ON THE ORDER OF THE CESÀRO MEANS OF FOURIER SERIES AND ITS SUCCESSIVELY DERIVED SERIES

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1.1. Let $f(\theta)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let the Fourier series associated with $f(\theta)$ be

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \qquad \ldots \qquad \ldots \qquad (1.1.1)$$

Then the rth derived series of the Fourier series (1.1.1) is

$$\sum_{n=1}^{\infty} \frac{d^r}{d\theta^r} (a_n \cos n\theta + b_n \sin n\theta). \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1.1.2)$$

We write

$$\Phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\};$$

$$P(t) = \sum_{p=0}^{r-1} d_p \frac{t^p}{p!},$$

d_p's being constants independent of t;

$$g(t) = \frac{1}{2t^r}\left[\{f(x+t) - P(t)\} + (-1)^r\{f(x-t) - P(-t)\}\right];$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \ (\alpha > 0);$$

$$\Phi_0(t) = \phi(t);$$

$$\phi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t);$$

and define $g_\alpha(t)$ in a similar way.

Also we write

$$V_\alpha(x) = \int_0^1 (1-t)^{\alpha-1} \cos xt\, dt, \ (\alpha > 0);$$

$$J_\beta^\alpha(x) = \frac{(-1)^{h+1} x^{h+1}}{\Gamma(1+h-\alpha)\Gamma(\alpha+1)} \int_1^\infty (t-1)^{h-\alpha} V_{1+\beta}^{(h+1)}(xt)\, dt, \ (\beta \geqslant \alpha \geqslant 0),$$

where h is the greatest integer not greater than α.

Suppose that there exists a polynomial $P(t)$ of $(r-1)$th degree in t, such that $g(t)$, defined as above, is integrable (L) in $(-\pi, \pi)$, and is defined by periodicity outside this range, then the rth generalized derivative of $f(\theta)$, at $\theta = x$, has been defined (Zygmund, 1935) to be the limit of $r!\,g(t)$, as $t \to 0$, and is equal to the rth derivative of $f(\theta)$, namely $f^r(\theta)$, at $\theta = x$, in case the latter exists and is finite.

1.2. DEFINITION (see Hardy and Riesz, 1952). Let $\Sigma_1^\infty u_n$ be a given infinite series, and λ_n a positive steadily increasing function of n, tending to infinity with n. We write

$$A_\lambda(w) = A_\lambda^0(w) = \sum_{\lambda_n \leq w} u_n,$$

$$A_\lambda^r(w) = \sum_{\lambda_n \leq w} (w-\lambda_n)^r u_n$$

and

$$C_\lambda^r(w) = A_\lambda^r(w)/w^r.$$

The series $\Sigma_1^\infty u_n$ is said to be summable (R, λ, r), $r \geqslant 0$, if $\lim_{w\to\infty} c_\lambda^r(w)$ exists and is finite.

The summability (R, λ, r), for the particular cases $\lambda_n = n$ and $\lambda_n = \log n$, is equivalent to the summabilities (c, r) and $(R, \log n, r)$, respectively. (Hardy and Riesz, 1952.)

1.3. In the year 1909 Lebesgue (see Titchmarsh, 1949) investigated the order of the partial sums of Fourier series. He proved that if $f(x)$ is continuous, then $S_n = o(\log n)$, S_n being the nth partial sum of the Fourier series of $f(x)$. By the help of Fejér's (1910 *a*, *b*, *c*) examples it was found that *no more is true* in the above statement, since if $\epsilon(n)$ is a function which decreases steadily to zero, as $n \to \infty$, however slowly, there is a Fourier series of a continuous function for which $S_n > \epsilon(n) \log n$, for arbitrary large values of n. The result of Lebesgue for the order of the partial sums of Fourier series has been extended by Sunouchi (1951) so as to be applicable to Cesàro means of partial sums. In § 3 of this paper we have shown that Sunouchi's result can be obtained under less restrictive conditions. In § 4 we have shown, by framing examples, that the result obtained on the orders of these means is *the best possible*.

In § 5 of this paper we have shown that from the knowledge of the order of the Cesàro means of Fourier series, the orders of the Cesàro means of its successively rth derived series can be obtained easily. Recently Mohanty and Nanda (1954) have published some results giving the orders of the Cesàro means of the first derived series of Fourier series, but this result can be easily derived either by putting the above-mentioned result of Lebesgue in a kernel established by Wang (1934), or still, in a rather generalized form, by putting $r = 1$ in Theorem 1 of this paper.

From the theorems of this paper in conjunction with certain known results, we have deduced like corollaries, a number of results which will throw some useful light on the theory of this subject.

2. We give some lemmas in this section which will be used in the proofs of the various theorems.

LEMMA 1 (Bosanquet, 1934). *The function* $J_\beta^\alpha(x)$, $\beta \geqslant \alpha > 0$, *and its derivatives are bounded for* $x \geqslant 0$, *and for large values of* x

$$J_\beta^\alpha(x) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)} \frac{\cos\left\{x - \frac{\pi}{2}(\alpha+\beta+1)\right\}}{x^{1+\beta-\alpha}} + O\left(\frac{1}{x^{2+\beta-\alpha}}\right) + O\left(\frac{1}{x^2}\right).$$

LEMMA 2 (Bosanquet, 1934). *If $S_\beta(w)$ denotes the* (C, β) *mean of the Fourier series of $f(\theta)$, at $\theta = x$, then*

$$S_\beta(w) - S = w\int_0^\eta \phi_\alpha(t) J_\beta^\alpha(wt)dt + o(1), \quad (\eta > 0, \beta \geqslant \alpha \geqslant 0).$$

LEMMA 3. *If*

$$\int_0^t |\phi_\alpha(u)|\,du = o\left\{t\left(\log\frac{1}{t}\right)^p\right\}, \quad (\alpha \geqslant 0),$$

as $t \to 0$, then we have

$$S_\alpha(w) = o(1) + o\{(\log w)^p\} + \int_{\frac{\pi\alpha}{w}}^{\pi} \phi_\alpha(t)\frac{\sin(wt - \pi\alpha)}{t}dt + s$$

$$= o\{(\log w)^{p+1}\}, \quad (-1 < p < \infty),$$

and

$$S_{\alpha+\delta}(w) - S = o\{(\log w)^p\}, \quad (\delta > 0, p \geqslant 0).$$

