi)} \text{ for the portion } \eta = \pi.$$

On integration we obtain

$$z = -\frac{24\nu}{2\Omega c} \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)}{\sinh^2 2\xi_0 \cosh \xi_0} \log\left(\frac{\cosh \xi_0 + \cosh \xi}{\cosh \xi_0 - \cosh \xi}\right) + \text{const. for } \eta = 0 \quad \text{(25a)}$$

$$= \frac{24\nu}{2\Omega c} \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)}{\sinh^2 2\xi_0 \cosh \xi_0} \log\left(\frac{\cosh \xi_0 + \cosh \xi}{\cosh \xi_0 - \cosh \xi}\right) + \text{const. for } \eta = \pi \quad \text{(25b)}$$

Similarly, substituting the values of w_0 from (15) and $\frac{\partial \psi_1}{\partial \xi}$ from (20) and putting $\xi = 0$ in the above differential equation of the stream lines we have

$$dz = \frac{24\nu}{\Omega c} \cdot \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)}{\sinh^2 2\xi_0} \cdot \frac{\sin \eta d\eta}{(\cosh^2 \xi_0 - \cos^2 \eta)}$$

which on integration gives

$$z = -\frac{24\nu}{2\Omega c} \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)}{\sinh^2 2\xi_0 \cosh \xi_0} \log\left(\frac{\cosh \xi_0 + \cos \eta}{\cosh \xi_0 - \cos \eta}\right) + \text{const.} \quad \text{(25c)}$$

Let us consider the particular stream line for which the constants of integration, which generally should be different for different stream lines, are both equal to zero. We see that at one focus, namely, $\xi = 0$, $\eta = 0$

$$z = -\frac{24\nu}{2\Omega c} \frac{(2\cosh^2 2\xi_0 + 2\cosh 2\xi_0 + 1)}{\sinh^2 2\xi_0 \cosh \xi_0} \log\left(\frac{\cosh \xi_0 + 1}{\cosh \xi_0 - 1}\right) = -z_0 \text{ say}$$

the other focus $\xi = 0$, $\eta = \pi$, $z = z_0$. We find that for $\eta = \pi$, z increases as ξ increases; since $z \to \infty$ as $\xi \to \xi_0$, the stream lines never reach the edge of the pipe at finite distance. For $\eta = 0$, the expression for z is the same with the sign reversed, and $z \to -\infty$ as $\xi \to \xi_0$. Similarly, if we take the constant of integration in z in equation (25c) to be zero, then on the part of the major axis of the ellipse joining the two foci $\xi = 0$, $z = 0$ when $\eta = \pi/2$. We note that putting the constants of integration zero in all the three above formulae (25a), (25b), (25c) the stream line corresponds to the stream line passing through the origin (Fig. 1).

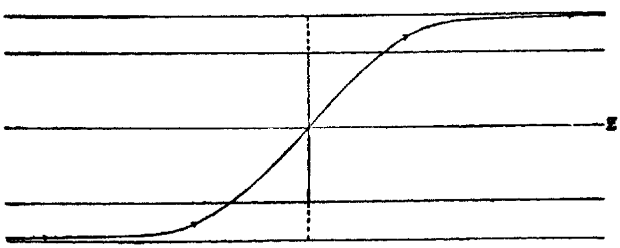

FIG. 1.

Other stream lines may be drawn in this manner by choosing non-zero values for the constants. For a given value of ξ or η, z varies inversely as 2Ω. So as the angular velocity of the pipe increases z diminishes. It may also be noted that to the first order of approximation the foregoing relations do not depend on the pressure gradient c' along the pipe. The differential equation for any stream line is given by

$$\frac{d\xi}{\sqrt{Q}u_1} = \frac{d\eta}{\sqrt{Q}v_1} = \frac{dz}{w}$$

From the first equality above, we get on substituting for u_1 and v_1 from (20)

$$d\psi_1 = 0$$

which on integration gives

$$\psi_1 = A'$$

or

$$\sin\eta(A_1 \sinh\xi + (B_1 - B - 2B_2)\sinh 3\xi - B' \sinh 5\xi)$$
$$+\sin 3\eta((B_1 - B - 2B_2)\sinh\xi + A_2 \sinh 3\xi + B_2 \sinh 5\xi)$$
$$+\sin 5\eta(-B' \sinh\xi + B_2 \sinh 3\xi + A_3 \sinh 5\xi) = A'$$

where A' is a constant of integration.

