WAVE RESISTANCE IN DEEP WATER DUE TO THE ACCELERATED MOTION OF A PRESSURE SYSTEM

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1. We have seen in a previous paper (Bhattacharya, 1956) which will henceforth be called paper I, that when a pressure system $H(\tilde{\omega})$, symmetrical about the origin starts with an initial velocity c, and moves over the surface of deep water with uniform acceleration f, the surface elevation of the waves that are generated is given by

$$\pi g\rho y = \int_0^\infty kV\psi(k)\exp(ik\tilde{\omega})dk\int_0^t \sin kV\tau \exp\left[-\mu\tau+ik\{c\tau+\tfrac{1}{2}f\tau(2t-\tau)\}\right]\times d\tau \quad \ldots \quad (1.1)$$

where

$$\psi(k) = \int_{-\infty}^\infty H(\tilde{\omega})\exp(-ik\tilde{\omega})\,d\tilde{\omega}$$

$$= 2\int_0^\infty H(\tilde{\omega})\cos k\tilde{\omega}\,d\tilde{\omega}. \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

We define, after Havelock (1917), the wave resistance as the total horizontal component of the pressure over the wetted area of the ship, which by our hypothesis is represented by the moving pressure system.

In Fig. 1, $\tilde{\omega}$-axis represents the undisturbed surface, y the vertical axis, and the curve a depression of the free surface.

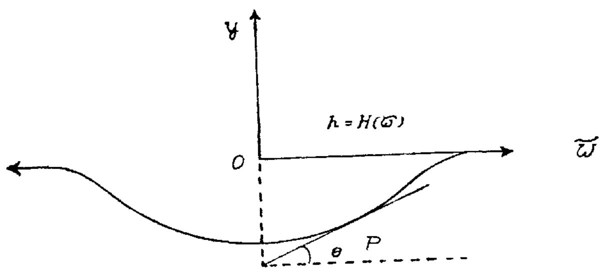

FIG. 1

The horizontal component of the pressure at a point of the free surface is

$$-p\sin\theta = -p\tan\theta = -H(\tilde{\omega})\frac{\partial y}{\partial\tilde{\omega}},$$

for θ is small.

We define the wave resistance by

$$R = \int_{-\infty}^{\infty} H(\tilde{\omega}) \frac{\partial y}{\partial \tilde{\omega}} d\tilde{\omega}. \quad \quad (1.3)$$

In this integral y will be replaced by its three parts,

$$y = y_1 + y_2 + y_3 \quad \quad (1.4)$$

worked out in paper I referred to above and the parts of the resistance arising out of y_1, y_2 and y_3 will be called R_1, R_2 and R_3 respectively, so that the total resistance will be given by

$$R = R_1 + R_2 + R_3. \quad \quad (1.5)$$

2. *Calculation of* R_1

This is simple and straightforward. It has been shown in paper I that

$$\pi g \rho y_1 = f(\tilde{\omega}), \text{ for } \tilde{\omega} > 0$$

and

$$= 2\pi k_0 \psi(k_0) \sin k_0 \tilde{\omega} + f(-\tilde{\omega}), \text{ for } \tilde{\omega} < 0,$$

with

$$k_0 = g/v^2.$$

Substituting y_1 for y in (1.3) and noticing that the symmetrical part $f(\tilde{\omega})$ of y_1 contributes nothing to R_1, we have

$$\pi g \rho R_1 = \int_{-\infty}^{0} H(\tilde{\omega}).\, 2\pi k_0^2\, \psi(k_0) \cos k_0 \tilde{\omega}\, d\tilde{\omega}$$

$$= 2\pi k_0^2\, \psi(k_0) \int_0^{\infty} H(\tilde{\omega}) \cos k_0 \tilde{\omega}\, d\tilde{\omega}$$

$$= \pi k_0^2 \{\psi(k_0)\}^2,$$

by (1.2) and the symmetry of $H(\tilde{\omega})$ about the origin.
Then

$$R_1 \doteq \frac{k_0^2}{g\rho} \left\{\psi(k_0)\right\}^2$$

$$= \frac{g}{\rho v^4} \left\{\psi(k_0)\right\}^2. \quad \quad (2.1)$$

3. *Calculation of* R_2

In obtaining R_2 we might proceed as in the case of R_1. But the expression for y_2 is much too complicated for further differentiation and integration. We therefore fall back upon the earlier integral given in (1.1). We note that the time integral involved in (1.1) is

$$\int_0^t \sin kV\tau.\ \exp\,[-\mu\tau + ik\{c\tau + \tfrac{1}{2} f\tau(2t-\tau)\}]d\tau. \quad \quad (3.1)$$