Proof. The proof of the first part of this lemma for the case ($\alpha > 0$, $-1 < p < \infty$) is given by Sunouchi (1951) and for the case ($\alpha = 0$, $p \geqslant 0$) is given by F. T. Wang (1935-36). The proof for the case ($\alpha = 0$, $-1 < p < 0$) follows from the arguments similar to those of Wang. For the second case we have from Lemma 2 and Lemma 1

$$S_{\alpha+\delta}(w) - S = w\int_0^\eta \phi_\alpha(t) J_{\alpha+\delta}^\alpha(wt)\,dt + o(1)$$

$$= O\left(w\int_0^{\frac{1}{w}} |\phi_\alpha(t)|\,dt\right) + O\left(w\int_{\frac{1}{w}}^{\eta} \frac{|\phi_\alpha(t)|}{(wt)^{1+\delta}}\,dt\right)$$

$$= o\{(\log w)^p\} + O\left[\frac{1}{w^\delta}\left\{\phi_\alpha^*(t)\cdot\frac{1}{t^{1+\delta}}\right\}_{t=\frac{1}{w}}^{\pi} + \frac{1+\delta}{w^\delta}\int_{\frac{1}{\omega}}^{\pi}\frac{\phi_\alpha^*(t)}{t^{2+\delta}}\,dt\right],$$

where

$$\phi_\alpha^*(t) = \int_\sigma^t |\phi_\alpha(t)|\,dt.$$

Or

$$S_{\alpha+\delta}(w) - S = o\{(\log w)^p\},$$

which completes the proof.

LEMMA 4 (Wang, 1934). *If $B_r^k(w)$ denotes the* (c, *k*) *mean, for $k \geqslant r$, of the rth derived series of the Fourier series of $f(\theta)$, at $\theta = x$, and $A_r^{k-i}(w)$ denotes the* (c, *k*$-$*i*) *mean of the Fourier series associated with $g(t)$, at $t = 0$, then we have*

$$B_r^k(w) - r!\,S = \sum_{i=0}^{r}(-1)^i C_i^r \frac{\Gamma(1+k+r-i)}{\Gamma(1+k-i)}\left\{A_r^{k-i}(w) - S\right\} + o(1),$$

where

$$C_i^r = \frac{(r-i+1)(r-i+2)\ \ldots\ (r)}{\Gamma(i+1)}.$$

LEMMA 5 (Hardy, 1949). *If*

$$\sum_{r=1}^{\infty} u_r$$

is summable (c, $k+1$), $k > -1$, *then a necessary and sufficient condition that it should be summable* (c, k) *is that*

$$\sum_{r=1}^{n} A_{n-r}^{k} r u_r = o(n^{k+1}),$$

as $n \to \infty$, *where*

$$A_m^k = \frac{(k+1)(k+2)\ \ldots\ (k+m)}{m\,!}.$$

LEMMA 6. (*a*) *If* (c, α) *means of the series*

$$\sum_{r=1}^{\infty} u_r$$

are o (log n), *as* $n \to \infty$, *for* $\alpha \geqslant 0$, *and if its* (c, $\alpha+1$) *mean exists, then the series*

$$\sum_{r=1}^{\infty} \frac{u_r}{\log (r+1)}$$

is summable (c, α), *and*

(*b*) *If the partial sums of the series*

$$\sum_{r=1}^{\infty} u_r$$

are $o\{(\log n)^{p+1}\}$ *and its* (c, 1) *means are* $s+o\{(\log n)^p\}$, *as* $n \to \infty$, *for* $-1 < p < \infty$, *then the series*

$$\sum_{r=1}^{\infty} \frac{u_r}{\{\log (r+1)\}^{p+1}}$$

is convergent.

Proof. For the proof of part (*a*), it will be sufficient to show, by Lemma 5, that

$$J = \sum_{r=1}^{n} A_{n-r}^{\alpha} \frac{r u_r}{\log (r+1)} = o(n^{\alpha+1}),\ (\alpha \geqslant 0),$$

as $n \to \infty$.
Applying Abel's transformation twice we get, denoting $\mu_n - \mu_{n+1}$ by $\Delta \mu_n$,

$$J = \sum_{r=1}^{n-2} \tau_r^{\alpha+1} \Delta^2 \left\{ \frac{r}{\log (r+1)} \right\} + \tau_n^{\alpha} \frac{n}{\log (n+1)} + \tau_{n-1}^{\alpha+1} \Delta \left(\frac{n-1}{\log n} \right),$$

where

$$\tau_r^{\alpha} = \sum_{p=1}^{r} A_{n-p}^{\alpha} u_p,\ (r \leqslant n),$$

and

$$\tau_r^{\alpha+1} = \sum_{p=1}^{r} \tau_p^{\alpha}.$$

Now it is easy to see that

$$\Delta\left(\frac{r}{\log (r+1)}\right) = O\left\{\frac{1}{\log (r+1)}\right\},$$

$$\Delta^2\left(\frac{r}{\log (r+1)}\right) = O\left\{\frac{1}{r (\log (r+1))^2}\right\},$$

$$\tau_n^{\alpha} = o(n^{\alpha} \log n),$$

$$\tau_n^{\alpha+1} = O(n^{\alpha+1}),$$

$$\tau_r^{\alpha+1} = o(n^{\alpha} \log n \,.\, r).$$

Thus we get

$$J = O\left[\sum_{r=1}^{n} |\tau_r^{\alpha+1}| \Big/ (r\{\log (r+1)\}^2)\right] + o(n^{\alpha+1}) + O\left(\frac{n^{\alpha+1}}{\log n}\right)$$

$$= o\left[n^{\alpha} \log n \sum_{r=1}^{n} \frac{r}{r\{\log (r+1)\}^2}\right] + o(n^{\alpha+1})$$

$$= o(n^{\alpha+1}),$$

which proves the part (*a*) of the lemma.