Taking $c = 1$ and eccentricity of the ellipse to be $\dfrac{1}{\sqrt{2}}$ and transforming this equation in terms of Cartesian system of co-ordinates and considering the stream line in the case when

$$\frac{2A'}{A_3 - B_2 - B'} = -\frac{7200\nu^2}{c'} = c_p \quad \text{say,}$$

we obtain

$$32y[1 - x^2 - 2y^2 + x^2y^2 + \tfrac{1}{4}x^4 + y^4] = c_p$$

The projection of the stream lines on a cross-section of the pipe in the case when $c_p = 1$ has been plotted graphically and shown in Fig. 2. For different values of

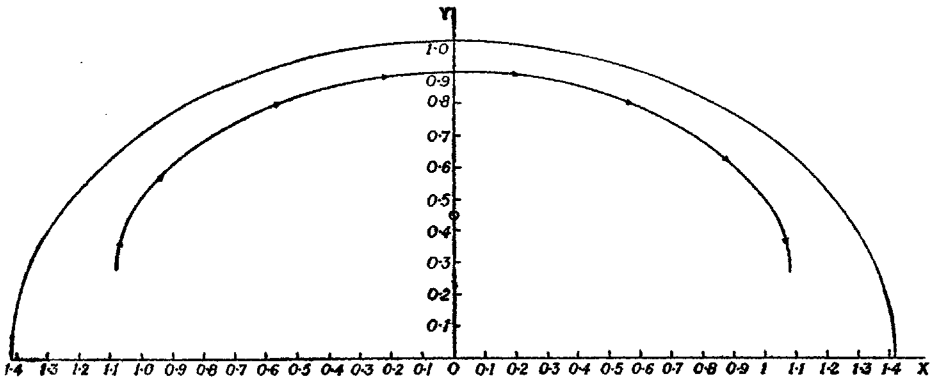

FIG. 2.

c_p we get a similar set of curves. For equal and opposite value of c_p, we get a curve which is the reflection of the former curve on the major axis of the cross-section of the tube. The central plane $y = 0$ is a plane of symmetry and the motions in the upper and lower halves of the cylinder are reflections of one other. Further it can be seen that on the y-axis, i.e. minor axis of the ellipse, the velocity v of the fluid element parallel to the y-axis is always zero and u, the velocity parallel to the x-axis, is also zero only when

$$y = \pm\frac{1}{\sqrt{5}} = \pm 0{\cdot}447 \text{ (approx.)},$$

Hence to the first approximation the stream lines of the relative motion through the points

$$\left(x=0,\, y=\frac{1}{\sqrt{5}}\right) \text{ and } \left(x=0,\, y=-\frac{1}{\sqrt{5}}\right)$$

are straight lines and the velocity on any of them is the same as when there is no rotation. The secondary motion due to rotation consists in slight curving of the original straight stream line round the above two undisturbed stream lines.