It has been shown in paper I that the part of the integral (3.1) which yields y_2 is given by

$$-\frac{1}{2k}\exp[ik(vt-\tfrac{1}{2}ft^2)]\left\{\frac{\exp(ikVt)}{v+V}-\frac{\exp(-ikVt)}{v-V}\right\} \quad \ldots \quad (3.2)$$

when $\mu \to 0$.

This will be responsible for R_2. To calculate this, however, we go back to our original equation (1.1), and the definition of total resistance R in (1.3). These two equations together give

$$\pi g\rho R = -i\int_0^\infty k^2V\{\psi(k)\}^2 dk\int_0^t \sin kV\tau \times$$

$$\exp[-\mu\tau+ik\{c\tau+\tfrac{1}{2}f\tau(2t-\tau)\}]\,d\tau. \quad \ldots \quad (3.3)$$

Now if in this integral we replace the time-integral part of it by (3.2), we obtain

$$\pi g\rho R_2 = \frac{i}{2}\int_0^\infty kV\{\psi(k)\}^2\exp(ikd)\left\{\frac{\exp(ikVt)}{v+V}-\frac{\exp(-ikVt)}{v-V}\right\}d\tau$$

$$= \frac{1}{2}i(A-B) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

where

$$A = \int_0^\infty \frac{kV\{\psi(k)\}^2}{v+V}\exp\{ik(d+Vt)\}dk, \quad \ldots \quad \ldots \quad (3.5)$$

$d = vt-\frac{1}{2}ft^2$
$= ct+\frac{1}{2}ft^2$
= distance travelled by the moving pressure system in time t, and

$$B = \int_0^\infty \frac{kV\{\psi(k)\}^2}{v-V}\exp\{ik(d-Vt)\}dk. \quad \ldots \quad \ldots \quad (3.6)$$

Now after a little reduction we obtain

$$A = 2(k_0)^{\frac{1}{2}}\int_0^\infty \phi(u)e^{if(u)}\,du, \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

where

$$\phi(u) = u^3\{\psi(u^2)\}^2/(u+\sqrt{k_0}),$$

$$f(u) = d.u^2+pu \text{ and } p = g^{\frac{1}{2}}t.$$

This gives

$$f'(u) = 2du+p,$$

so that

$$\alpha = (\text{a root of } f'(u)=0) = -p/2d = -Ve.$$

By Kelvin's group approximation method (Lamb, 1932) contribution to the integral (3.7) comes only from α, and as α lies outside the range of integration, $A = 0$. Hence

$$\pi g\rho R_2 = -\tfrac{1}{2}iB$$

where after some reduction we obtain

$$B = 2(k_0)^{\frac{1}{2}} \int_0^\infty \phi(u) e^{if(u)}\, du, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

where

$$\phi(u) = u^3\{\psi(u^2)\}^2/(u-\sqrt{k_0}),$$

and

$$f(u) = du^2 - pu\,.$$

It may be observed that the principal value of this integral exists and can be evaluated by Kelvin's group approximation method; the contribution comes from $\alpha = p/2d$, which is a root of $f'(u) = 0$.
Thus we get

$$B = 2\sqrt{k_0}\,.\, \frac{\sqrt{\pi}\cdot\phi(\alpha)}{\sqrt{|\frac{1}{2}f''(\alpha)|}} \exp\left[i\left\{f(\alpha) \pm \frac{\pi}{4}\right\}\right],$$

the upper or the lower sign in the exponential being taken according as $f''(\alpha)$ is +ve or −ve respectively.