For part (*b*) it is sufficient to prove that

$$I = \sum_{r=n}^{m} \frac{u_r}{\{\log (r+1)\}^{p+1}} = o(1),$$

as $n \to \infty$, $m \to \infty$.

Now putting

$$\sum_{r=1}^{k} u_r = S_k \quad \text{and} \quad \sum_{r=1}^{k} S_r = S_k^1,$$

we have by applying Abel's transformation twice

$$I = \sum_{r=n}^{m-2} S_r^1 \, \Delta^2\left[\frac{1}{\{\log (r+1)\}^{p+1}}\right] - \frac{S_{n-1}}{(\log n)^{p+1}} + \frac{S_m}{\{\log (m+1)\}^{p+1}}$$

$$-S_{n-1}^1 \, \Delta\left[\frac{1}{(\log n)^{p+1}}\right] + S_{m-1}^1 \, \Delta\left[\frac{1}{(\log m)^{p+1}}\right]$$

$$= O\left[\sum_{r=n}^{m-2} \frac{1}{r (\log r)^2}\right] + o(1)$$

$$= o(1),$$

which completes the proof.

LEMMA 7 (Zygmund, 1925). *If* $\Sigma u_n = o(1)$ (R, n, β), *for some value of* $\beta > 0$, *and if* $S_\alpha(w) = o\left[\{p(w)\}^\alpha\right]$, $\alpha > 0$, *where* $p(w)$ *is an increasing function which tends to infinity as* $w \to \infty$, *and* $S_\alpha(w)$ *denotes the* (R, w, α) *mean of the series* Σu_n, *then* $\Sigma u_n = o(1)$ $[R, u(w), \alpha]$, where

$$u(w) = \exp\left[\int^{w} \frac{dt}{tp(t)}\right].$$

3.1. We now establish the following theorem:

THEOREM 1. *If*

$$\int_{t}^{\pi} \frac{|\phi_\alpha(t)|}{t}\,dt = o\left[\left(\log\frac{1}{t}\right)^{p+1}\right], \ (\alpha \geqslant 0, \ -1 < p < \infty),$$

as $t \to 0$, *then we have*

$$S_\alpha(w) = o[(\log w)^{p+1}],$$

as $w \to \infty$, $S_\alpha(w)$ *being the* (c, α) *mean of the Fourier series of* $f(\theta)$, *at* $\theta = x$.

Proof. We have, from Lemma 2,

$$S_\alpha(w) - S = w\int_0^{\eta} \phi_\alpha(t) J_\alpha^\alpha(wt)\,dt + o(1), \ (\eta > 0)$$

$$= w\int_0^{\frac{1}{w}} + w\int_{\frac{1}{w}}^{\eta} + o(1)$$

$$= Q_1(w) + Q_2(w) + o(1), \qquad \ldots \quad (3.1.1)$$

say. We have

$$Q_1(w) = O\left[w\int_0^{\frac{1}{w}} \frac{|\phi_\alpha(t)|}{t}\,.\,t\,dt\right]$$

$$= O\left[-\left\{wt\int_t^{\pi}\frac{|\phi_\alpha(u)|}{u}\,du\right\}_{t=0}^{\frac{1}{w}} + w\int_0^{\frac{1}{w}}\left\{\int_t^{\pi}\frac{|\phi_\alpha(u)|}{u}\,du\right\}dt\right]$$

$$= o[(\log w)^{p+1}] + o\left[w\int_0^{\frac{1}{w}}\left(\log\frac{1}{t}\right)^{p+1}dt\right]$$

$$= o[(\log w)^{p+1}]. \qquad \ldots \quad (3.1.2)$$

Also by Lemma 1 we have

$$Q_2(w) = \int_{\frac{1}{w}}^{\eta} \phi_\alpha(t)\frac{\sin(wt - \pi\alpha)}{t}\,dt + O\left[\frac{1}{w}\int_{\frac{1}{w}}^{\eta}\frac{|\phi_\alpha(t)|}{t^2}\,dt\right]$$

$$= Q_{2,1}(w) + Q_{2,2}(w), \qquad \ldots \quad (3.1.3)$$

say. Now

$$Q_{2,1}(w) = O\left\{ \int_{\frac{1}{w}}^{\eta} \frac{|\phi_\alpha(t)|}{t} dt \right\}$$

$$= o\{(\log w)^{p+1}\}, \qquad \ldots \quad \ldots \quad \ldots \quad (3.1.4)$$

and

$$Q_{2,2}(w) = O\left\{ \int_{\frac{1}{w}}^{\eta} \frac{|\phi_\alpha(t)|}{t} dt \right\}$$

$$= o\{(\log w)^{p+1}\}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.1.5)$$

Thus we get from (3.1.1),........(3.1.5),

$$S_\alpha(w) = o\{(\log w)^{p+1}\},$$

which completes the proof.

4.1. We shall now show that the result of the above theorem is the *best possible* of its kind. We shall discuss this from two points of view. Firstly, we shall show that:

There is an even periodic integrable function $\phi(t)$ *such that*

$$\phi_\alpha(t) = o\left\{\left(\log \frac{1}{t}\right)^p\right\}, \; (\alpha \geqslant 0, \; -1 < p < \infty),$$

as $t \to 0$, *but, at* $t = 0$,

$$S_\alpha(w) > \xi_w (\log w)^{p+1}, \qquad \ldots \quad \ldots \quad \ldots \quad (4.1.1)$$

for any ξ_w *tending to zero as* $w \to \infty$, *however slowly and for arbitrary large values of* w.