4. Calculation of resistance coefficient

In order to calculate the resistance coefficient, we shall now calculate the second order terms in 2Ω in (16) and (17). Equating the coefficient of $(2\Omega)^2$ we get from (13)

$$\nu\nabla'^2(Q\nabla'^2\psi_2) = \frac{\partial(\psi_1, Q\nabla'^2\psi_1)}{\partial(\xi,\eta)} - c\left(\cosh\xi\sin\eta\frac{\partial}{\partial\xi}+\sinh\xi\cos\eta\frac{\partial}{\partial\eta}\right)w_1 \cdot$$

Substituting for ψ_1 and w_1 from (20) and (24) respectively and simplifying we get

$$\left.\begin{aligned}\nu\nabla'^2(Q\nabla'^2\psi_2) = {} & \sin 2\eta(0+D_{11}\sinh 4\xi+D_{12}\sinh 6\xi+D_{13}\sinh 8\xi)\\ & +\sin 4\eta(-D_{11}\sinh 2\xi+0+D_{14}\sinh 6\xi+D_{15}\sinh 8\xi)\\ & +\sin 6\eta(-D_{12}\sinh 2\xi-D_{14}\sinh 4\xi+0+D_{16}\sinh 8\xi)\\ & +\sin 8\eta(-D_{13}\sinh 2\xi-D_{15}\sinh 4\xi-D_{16}\sinh 6\xi+0)\end{aligned}\right\} \quad (26)$$

where

$$\left.\begin{aligned}D_{11} = {} & \frac{4}{c^2}\{4(B_1-B-2B_2)(\overline{B_1-B-2B_2}+B_2+6B')-144B_2B'+12A_2B_1\\ & +12A_1B'+120B'^2+60A_3B'-8B_2^2-24A_1B_2\}\\ & -\frac{c}{2}(b_1-3c_3+3b_2-b_4)\\ D_{12} = {} & \frac{4}{c^2}\{-8(B'+4B_2)(\overline{B_1-B-2B_2}-4B_2)-12B'(10A_3+10B'+6A_2)\}\\ & -\frac{c}{2}(2b_2-4b_4+4b_3-2b_5)\\ D_{13} = {} & \frac{4}{c^2}\{12(B'^2-B_2B'+10B_2A_3)\}-\frac{c}{2}(3b_3-5b_5)\\ D_{14} = {} & \frac{4}{c^2}\{28B_2(B_1-B-2B_2)+20A_3B_1+8B_2^2+84B'^2-96B_2B'+36A_2B'\}\\ & -\frac{c}{2}(b_4-5c_5+5b_5-b_6)\\ D_{15} = {} & \frac{4}{c^2}\{-24B'(B_2+5A_3)\}-\frac{c}{2}(2b_5-6b_6)\\ D_{16} = {} & \frac{4}{c^2}\{6(10A_3B'+4B_2^2)\}-\frac{c}{2}(b_6-7c_7)\end{aligned}\right\} \quad (27)$$

Equation (26) can be written in the form

$$\left[(\cosh 2\xi-\cos 2\eta)\nabla'^2+4(\cosh 2\xi+\cos 2\eta)-4\left(\sinh 2\xi\frac{\partial}{\partial\xi}+\sin 2\eta\frac{\partial}{\partial\eta}\right)\right]\nabla'^2\psi_2$$
$$=\frac{c^2}{2\nu}(\cosh 2\xi-\cos 2\eta)^2[\sin 2\eta(0+D_{11}\sinh 4\xi+D_{12}\sinh 6\xi+D_{13}\sinh 8\xi)$$
$$+\sin 4\eta(-D_{11}\sinh 2\xi+0+D_{14}\sinh 6\xi+D_{15}\sinh 8\xi)$$
$$+\sin 6\eta(-D_{12}\sinh 2\xi-D_{14}\sinh 4\xi+0+D_{16}\sinh 8\xi)$$
$$+\sin 8\eta(-D_{13}\sinh 2\xi-D_{15}\sinh 4\xi-D_{16}\sinh 6\xi+0)]$$