From the above formula, taking the real part only, we have

$$\rho R_2 = -\frac{t^2\sqrt{g}}{4c\pi^{\frac{1}{2}}d^{\frac{3}{2}}}\left\{\psi(p^2/4d^2)\right\}^2 \sin\left(\frac{\pi}{4}-\frac{p^2}{4d}\right). \quad \ldots \quad \ldots \quad (3.9)$$

4. *Calculation of R_3*

To get R_3 we proceed as in the case of calculation of R_2. The part of the time-integral in (1.1) which gives rise to y_3 and hence to R_3 is given by

$$-\frac{1}{2i}\left(\frac{2\pi}{kf}\right)^{\frac{1}{2}} \exp\left[i\left\{\frac{k(v^2+V^2)}{2f} - \frac{\pi}{4}\right\}\right] \exp\left(-ikvV/f\right). \quad \ldots \quad (4.1)$$

Again replacing the time-integral part of (3.3) by (4.1) we get

$$\pi g\rho R_3 = \frac{1}{2}\left(\frac{2\pi g}{f}\right)^{\frac{1}{2}} \int_0^\infty k\{\psi(k)\}^2 \exp\left[i\left\{\frac{k(v^2+V^2)}{2f} - \frac{\pi}{4}\right\}\right]$$
$$\times \exp\left(-ikvV/f\right) dk\,.$$

Putting $V^2 = g/k$, the above expression reduces to

$$\frac{1}{2}\left(\frac{2\pi g}{f}\right)^{\frac{1}{2}} e^{i\left(\frac{g}{2f}-\frac{\pi}{4}\right)} \int_0^\infty k\{\psi(k)\}^2 e^{i\left(\frac{v^2}{2f}k - \frac{vg^{\frac{1}{2}}}{f}k^{\frac{1}{2}}\right)} dk$$

$$= \left(\frac{2\pi g}{f}\right)^{\frac{1}{2}} e^{i\left(\frac{g}{2f}-\frac{\pi}{4}\right)} \int_0^\infty u^3\{\psi(u^2)\}^2 e^{i\left(d_1u^2 - p_1u\right)} du$$

$$= \left(\frac{2\pi g}{f}\right)^{\frac{1}{2}} e^{i\left(\frac{g}{2f}-\frac{\pi}{4}\right)} \int_0^\infty \phi(u) e^{if(u)} du. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

where we have put

$$d_1 = v^2/2f,\ p_1 = vg^{\frac{1}{2}}/f,\ \phi(u) = u^3\{\psi(u^2)\}^2,$$

and

$$f(u) = d_1u^2 - p_1u.$$

Evaluating (4.2) again by Kelvin's group approximation method we have

$$\pi g\rho R_3 = \left(\frac{2\pi g}{f}\right)^{\frac{1}{2}} e^{i\left(\frac{g}{2f}-\frac{\pi}{4}\right)} \frac{\sqrt{\pi}\cdot\phi(\alpha)}{\sqrt{|\frac{1}{2}f''(\alpha)|}} e^{i\left\{f(\alpha)+\frac{\pi}{4}\right\}}$$

where $\alpha = p_1/2d_1$ is obtained from $f'(\alpha) = 0$.
Further as

$$f''(\alpha) = 2d_1, \quad f(\alpha) = -p_1^2/4d_1,$$

and

$$\phi(\alpha) = p_1^3\{\psi(p_1^2/4d_1^2)\}^2/8d_1^3$$

we get after some simplification

$$\rho R_3 = \frac{2g}{v^4}\left\{\psi(p_1^2/4d_1^2)\right\}^2 \exp\left\{i\left(\frac{g}{2f} - \frac{p_1^2}{4d_1}\right)\right\}$$

$$= \frac{2g}{v^4}\left\{\psi(g/v^2)\right\}^2 = 2k_0^2\{\psi(k_0)\}^2$$

$$= 2\rho R_1.$$

$$\therefore R_3 = 2R_1. \qquad \ldots \quad (4.3)$$

5. *Numerical tables for R_1, R_2 and R_3*

Let us choose

$$\psi(k) = Av^2 \exp(-\gamma k). \qquad \ldots \quad (5.1)$$

Then from (2.1)