Secondly, we show that:

There is an even periodic integrable function $\phi(t)$ *such that*

$$\int_0^t |\phi_\alpha(u)|\, du = O\left\{t\left(\log \frac{1}{t}\right)^p\right\}, \; (\alpha \geqslant 0, \; -1 < p < \infty),$$

and

$$\int_0^t \phi_\alpha(u) du = o\left\{t\left(\log \frac{1}{t}\right)^p\right\},$$

as $t \to 0$, *but, at* $t = 0$,

$$S_\alpha(w) \neq o\{(\log w)^{p+1}\}, \qquad \ldots \quad \ldots \quad \ldots \quad (4.1.2)$$

for arbitrary large values of w.

We shall demonstrate these by constructing examples.

4.2. To demonstrate (4.1.1) we construct the following example:

EXAMPLE 1. *Consider first* α *to be a non-integral number and let* $[\alpha]$ *and* $\{\alpha\}$ *denote its integral and fractional parts, respectively. Let* $h = [\alpha]+1$. *Take the sequence of integers* $r_0, r_1, r_2, \ldots\ldots\ldots\ldots$ *monotonically tending to infinity such that* $r_0 = A$, *where* A *is a sufficiently large integer, and*

$$r_i = \left(\frac{\left[(r_{i-1})^{(\{\alpha\}-\epsilon)/(1+\alpha)}\right]}{\alpha} + 1\right) r_{i-1}, \; (0 < \epsilon < \{\alpha\}),$$

where the inner brackets [ ] *denote the integral part of their content.*

Let

$$\alpha_i = \frac{\pi\alpha}{r_i},$$

and $\{C_i\}$ *be a sequence tending to zero as* $i \to \infty$, *such that*

$$\frac{\xi_i}{C_i} \to 0,$$

for any given sequence $\{\xi_i\}$ *tending to zero as* $i \to \infty$, *however slowly. We denote the interval* $[\alpha_i, \alpha_{i-1}]$ *by* I_i, *and define an even periodic function* $\sigma(t)$ *such that*

$$\sigma(t) = \frac{C_{n_i}}{\Gamma(\alpha+1)} t^{\alpha}\left(\log\frac{1}{t}\right)^p \sin\,(r_{n_i}t - \pi\alpha),$$

for values of t lying in the interval In_i, *and*

$$\sigma(t) = 0,$$

for t lying everywhere else in (o, π), *the sequence* $\{n_i\}$ *consisting of positive integers monotonically tending to infinity taken with sufficient large gaps.* †

We define

$$\phi(t) = \frac{1}{\Gamma(1-\{\alpha\})}\int_0^t (t-u)^{-\{\alpha\}}\sigma^{(h)}(u)\,du.$$

In order to show that $\phi(t)$ is integrable (L) it will be sufficient to show that $\sigma^{(h)}(t)$ is integrable (L).

We have, for t lying in I_{n_i},

$$\Gamma(\alpha+1)\frac{\sigma(t)}{C_{n_i}} = t^{\alpha}\left(\log\frac{1}{t}\right)^p \sin\,(r_{n_i}t - \pi\alpha),$$

$$\Gamma(\alpha+1)\frac{\sigma^{(1)}(t)}{C_{n_i}} = t^{\alpha-1}\sin\,(r_{n_i}t-\pi\alpha)\left\{\alpha\left(\log\frac{1}{t}\right)^p - p\left(\log\frac{1}{t}\right)^{p-1}\right\}$$

$$-r_{n_i}t^{\alpha}\left(\log\frac{1}{t}\right)^p \sin\left(r_{n_i}t-\pi\alpha-\frac{\pi}{2}\right)$$

$$= O(t^{\alpha-1-\epsilon_1}) + O(r_{n_i}t^{\alpha-\epsilon_1}),$$

where ϵ_1 is as small as we please but always greater than zero. Differentiating again $\sigma^1(t)$, $(h-1)$ times, we get finally

$$\frac{\sigma^{(h)}(t)}{C_{n_i}} = O\{(r_{n_i})^h t^{\alpha-\epsilon_1}\} + O\{(r_{n_i})^{h-1}t^{\alpha-1-\epsilon_1}\} + \ldots .$$

$$+ O\{(r_{n_i})^{h-m}t^{\alpha-m-\epsilon_1}\} + \ldots . + O(t^{\alpha-h-\epsilon_1}).$$

Or

$$\sigma^{(h)}(t) = P_1 + P_2 + \ldots . + P_{m+1} + \ldots . + P_{h+1},$$

† It is easy to see that

$$\sigma(\alpha_{n_i}) = \sigma(\alpha_{n_{i-1}}) = 0.$$

say. Now

$$\int_{\alpha_{n_i}}^{\alpha_{n_i-1}} P_{m+1}\,dt = O\left\{\int_{\pi\alpha/r_{n_i}}^{\pi\alpha/r_{n_i-1}} C_{n_i}(r_{n_i})^{h-m} t^{\alpha-m-\epsilon_1}\,dt\right\}$$

$$= O\left\{\frac{C_{n_i}}{(r_{n_i})^{\{\alpha\}-\epsilon_1}}\left(\frac{r_{n_i}}{r_{n_i-1}}\right)^{\alpha+1-\epsilon_1}\right\}$$

$$= O\left\{\left(\frac{r_{n_i-1}}{r_{n_i}}\right)^{\{\alpha\}-\epsilon} C_{n_i}\right\}$$

$$= O\left\{\frac{C_{n_i}}{(r_{n_i-1})^{(\{\alpha\}-\epsilon)^2/(1+\alpha)}}\right\}.$$