Of which the solution is of the form

$$\psi_2=\sin 2\eta\{L_1\sinh 2\xi+(\beta_1+T_1-5T_2)\sinh 4\xi+\beta_2\sinh 6\xi+\beta_3\sinh 8\xi+\beta_4\sinh 10\xi\}$$
$$+\sin 4\eta\{(\beta_1+T_1-5T_2)\sinh 2\xi+L_2\sinh 4\xi+(3T_2-7T_3-3\beta_3)\sinh 6\xi+\beta_5\sinh 8\xi+\beta_6\sinh 10\xi\}$$
$$+\sin 6\eta\{\beta_2\sinh 2\xi+(3T_2-7T_3-3\beta_3)\sinh 4\xi+L_3\sinh 6\xi+(5T_3-9T_4-3\beta_6)\sinh 8\xi+\beta_7\sinh 10\xi\}$$
$$+\sin 8\eta\{\beta_3\sinh 2\xi+\beta_5\sinh 4\xi+(5T_3-9T_4-3\beta_6)\sinh 6\xi+L_4\sinh 8\xi+7T_4\sinh 10\xi\}$$
$$+\sin 10\eta\{\beta_4\sinh 2\xi+\beta_6\sinh 4\xi+\beta_7\sinh 6\xi+7T_4\sinh 8\xi+L_5\sinh 10\xi\} \quad \ldots \quad (28)$$

where L_1, L_2, L_3, L_4, L_5, T_1, T_2, T_3, T_4 are arbitrary constants and

$$\beta_1=\frac{c^2}{4\nu}\cdot\frac{D_{12}}{32\times 12};\ \beta_2=\frac{c^2}{4\nu}\cdot\frac{1}{32}\left(\frac{D_{11}}{12}+\frac{D_{13}}{60}-\frac{D_{14}}{20}\right);\ \beta_3=\frac{c^2}{4\nu}\frac{1}{60}\left(\frac{D_{12}}{32}-\frac{D_{15}}{48}\right)$$

$$\beta_4=\frac{c^2}{4\nu}\frac{D_{13}}{60\times 96};\ \beta_5=\frac{c^2}{4\nu}\cdot\frac{1}{48}\left(\frac{D_{14}}{20}-\frac{D_{13}}{60}-\frac{D_{16}}{28}\right);\ \beta_6=\frac{c^2}{4\nu}\frac{D_{15}}{48\times 84};$$

$$\beta_7=\frac{c^2}{4\nu}\cdot\frac{D_{16}}{28\times 64}. \quad \ldots \quad (29)$$

The boundary conditions $\dfrac{\partial\psi_2}{\partial\xi}=\dfrac{\partial\psi_2}{\partial\eta}=0$ on $\xi=\xi_0$, for all η give

$$L_1\sinh 2\xi_0+(\beta_1+T_1-5T_2)\sinh 4\xi_0+\beta_2\sinh 6\xi_0+\beta_3\sinh 8\xi_0+\beta_4\sinh 10\xi_0=0 \quad \ldots \quad (1')$$

$$(\beta_1+T_1-5T_2)\sinh 2\xi_0+L_2\sinh 4\xi_0+(3T_2-7T_3-3\beta_3)\sinh 6\xi_0+\beta_5\sinh 8\xi_0+\beta_6\sinh 10\xi_0=0 \quad \ldots \quad (2')$$

$$\beta_2\sinh 2\xi_0+(3T_2-7T_3-3\beta_3)\sinh 4\xi_0+L_3\sinh 6\xi_0+(5T_3-9T_4-3\beta_6)\sinh 8\xi_0+\beta_7\sinh 10\xi_0=0 \quad \ldots \quad (3')$$

$$\beta_3\sinh 2\xi_0+\beta_5\sinh 4\xi_0+(5T_3-9T_4-3\beta_6)\sinh 6\xi_0+L_4\sinh 8\xi_0+7T_4\sinh 10\xi_0=0 \quad \ldots \quad (4')$$

$$\beta_4\sinh 2\xi_0+\beta_6\sinh 4\xi_0+\beta_7\sinh 6\xi_0+7T_4\sinh 8\xi_0+L_5\sinh 10\xi_0=0 \quad (5')$$