$$\rho R_1 = A^2 g \exp(-2\gamma k_0). \qquad \ldots \quad (5.2)$$

From (3.9) and (5.1) we have

$$\rho R_2 = -\frac{Ag^{\frac{1}{2}}t^2v^4}{4c\pi^{\frac{1}{2}}d^{\frac{5}{2}}} \exp(-\gamma p^2/2d^2) \sin\left(\frac{\pi}{4} - \frac{p^2}{4d}\right). \qquad \ldots \quad (5.3)$$

Taking $A = 1$, $\gamma = 1$, we have

$$\rho R_2 = -\frac{8(c+ft)^4 \exp(-p^2/2d^2)}{c\pi^{\frac{1}{2}}t^{\frac{1}{2}}(2c+ft)^{\frac{5}{2}}} \sin\left(\frac{\pi}{4} - \frac{p^2}{4d}\right)$$

$$= -\rho \underline{R}_2 \sin\left(\frac{\pi}{4} - \frac{p^2}{4d}\right), \qquad \ldots \quad (5.3a)$$

where $\underline{R}_2$ is the amplitude of R_2, so that

$$\rho \underline{R}_2 = 8(c+ft)^4 \exp(-p^2/4d^2)/c\sqrt{\pi t}(2c+ft)^{\frac{5}{2}}. \qquad \ldots \quad (5.4)$$

R_3 is given in terms of R_1 by (4.3).

In addition to $A = 1$, and $\gamma = 1$, let us take $c = 2{\cdot}5$ ft./sec., $f = \frac{1}{4}$ ft./sec.^{2}, and tabulate R_1, $\underline{R}_2$ and R_3 as given by (5.2), (5.4) and (4.3) between the times 5 and 60 seconds.

TABLE I

For ρR_1, $\rho \underline{R}_2$, $\rho(R_1+R_3)$

t	5	10	15	20	25	30	35	40	45	50	55	60
ρR_1	·34	2·47	6·19	10·26	13·87	16·87	19·30	21·25	22·81	24·08	25·11	25·97
$\rho \underline{R}_2$	1·38	2·18	2·99	3·83	4·85	5·84	6·85	7·88	8·79	9·98	11·04	12·12
$\rho(R_1+R_3)$	1·02	7·41	18·57	30·78	41·61	50·61	57·90	63·75	68·43	72·24	75·33	77·91

Since $R_3 = 2R_1$, $R_1+R_3 = 3R_1$, the entries in the last row have been obtained by multiplying the corresponding entries in the first row by 3. The actual graph of ρR_2 is an oscillating curve of moderate frequency lying between $\pm\rho \underline{R}_2$. The graph of $\pm\rho \underline{R}_2$ is shown by the bounding curves in Fig. 2. From the tables it is clear that $\underline{R}_2$ is small in comparison with the other parts.

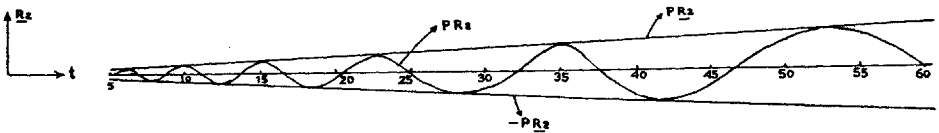

FIG. 2

Both R_1 and R_3 go on increasing with time but R_3 dominates over the other components. R_3, which is the special contribution of the acceleration, lasts as long as the acceleration lasts. Although it is the special effect of acceleration, it does not depend purely on the acceleration but, like R_1, is determined by the instantaneous velocity. But for reasons given in paper I (p. 159), R_3 would disappear if the acceleration vanishes while R_1 would persist in that case also.