Hence we get for all $o \leqslant m \leqslant h$,

$$\int_0^{\pi} P_{m+1}dt = O\left\{\sum_1^{\infty}\frac{C_{n_i}}{(r_{n_i-1})^{(\{\alpha\}-\epsilon)^2/(1+\alpha)}}\right\}$$

$$= O(1),$$

which shows that

$$\int_0^{\pi} |\sigma^{(h)}(t)|\,dt = O\left\{\int_0^{\pi}(P_1+P_2+\ldots\ldots+P_{h+1})dt\right\} = O(1),$$

and thus $\sigma^{(h)}(t)$ is integrable (L) and consequently $\phi(t)$ is also integrable (L). Again, by a theorem of fractional integrals, we have

$$\Phi_{\alpha}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\left\{\frac{1}{\Gamma(1-\{\alpha\})}\int_0^u (u-v)^{-\{\alpha\}}\sigma^{(h)}(v)\,dv\right\}du$$

$$= \frac{1}{\Gamma(\alpha)}\cdot\frac{1}{\Gamma(1-\{\alpha\})}\int_0^t \sigma^{(h)}(v)\,\frac{\Gamma(\alpha)\Gamma(1-\{\alpha\})}{\Gamma(\alpha+1-\{\alpha\})}(t-v)^{\alpha-\{\alpha\}}\,dv$$

$$= \frac{1}{\Gamma(h)}\int_0^t (t-v)^{h-1}\sigma^{(h)}(v)\,dv$$

$$= \sigma(t).$$

Thus we get

$$\phi_{\alpha}(t) = \Gamma(\alpha+1)t^{-\alpha}\Phi_{\alpha}(t)$$

$$= C_{n_i}\left(\log\frac{1}{t}\right)^p \sin(r_{n_i}t-\pi\alpha),$$

for t lying in I_{n_i}, and

$$\phi_{\alpha}(t) = 0,$$

everywhere else in (o, π). It is easily seen that

$$\phi_\alpha(t) = o\left\{\left(\log \frac{1}{t}\right)^p\right\}.$$

Thus we have framed an integrable and periodic function $\phi(t)$ such that

$$\phi_\alpha(t) = o\left\{\left(\log \frac{1}{t}\right)^p\right\}.$$

We now show that

$$S_\alpha(w) > \xi_w (\log w)^{p+1}.$$

By Lemma 3, we have

$$S_\alpha(r_{n_i}) = o(1)+o\{(\log (r_{n_i}))^p\}+\int_{\pi\alpha/r_{n_i}}^{\pi} \phi_\alpha(t) \frac{\sin (r_{n_i}t-\pi\alpha)}{t} dt$$

$$= o(1)+o\{(\log (r_{n_i}))^p\}+J, \quad \ldots \quad (4.2.1)$$

say. Now

$$J = \int_{\pi\alpha/r_{n_i}}^{\pi\alpha/r_{n_i-1}} \frac{\phi_\alpha(t) \sin (r_{n_i}t-\pi\alpha)}{t} dt+\int_{\pi\alpha/r_{n_i-1}}^{\pi} \phi_\alpha(t) \frac{\sin (r_{n_i}t-\pi\alpha)}{t} dt$$

$$= J_1+J_2,$$

say. We have

$$J_1 = \frac{C_{n_i}}{2}\int_{\pi\alpha/r_{n_i}}^{\pi\alpha/r_{n_i-1}} \frac{1}{t}\left(\log \frac{1}{t}\right)^p dt+\frac{C_{n_i}}{2}\int_{\pi\alpha/r_{n_i}}^{\pi\alpha/r_{n_i-1}} \left(\log \frac{1}{t}\right)^p \frac{\cos (2r_{n_i}t-2\pi\alpha)}{t} dt$$

$$= J_{1,1}+J_{1,2}, \quad \ldots \quad (4.2.3)$$

say. Now we have

$$J_{1,1} = \frac{C_{n_i}}{2(p+1)}\left\{\left(\log \frac{r_{n_i}}{\pi\alpha}\right)^{p+1}-\left(\log \frac{r_{n_i-1}}{\pi\alpha}\right)^{p+1}\right\}$$

$$= \frac{C_{n_i}}{2(p+1)}\left\{\left(\log \frac{r_{n_i}}{\pi\alpha}\right)^{p+1}(A+0(1))\right\}$$

$$> \xi_{n_i}(\log r_{n_i})^{p+1}. \quad \ldots \quad (4.2.4)$$

Also we have, by second mean value theorem

$$J_{1,2} = C_{n_i}\cdot\frac{r_{n_i}}{2}\left(\log \frac{r_{n_i}}{\pi\alpha}\right)^p\int_{\pi\alpha/r_{n_i}}^{\zeta} \cos (2r_{n_i}t-2\pi\alpha)dt, \left(\frac{\pi\alpha}{r_{n_i}}<\zeta<\frac{\pi\alpha}{r_{n_i-1}}\right),$$

$$= O\{C_{n_i}(\log r_{n_i})^p\}$$

$$= o\{(\log r_{n_i})^p\}. \quad \ldots \quad (4.2.5)$$

Thus we get from (4.2.3), (4.2.4) and (4.2.5),

$$J_1 > \xi_{n_i}(\log r_{n_i})^{p+1}. \quad \ldots \quad (4.2.6)$$

Again we break J_2 as follows:

$$J_2 = \int_{\pi\alpha/r_{n_i}}^{\pi} \phi_\alpha(t) \frac{\sin(r_{n_i}t-\pi\alpha)\, dt}{t}$$

$$= \int_{\pi\alpha/r_{n_i}-1}^{\pi\alpha/r_{n_{i-1}}} + \int_{\pi\alpha/r_{n_{i-1}}}^{\pi}$$

$$= J_{2,1}+J_{2,2}, \quad \ldots \quad (4.2.7)$$

say. As $\phi_\alpha(t)$ is equal to zero for t lying in

$$\left[\frac{\pi\alpha}{r_{n_i}-1}, \frac{\pi\alpha}{r_{n_{i-1}}}\right],$$

we have

$$J_{2,1} = 0. \quad \ldots \quad (4.2.8)$$

Also

$$J_{2,2} = \sum_{l=1}^{i-1} \int_{\pi\alpha/r_{n_l}}^{\pi\alpha/r_{n_l}-1} C_{n_l} \frac{\sin(r_{n_l}t-\pi\alpha).\sin(r_{n_i}t-\pi\alpha)}{t} \left(\log\frac{1}{t}\right)^p dt$$

$$= \sum_{l=1}^{i-1} \int_{\pi\alpha/r_{n_l}}^{\pi\alpha/r_{n_l}-1} C_{n_l} \frac{\cos(r_{n_i}-r_{n_l})t-\cos\{(r_{n_i}+r_{n_l})t-2\pi\alpha\}}{t} \times \left(\log\frac{1}{t}\right)^p dt$$

$$= O\left(\sum_{l=1}^{i-1} \frac{r_{n_l}(\log r_{n_l})^p}{r_{n_i}-r_{n_l}}\right)$$

$$= O\left\{\frac{i(\log r_{n_{i-1}})^p}{\left(\dfrac{r_{n_i}}{r_{n_{i-1}}}-1\right)}\right\}$$

$$= o(1). \quad \ldots \quad (4.2.9)$$

Collecting all our results (4.2.2), (4.2.6) and (4.2.9), we get

$$S_\alpha(r_{n_i}) > \xi_{n_i}(\log r_{n_i})^{1+p},$$

which establishes our statement (4.1.1) for 'α' non-integral.