$$L_1\cosh 2\xi_0+2(\beta_1+T_1-5T_2)\cosh 4\xi_0+3\beta_2\cosh 6\xi_0+4\beta_3\cosh 8\xi_0+5\beta_4\cosh 10\xi_0=0 \quad \ldots \quad (6')$$

$$(\beta_1+T_1-5T_2)\cosh 2\xi_0+2L_2\cosh 4\xi_0+3(3T_2-7T_3-3\beta_3)\cosh 6\xi_0$$
$$+4\beta_5\cosh 8\xi_0+5\beta_6\cosh 10\xi_0=0 \quad \ldots \quad (7')$$

$$\beta_2\cosh 2\xi_0+2(3T_2-7T_3-3\beta_3)\cosh 4\xi_0+3L_3\cosh 6\xi_0$$
$$+4(5T_3-9T_4-3\beta_6)\cosh 8\xi_0+5\beta_7\cosh 10\xi_0=0 \quad \ldots \quad (8')$$

$$\beta_3\cosh 2\xi_0+2\beta_5\cosh 4\xi_0+3(5T_3-9T_4-3\beta_6)\cosh 6\xi_0+4L_4\cosh 8\xi_0$$
$$+35T_4\cosh 10\xi_0=0 \quad \ldots \quad (9')$$

$$\beta_4\cosh 2\xi_0+2\beta_6\cosh 4\xi_0+3\beta_7\cosh 6\xi_0+28T_4\cosh 8\xi_0$$
$$+5L_5\cosh 10\xi_0=0 \quad \ldots \quad (10')$$

It can be verified that these ten equations (1′)—(10′) containing nine arbitrary constants are consistent. Solving for L_1, $(\beta_1+T_1-5T_2)$, etc. we obtain

$$L_1=\beta_2(1+4\cosh^2 2\xi_0)+16\beta_3\cosh^3 2\xi_0+\beta_4\{5+6\cosh 4\xi_0(1+4\cosh^2 2\xi_0)\};$$
$$\beta_1+T_1-5T_2=-4\beta_2\cosh 2\xi_0-2\beta_3(2+3\cosh 4\xi_0)-4\beta_4\cosh 2\xi_0(1+4\cosh 4\xi_0),$$
$$\text{etc.} \quad \ldots \quad (30)$$

Equating the coefficient of $(2\Omega)^2$, we obtain from (14)

$$\nu\nabla'^2 w_2=c\left[\cosh\xi\sin\eta\frac{\partial}{\partial\xi}+\sinh\xi\cos\eta\frac{\partial}{\partial\eta}\right]\psi_1+\frac{\partial(\psi_2,w_0)}{\partial(\xi,\eta)}+\frac{\partial(\psi_1,w_1)}{\partial(\xi,\eta)}$$

or

$$\nabla'^2 w_2=\frac{1}{\nu}\Big[\alpha_1\cosh 2\xi+\alpha_2\cosh 4\xi+\alpha_3\cosh 6\xi+\alpha_4\cosh 8\xi+\alpha_5\cosh 10\xi$$
$$+\alpha_6\cosh 12\xi+\cos 2\eta\{-\alpha_1+\alpha_7\cosh 4\xi+\alpha_8\cosh 6\xi$$
$$+\alpha_9\cosh 8\xi+\alpha_{10}\cosh 10\xi+\alpha_{11}\cosh 12\xi\}$$
$$+\cos 4\eta\{\cdots\cdots\cdots\}$$
$$+\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$+\cos 12\eta\{\qquad\qquad\}\Big]^* \quad \ldots \quad (31)$$

where

$$\alpha_1=\frac{c}{2}(A_1+\overline{B_1-B-2B_2})+\frac{c^2c'}{8\nu\cosh 2\xi_0}(\overline{\beta_1+T_1-5T_2}-2L_1\cosh 2\xi_0)$$
$$-\tfrac{1}{2}\{A_1(c_1-b_1)+(B_1-B-2B_2)(c_1-b_2+3b_1-3c_3)$$
$$+B'(b_3-b_1+5b_4-5b_2)+3A_2(b_1-b_4)+B_2(3c_3-3b_5+5b_2-5c_5)$$
$$+5A_3(b_4-b_6)\}$$