6. *Zeros, humps and hollows of* R_2

We have from (5.3*a*) and (5.4)

$$R_2 = \underline{R}_2 \sin\theta$$

where $\underline{R}_2$ has been graphed before and

$$\theta = \frac{p^2}{4d} - \frac{\pi}{4} = \frac{64t}{20+t} - \frac{\pi}{4},$$

where use has been made of

$$p = g^{\frac{1}{2}}t,\ d = ct + \tfrac{1}{2}ft^2,\ c = 2{\cdot}5 \text{ ft./sec.}$$

and $$f = \tfrac{1}{4} \text{ ft./sec.}^2$$

Thus $$\theta \longrightarrow 64 - \frac{\pi}{4}, \text{ as } t \longrightarrow \infty.$$

The zeros of R_2 are given by

$$\sin\theta = 0, \text{ i.e. } \theta = n\pi,$$

or

$$\frac{64t}{20+t} = \left(n+\frac{1}{4}\right)\pi = K\pi,$$

where

$$K = n+\tfrac{1}{4}.$$

This gives

$$t = 20\,K\pi/(64-K\pi).$$

The following table shows the results of numerical calculation:

TABLE II

Zeros of R_2

n	4	5	6	7	8	9	10	11	12	13	14	15
t	5·3	6·9	8·8	11	13·6	16·6	20·2	24·7	32·2	37·2	46·5	59·5

The number of fluctuations in a specified interval of time gradually diminishes with time and ultimately for a large time the motion is without fluctuation. The wave length of the fluctuation increases with time.

R_2 is positive between 5·3 and 6·9 seconds, and negative between 6·9 and 8·8 seconds, etc. This is shown by signs like $\overset{+}{\cap}$ and $\underset{-}{\cup}$. Assuming that maximum positive and negative amplitudes occur roughly half-way between two consecutive zeros and reading these extremum values from the graphs of $\pm R_2$ we get the following table for the humps and hollows of R_2.

TABLE III

Humps and Hollows of R_2

t	6·1	7·9	9·9	12·3	15·1	18·4	22·5	27·5	33·7	41·9	53
R_2 (extremum)	+1·6	−1·8	+2·2	−2·5	+3	−3·5	+4·2	−5·2	+6·6	−8·1	+10·6

With the help of this table and table II the actual graph of R_2 can be drawn. This is shown in Fig. 2.

7. *Total resistance*

The total resistance is

$$R = R_1+R_2+R_3.$$

From Table I the graph of R_1+R_3 is drawn (shown in Fig. 3). R_2 has been superposed with the help of tables II and III on the graph of R_1+R_3. R_2 represents a sort of perturbation effect on R_1+R_3.

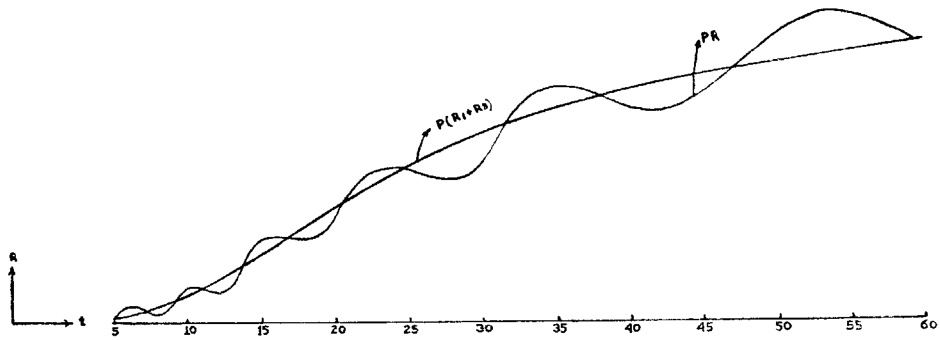

FIG. 3.

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ABSTRACT

In an earlier paper the surface wave produced by the uniformly accelerated motion of a pressure system over the undisturbed surface of deep water was worked out in the form of an infinite repeated integral which was evaluated by Kelvin's method of group approximation. The wave resistance corresponding to these waves has been worked out in the present paper. It consists of three parts, R_1, R_2 and R_3. R_3 represents the direct effect of acceleration; R_1 and R_2 are analogous to Havelock's results (1917) for uniform motion of a pressure system. R_2 in any case is small, while R_3, due to acceleration, is more prominent than the other two components.

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