For the case when 'α' is an integer we have $\alpha = h$, and we put this time

$$r_i = \left\{\frac{[(r_{i-1})^{(1-\epsilon)/(1+\alpha)}]}{\alpha}+1\right\} r_{i-1}, \quad (0<\epsilon<1),$$

and define the function $\phi(t)$ as before. The remaining analysis follows similarly.

Thus the statement (4.1.1) is completely demonstrated.

4.3. To demonstrate the statement (4.1.2) we construct the following example:

EXAMPLE. *First consider 'α' to be a non-integral number and let $[\alpha]$, $\{\alpha\}$ denote its integral and fractional parts, respectively, and let $h = [\alpha]+1$. Let $\{x_i\}$ be a sequence of odd integers monotonically tending to infinity, sufficiently slowly. Also take the sequence of integers $r_0, r_1, r_2, \ldots\ldots$ monotonically tending to infinity such that*

$$r_0 = k$$

where k is a sufficiently large integer, and

$$r_i = \{2[[(r_{i-1})^{(\{\alpha\}-\delta)/h}]]/\alpha+1\}r_{i-1},\ (0<\delta<\{\alpha\}),$$

where by $[[p]]$ *we denote the nearest integer greater than* p *such that* $[[p]]/\alpha$ *is also an integer.*

We write

$$\mu_i = \frac{x_i}{\alpha}+2,$$

$$M_i = \mu_i r_i,$$

$$\alpha_i = \frac{\pi\alpha}{2r_i},$$

$$C_i = \left(\log\frac{2r_{i-1}}{\pi\alpha}\right)^p,$$

and denote the interval $[\alpha_i, \alpha_{i-1}]$ *by* I_i.

Let $\sigma(t)$ *be an even periodic function such that*

$$\sigma(t) = C_{n_i} t^{\alpha}\sin(M_{n_i}t-\pi\alpha),$$

for t lying in I_{n_i}, *where* n_i *is a sequence of positive increasing integers taken with sufficient large gaps, and*

$$\sigma(t) = 0,$$

everywhere else in $(0, \pi)$.

We define

$$\phi(t) = \frac{1}{\Gamma(1-\{\alpha\})}\int_0^t (t-u)^{-\{\alpha\}}\sigma^{(h)}(u)du.$$

We now show that $\phi(t)$ is integrable (L) by showing that $\sigma^h(t)$ is integrable (L). Proceeding as in the previous example we have, for t lying in I_{n_i},

$$\frac{\sigma^{(h)}(t)}{C_{n_i}} = O\{t^{\alpha}(M_{n_i})^h\}+O\{t^{\alpha-1}(M_{n_i})^{h-1}\}+\cdots$$
$$+O\{t^{\alpha-m}(M_{n_i})^{h-m}\}+\cdots+O(t^{\alpha-h})$$
$$= P_1+P_2+\cdots+P_{m+1}+\cdots+P_{h+1},$$

say. Now

$$\int_{\alpha_{n_i}}^{\alpha_{n_i-1}} C_{n_i}P_{m+1}\,dt = O\left\{\int_{\pi\alpha/(2r_{n_i})}^{\pi\alpha/(2r_{n_i-1})} C_{n_i}(M_{n_i})^{h-m}t^{\alpha-m}\,dt\right\}$$
$$= O\{C_{n_i}(M_{n_i})^{h-m}(1/r_{n_i-1})^{\alpha+1-m}\}$$
$$= O\left\{C_{n_i}\left(\frac{r_{n_i}}{r_{n_i-1}}\right)^{h-m}\frac{(\mu_{n_i})^{h-m}}{(r_{n_i-1})^{\{\alpha\}}}\right\}$$
$$= O\left\{\frac{C_{n_i}(\mu_{ni})^h}{(r_{n_i-1})^{\delta}}\right\}$$
$$= O\left\{\frac{1}{(r_{n_i-1})^{\delta_1}}\right\},\ (0<\delta_1<\delta),$$

for all values of m.

Thus we have

$$\int_0^\pi |\sigma^{(h)}(t)|\,dt = O\left\{\sum_{i=1}^\infty \frac{1}{(r_{n_i-1})^{\delta_1}}\right\}$$
$$= O(1),$$

which shows that $\sigma^{(h)}(t)$ and so consequently $\phi(t)$ is integrable (L). Also we obtain, as in the previous example

$$\phi_\alpha(t) = C_{n_i}\sin(M_{n_i}t-\pi\alpha),$$

for t lying in I_{n_i}, and

$$\phi_\alpha(t) = 0,$$

everywhere else in $(0, \pi)$.