$$\alpha_2=c(\overline{B_1-B-2B_2}-B')+\frac{c^2c'}{4\nu\cosh 2\xi_0}(L_1+\beta_2-2(\beta_1+T_1-5T_2)\cosh 2\xi_0)$$
$$-\{A_1(b_1-b_2)+(B_1-B-2B_2)(c_1-b_3+3c_3-3b_4)-B'(c_1+5b_4-5c_5)$$
$$+3A_2(b_1-b_5)+5A_3b_2+B_2(3b_1+5b_2-5b_6)\}$$

$$\alpha_3=-\frac{3cB'}{2}+\frac{3c^2c'}{8\nu\cosh 2\xi_0}(\overline{\beta_1+T_1-5T_2}+\beta_3-2\beta_2\cosh 2\xi_0)$$
$$-\tfrac{3}{2}\{A_1(b_2-b_3)+(B_1-B-2B_2)(b_1+3b_4-3b_5)-B'(c_1+5c_5-5b_6)$$
$$+3A_2c_3+B_2(3b_1+5b_4)+5A_3b_2\}$$

* Terms not written in full in (30), (31), and (33) do not make any contribution in the calculation of the flux integral (34).

$$\alpha_4 = \frac{c^2c'}{2\nu\cosh 2\xi_0}(\beta_2+\beta_4-2\beta_3\cosh 2\xi_0)$$
$$-2\{A_1b_3+(B_1-B-2B_2)(b_2+3b_5)-B'(b_1+5b_6)$$
$$+3A_2b_4+B_2(3c_3+5c_5)+5A_3b_4\}$$

$$\alpha_5 = \frac{5c^2c'}{8\nu\cosh 2\xi_0}(\beta_3-2\beta_4\cosh 2\xi_0)$$
$$-\tfrac{5}{2}\{(B_1-B-2B_2)b_3-B'b_2+3A_2b_5+B_2(3b_4+5b_6)+5A_3c_5\}$$

$$\alpha_6 = \frac{6c^2c'}{8\nu\cosh 2\xi_0}\beta_4\cosh 2\xi_0-3\{-B'b_3+3B_2b_5+5A_3b_6\}$$

$$\alpha_7 = \frac{c}{2}(3B'+B_2+3A_2-\overline{B_1-B-2B_2})$$
$$+\frac{c^2c'}{8\nu\cosh 2\xi_0}\big(3(\beta_1+T_1-5T_2)+\overline{3T_2-7T_3-3\beta_3}$$
$$-2(2L_2+L_1-\beta_2)\cosh 2\xi_0\big)-\tfrac{1}{2}\{A_1(b_1+3c_3-3b_2-b_4)$$
$$+(B_1-B-2B_2)(10b_1-c_1+7b_4+b_5-7b_2-5c_5-5b_3)$$
$$+B'(7b_1-3c_1+9c_3+11b_5-11b_4-9b_6)+A_2(3c_1+9b_2-9b_3-3b_6)$$
$$+5A_3(b_1+3b_3)+B_2(c_1+11b_2+7b_1+13b_3-13b_5-7c_7)\} \quad \ldots \quad (32)$$

$$\alpha_8 = \frac{c}{2}(4B_2+2B')+\frac{c^2c'}{8\nu\cosh 2\xi_0}\big(4L_2+2\beta_5-2(3(3T_2-7T_3-3\beta_3)$$
$$+\overline{\beta_1+T_1-5T_2}-\beta_3)\cosh 2\xi_0\big)-\tfrac{1}{2}\{A_1(2b_2+4b_4-4b_3-2b_5)$$
$$+(B_1-B-2B_2)(6c_3+8b_2+10c_5-10b_3-8b_6)$$
$$-B'(8b_1-2c_1+14b_4+16b_6-16b_5-14c_7)+A_2(6b_1+12b_4)$$
$$+A_3(10b_1+20b_3)+B_2(4c_1+14b_2+12c_3+18b_5)\}$$

$$\alpha_9 = \frac{c^2c'}{8\nu\cosh 2\xi_0}\{5(3T_2-7T_3-3\beta_3)+3\beta_6-2(4\beta_5+\beta_2-\beta_4)\cosh 2\xi_0\}$$
$$-\tfrac{1}{2}\{A_1(3b_3+5b_5)+(B_1-B-2B_2)(7b_4+b_2+11b_3+13b_6)$$
$$-B'(9c_3-b_1+19b_5+21c_7)+5A_3(3c_3+5b_5)$$
$$+B_2(7b_1+34b_4+23b_6)\}$$

$$\alpha_{10} = \frac{c^2c'}{8\nu\cosh 2\xi_0}(6\beta_5-2(5\beta_6+\beta_3)\cosh 2\xi_0)$$
$$-\tfrac{1}{2}\{(B_1-B-2B_2)(2b_3+8b_5)-10B'b_4+A_2(12b_3+18b_6)$$
$$+B_2(10b_2+20c_5+22b_5+28c_7)+A_3(20b_4+30b_6)\}$$

$$\alpha_{11} = \frac{c^2c'}{8\nu\cosh 2\xi_0}(7\beta_6-2\beta_4\cosh 2\xi_0)$$
$$-\tfrac{1}{2}\{-B'(11b_5+b_3)+B_2(13b_3+23b_6)+A_3(25b_5+35c_7)\}.$$

The solution of this equation appropriate to the boundary condition $w_2 = 0$ on $\xi = \xi_0$ for all η is

$$w_2 = \Big[\frac{\alpha_1}{4}(\cosh 2\xi-\cosh 2\xi_0)+\frac{\alpha_2}{16}(\cosh 4\xi-\cosh 4\xi_0)+\frac{\alpha_3}{36}(\cosh 6\xi-\cosh 6\xi_0)$$
$$+\frac{\alpha_4}{64}(\cosh 8\xi-\cosh 8\xi_0)+\frac{\alpha_5}{100}(\cosh 10\xi-\cosh 10\xi_0)$$

$$+\frac{\alpha_6}{144}(\cosh 12\xi-\cosh 12\xi_0)+\frac{\cos 2\eta}{\cosh 2\xi_0}\left\{\frac{\alpha_1}{4}(\cosh 2\xi_0-\cosh 2\xi)\right.$$

$$+\frac{\alpha_7}{12}(\cosh 4\xi\cosh 2\xi_0-\cosh 2\xi\cosh 4\xi_0)$$

$$+\frac{\alpha_8}{32}(\cosh 6\xi\cosh 2\xi_0-\cosh 2\xi\cosh 6\xi_0)$$

$$+\frac{\alpha_9}{60}(\cosh 8\xi\cosh 2\xi_0-\cosh 2\xi\cosh 8\xi_0)$$

$$+\frac{\alpha_{10}}{96}(\cosh 10\xi\cosh 2\xi_0-\cosh 2\xi\cosh 10\xi_0)$$

$$\left.+\frac{\alpha_{11}}{140}(\cosh 12\xi\cosh 2\xi_0-\cosh 2\xi\cosh 12\xi_0)\right\}$$

$$+\cos 4\eta\{\quad\}+\ldots.+\cos 12\eta\{\quad\}\Big] \quad \ldots \quad (33)$$