We have, if i belongs to the sequence $\{n_p\}$,

$$\int_{\pi\alpha/(2r_i)}^{\pi\alpha/(2r_{i-1})} \phi_\alpha(t) = \int_{\pi\alpha/(2r_i)}^{\pi\alpha/(2r_{i-1})} C_i \sin(M_it-\pi\alpha)dt$$
$$= \frac{C_i}{M_i}\left\{\cos\left(\frac{M_i\pi\alpha}{2r_{i-1}}-\pi\alpha\right)-\cos\left(\frac{M_i\pi\alpha}{2r_i}-\pi\alpha\right)\right\}$$
$$= 0.$$

Hence, if t lies in I_{n_l}, then

$$\int_0^t \phi_\alpha(u)\,du = \int_{\pi\alpha/(2r_{n_l})}^t C_{n_l}\sin(M_{n_l}t-\pi\alpha)+0$$
$$= -\frac{C_{n_l}}{M_{n_l}}\cos(M_{n_l}t-\pi\alpha)$$
$$= o\left\{t\left(\log\frac{1}{t}\right)^p\right\},$$

taking $\frac{1}{\mu_{n_l}}$ tending to zero as $l \to \infty$. This, however, is always possible. Again, since

$$\phi_\alpha(t) = O(C_{n_i})$$
$$= O\left\{\left(\log\frac{1}{t}\right)^p\right\},$$

we have

$$\int_0^t |\phi_\alpha(u)|\,du = O\left\{t\left(\log\frac{1}{t}\right)^p\right\}.$$

We now show that

$$S_\alpha(w) \neq o\{(\log w)^{p+1}\}.$$

Proceeding as in the proof of Sunouchi's result and Lemma 3 we have

$$S_\alpha(w) = o(1)+O\{(\log w)^p\}+w\int_{\pi\alpha/(2w)}^\pi \phi_\alpha(t)\frac{\sin(wt-\pi\alpha)}{wt}\,dt$$
$$= o\{(\log w)^{p+1}\}+H(w),$$

say. Now we have

$$H(M_{n_i}) = \int_{\pi\alpha/(2M_{n_i})}^{\pi} \phi_\alpha(t) \frac{\sin(M_{n_i}t-\pi\alpha)}{t}\, dt$$

$$= \int_{\pi\alpha/(2M_{n_i})}^{\pi\alpha/(2r_{n_i})} + \int_{\pi\alpha/(2r_{n_i})}^{\pi\alpha/(2r_{n_i}-1)} + \int_{\pi\alpha/(2r_{n_i}-1)}^{\pi}$$

$$= H_1+H_2+H_3, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3.2)$$

say. Now, as we take the sequence of integers $\{n_i\}$ with sufficient large gaps, we can take $n_{i+1} > n_i+2$ and thus take

$$\frac{r_{n_i}}{r_{n_{i+1}-1}} < \frac{r_{n_i}}{r_{n_i+1}} < \frac{1}{\mu_{n_i}}$$

or

$$\frac{\pi\alpha}{2r_{n_{i+1}-1}} < \frac{\pi\alpha}{2M_{n_i}},$$

which shows that $\frac{\pi\alpha}{2M_{n_i}}$ neither lies in the interval $I_{n_{i+1}}$ nor in the interval I_{n_i}. Thus we get $\phi_\alpha(t) = 0$ in the interval

$$\left[\frac{\pi\alpha}{2M_{n_i}}, \frac{\pi\alpha}{2r_{n_i}}\right],$$

and therefore

$$H_1 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3.3)$$

Next

$$H_2 = \frac{C_{n_i}}{2}\int_{\pi\alpha/(2r_{n_i})}^{\pi\alpha/(2r_{n_i}-1)} \frac{1}{t}\, dt - \frac{C_{n_i}}{2}\int_{\pi\alpha/(2r_{n_i})}^{\pi\alpha/(2r_{n_i}-1)} \frac{\cos 2(M_{n_i}t-\pi\alpha)}{t}\, dt$$

$$= \frac{C_{n_i}}{2}\cdot \log\left(\frac{r_{n_i}}{r_{n_i-1}}\right) + O\left\{C_{n_i}\frac{r_{n_i}}{M_{n_i}}\right\}$$

$$= A_0\{\log(r_{n_i-1})\}^{p+1}\{1+o(1)\}$$

$$= A_1(\log r_{n_i})^{p+1}\{1+o(1)\}$$

$$= A_2(\log M_{n_i})^{p+1}\{1+o(1)\},$$

A_0, A_1, A_2 being constants > 0. This shows that

$$H_2 \neq o\{(\log M_{n_i})^{p+1}\}. \qquad \ldots \quad \ldots \quad \ldots \quad (4.3.4)$$

Again proceeding as in the previous example we get

$$H_3 = \sum_{l=1}^{i-1} O\left\{\frac{C_{n_l} r_{n_l}}{M_{n_i}-M_{n_l}}\right\}$$

$$= O\left\{\frac{iC_{n_{i-1}} r_{n_i}}{M_{n_i}-M_{n_{i-1}}}\right\}$$

$$= O\left\{\frac{iC_{n_{i-1}}}{\mu_{n_i}}\right\}$$

$$= o\{(\log M_{n_i})^{p+1}\}. \qquad \ldots \quad \ldots \quad \ldots \quad (4.3.5)$$

Thus we have from (4.2.1) .... (4.2.5), putting $w = M_{n_i}$,

$$S_\alpha(w) \neq o\{(\log w)^{p+1}\},$$

which demonstrates our statement (4.1.2) for non-integral values of α.

For the case when α is an integer we have $\alpha = h$. We take this time

$$r_i = \{2[[(r_{i-1})^{(1-\delta)/\alpha}]]/\alpha+1\}r_{i-1}, \ (0 < \delta < 1),$$

where $[[p]]$ *has the same meaning*, and the remaining analysis follows similarly.

Thus the statement (4.1.2) is completely demonstrated.

4.4. It is easy to see that, if

$$\phi_\alpha(t) = o\left\{\left(\log \frac{1}{t}\right)^p\right\}, \ (-1 < p < \infty),$$

then we have

$$\int_0^t |\phi_\alpha(t)|\, dt = o\left\{t\left(\log \frac{1}{t}\right)^p\right\}$$

and not vice versa, and if

$$\int_0^t |\phi_\alpha(t)|\, dt = o\left\{t\left(\log \frac{1}{t}\right)^p\right\}, \ (-1 < p < \infty),$$

then we have

$$\int_t^\pi \frac{|\phi_\alpha(t)|}{t}\, dt = o\left\{\left(\log \frac{1}{t}\right)^{p+1}\right\},$$

and not vice versa.

Thus the two examples which have been framed, show that in Theorem 1, neither the order estimations of $S_\alpha(w)$ can be modified, nor the order conditions of $\phi_\alpha(t)$ can be replaced by less restrictive conditions of a certain type.

5. We now establish the following theorem for the successively rth derived series of the Fourier series.

THEOREM 2. *(a) If*

$$\int_t^\pi \frac{|g_\delta(t)-s|}{t}\, dt = o\left\{\left(\log \frac{1}{t}\right)^{p+1}\right\}, \ (\delta \geqslant 0, \ -1 < p < \infty),$$

as $t \to 0$, then we have

$$B_r^{r+\delta}(w) = o\{(\log w)^{p+1}\},$$

as $w \to \infty$, $B_r^{r+\delta}(w)$ being the $(c, r+\delta)$ mean of the rth derived series of the Fourier series of $f(\theta)$ at $\theta = x$, and

(b) if

$$\int_0^t |g_\delta(t)-s|\, dt = o\left\{t\left(\log \frac{1}{t}\right)^p\right\}, \ (\delta \geqslant 0, \ p \geqslant 0),$$

as $t \to 0$, then we have

$$B_r^{r+\delta_1}(w) - r!\, s = o\{(\log w)^p\}, \ (\delta_1 > \delta).$$

Proof. We have by lemma 4, putting $k = r+\delta$,

$$B_r^{r+\delta}(w) - r\,!\,s = \sum_{i=0}^{r} (-1)^i C_i^r \frac{\Gamma(1+2r+\delta-i)}{\Gamma(1+r+\delta-i)} \left\{ A_r^{r+\delta-i}(w) - S \right\}, \quad \ldots \quad (5.1)$$

where $A_r^{r+\delta-i}(w)$ denotes the $(c, r+\delta-i)$ mean of the Fourier series associated with $g(t)$, at $t = 0$. Now from Theorem 1 we have

$$A_r^{r+\delta-i}(w) - s = o\{(\log w)^{p+1}\}, \ (i = 0, 1, \ldots . r). \quad \ldots \quad \ldots \quad (5.2)$$

Hence we get from (5.1) and (5.2), for the case (a)

$$B_r^{r+\delta}(w) = o\{(\log w)^{p+1}\}.$$

Again putting 'δ_1' instead of 'δ' in (5.1.), we have for the case (b), from Lemma 4,

$$B_r^{r+\delta_1}(w) - r\,!\,s = o\{(\log w)^p\},$$

which proves the theorem.

6. We give some corollaries in this section which are obtained directly from Theorems 1 and 2 and Lemmas 3, 6 and 7.

COROLLARY 1. *If*

$$\int_0^t |\phi_\alpha(u)|\, du = o(t), \ (\alpha \geqslant 0),$$

as $t \to 0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\log(n+1)}$$

is summable (c, α), *and if*

$$\int_0^t |\phi(u)|\, du = o\left\{ t\left(\log \frac{1}{t}\right)^p \right\}, \ (0 \leqslant p \leqslant \infty)$$

as $t \to 0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{\{\log(n+1)\}^{p+1}}$$

is convergent.

We get the above result with the help of Lemmas 3 and 6.

COROLLARY 2. *If*

$$\int_0^t |g_\delta(t) - s|\, dt = o(t), \ (\delta \geqslant 0),$$

as $t \to 0$, *then the series*

$$\sum_{n=1}^{\infty} \frac{\dfrac{d^r}{d\theta^r}\{a_n \cos n\theta + b_n \sin n\theta\}}{\log(n+1)}$$

is summable $(c, r+\delta)$, *at* $\theta = x$.

We obtain the above result from Lemma 6 and Theorem 2. The particular case ($\delta = 0$, $r = 1$) is given by Mohanty and Nanda (1954).

COROLLARY 3. *(a) If*

$$\int_t^\pi \frac{|\phi_\alpha(t)|}{t}\,dt = o\left\{\left(\log \frac{1}{t}\right)^\alpha\right\}, \ (\alpha > 0),$$

and

$$\phi_k(t) = o(1),$$

as $t \to 0$, *for some value of* k, *then the Fourier series of* $f(\theta)$, *at* $\theta = x$, *is summable* $(R, \log \alpha)$, *to sum* S.

(b) If

$$\int_t^\pi \frac{|\phi_\alpha(t)|}{t}\,dt = o(1), \ (\alpha > 1),$$

and

$$\phi_k(t) = o(1),$$

at $t \to 0$, *for some value of* k, *then the Fourier series of* $f(\theta)$ *at* $\theta = x$, *is summable* (R, μ, α) *to sum* S, *where*

$$\mu(n) = \exp\{(\log n)^{1-1/\alpha}\}.$$

This result easily follows from Theorem 1 and Lemma 7.

COROLLARY 4. *(a) If*

$$\int_t^\pi \frac{|g_\delta(t)-s|}{t}\,dt = o\left\{\left(\log \frac{1}{t}\right)^{r+\delta}\right\}, \ (\delta \geqslant 0),$$

and

$$g_k(t)-s = o(1),$$

as $t \to 0$, *for some value of* k, *then the rth derived series of the Fourier series of* $f(\theta)$, *at* $\theta = x$, *is summable* $(R, \log, r+\delta)$ *to sum* $r!\,s$, *and (b) if*

$$\int_t^\pi \frac{|g_\delta(t)-s|}{t}\,dt = o(1), \ (r+\delta > 1, \ \delta \geqslant 0),$$

as $t \to 0$, *and*

$$g_k(t) = o(1),$$

for some value of k, *then the rth derived series of the Fourier series of* $f(\theta)$, at $\theta = x$, *is summable* $(R, \mu, r+\delta)$ *where*

$$\mu(n) = \exp\left\{(\log n)^{1-\frac{1}{r+\delta}}\right\}.$$

We can easily obtain this result from Theorem 2 and Lemma 7.

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