The flux through the pipe

$$=+\iint w\,dx\,dy=+\int_{\xi=0}^{\xi_0}\int_{\eta=0}^{2\pi} w(\xi,\eta)\frac{\partial(x,y)}{\partial(\xi,\eta)}d\xi\,d\eta$$

$$=+\frac{c^2}{2}\int_0^{\xi_0}\int_0^{2\pi}[w_0+(2\Omega)^2 w_2](\cosh 2\xi-\cos 2\eta)d\xi\,d\eta$$

$$=-\frac{\pi c^4 c'}{32\nu\cosh 2\xi_0}\sinh^3 2\xi_0\left[1+\frac{1}{c'}\left(\frac{2\Omega}{c}\right)^2\left\{2\left(\alpha_1+\frac{\alpha_3}{9}-\frac{\alpha_7}{3}\right)\right.\right.$$

$$+\frac{4}{3}\left(\alpha_2+\frac{\alpha_4}{4}-\frac{\alpha_8}{2}\right)\cosh 2\xi_0+2\left(\frac{\alpha_3}{9}+\frac{\alpha_5}{25}-\frac{\alpha_9}{15}\right)(2+3\cosh 4\xi_0)$$

$$+\frac{4}{5}\left(\frac{\alpha_4}{4}+\frac{\alpha_6}{9}-\frac{\alpha_{10}}{6}\right)\cosh 2\xi_0(1+4\cosh 4\xi_0)$$

$$+\frac{2}{3}\left(\frac{\alpha_5}{25}-\frac{\alpha_{11}}{35}\right)(9+16\cosh 4\xi_0+10\cosh 8\xi_0)$$

$$\left.\left.+\frac{4}{7}\cdot\frac{\alpha_6}{9}(6\cosh 2\xi_0+5\cosh 6\xi_0+3\cosh 10\xi_0)\right\}\right]$$

If we denote the mean velocity through a stationary elliptic pipe of semi-axes a and b under the same effective pressure gradient by w_m we have

$$\pi\frac{c^2}{2}\sinh 2\xi_0\,.\,w_m=-\frac{\pi c' c^4\sinh^3 2\xi_0}{32\nu\cosh 2\xi_0}\,.$$

Therefore

$$c'=-\frac{16\nu w_m\cosh 2\xi_0}{c^2\sinh^2 2\xi_0} \quad \ldots \quad (34)$$

Hence, the resistance coefficient becomes approximately

$$\frac{c_{Dr}}{c_D}=\left(\frac{F}{F_r}\right)^2=1-\frac{2}{c'}\left(\frac{2\Omega}{c}\right)^2\left\{2\left(\alpha_1+\frac{\alpha_3}{9}-\frac{\alpha_7}{3}\right)+\frac{4}{3}\left(\alpha_2+\frac{\alpha_4}{4}-\frac{\alpha_8}{2}\right)\cosh 2\xi_0\right.$$
$$+2\left(\frac{\alpha_3}{9}+\frac{\alpha_5}{25}-\frac{\alpha_9}{15}\right)(2+3\cosh 4\xi_0)$$
$$+\frac{4}{5}\left(\frac{\alpha_4}{4}+\frac{\alpha_6}{9}-\frac{\alpha_{10}}{6}\right)\cosh 2\xi_0(1+4\cosh 4\xi_0)$$
$$+\frac{2}{3}\left(\frac{\alpha_5}{25}-\frac{\alpha_{11}}{35}\right)(9+16\cosh 4\xi_0+10\cosh 8\xi_0)$$
$$\left.+\frac{4}{7}\cdot\frac{\alpha_6}{9}(6\cosh 2\xi_0+5\cosh 6\xi_0+3\cosh 10\xi_0)\right\}\quad (35)$$

where C_D and C_{Dr} stand for resistance coefficients for the cases when the pipe is stationary and when it is rotating respectively, and F and F_r stand for the flux through the pipe in those cases. The validity of the equation (35) requires that Ω should be fairly small.

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Summary

The secondary fluid motion in an infinite tube through which a fluid is flowing under constant pressure gradient while the tube rotates about a principal axis of the elliptic cross-section has been worked out. The character of the flow is the same as in a circular tube, though the details of calculations are very much complicated in the case of elliptic section. It is, however, possible to satisfy the boundary conditions accurately.

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