ANALYSIS OF CASCADE SHOWERS INDUCED BY PHOTONS

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I. INTRODUCTION

An accurate estimate of the size of a cascade shower produced by a photon of known energy in passing through materials of given thickness is of great importance in the interpretation of many observed results. In recent years several events have been recorded in photographic emulsion plates, many of which may possibly be explained as a $\pi^0 \rightarrow 2\gamma$ decay, but the possibilities of obtaining large photon multiplicities from a single event in the high energy region is also not ruled out. Thus to examine the various possibilities it is necessary to obtain a correct estimate of the average number of particles at the different stages of the cascade, having energies greater than a given minimum, depending on the mode of observation, and not the average number of particles. The frequency of large bursts produced by mesons also depends critically on the size of the photon excited shower. A comparison of the theoretical estimates of the frequency of bursts with those recorded under large thickness of material gives important informations on the nature of the mesons and particularly on the spin of the meson. Christy and Kusaka (1941), Chakrabarty (1942*a*), and Chakrabarty and Majumdar (1944) have obtained results which are widely different. Apart from the difference in the form of fluctuation assumed, the size of the shower associated with a photon of given energy assumed by these authors are different. In a recent paper (Chakrabarty and Gupta, 1956, henceforth denoted as A) we have derived an expression giving the average number of particles with energies greater than any given energy E (say) produced by a primary electron of energy E_0 and also have deduced the form of the energy spectrum of the shower particles. In the present paper we propose to derive the size as well as the spectrum of the particles in a cascade shower produced by a photon of known energy in traversing material of given thickness. A comparison of these results with similar results in the case of electron-initiated showers gives some interesting information on the materialisation of photons through pair production. In previous papers Chakrabarty (1942*b*), Snyder (1949) and Scott (1950) have studied this problem. While Chakrabarty derived the average number of particles neglecting ionization loss, the values derived by Snyder are much too low for non-zero values of E, the energy of the shower particles, and even for values of E near the critical energy of the material in which the shower is produced $\left(0 > \log_e \frac{E}{\beta} \geqslant -2\right)$ it gives absurd results. In a later section we have also compared the results relating to $\pi^0 \rightarrow 2\gamma$ decay derived from the analysis of the present paper with those based on observations.

II. MATHEMATICAL SOLUTION

Following the notations used in A and denoting by $P\,(E,\,t)\,dE$ the mean number of electrons positive and negative in the energy range $(E,\,E+dE)$, to be found in a cascade at depth t in radiation units and by $Q\,(E,\,t)$ the corresponding expression for the number of photons, the basic equations of cascade theory are (Bhabha and Chakrabarty, 1948; henceforth denoted as B)

$$\frac{\partial P(E,t)}{\partial t}-\beta\frac{\partial P(E,t)}{\partial E}=\int_E^\infty P(E',t)R(E',E'-E)\frac{E'-E}{E'^2}\times$$

$$\times dE'-P(E,t)\int_0^E R(E,E')\frac{E'dE'}{E^2}+2\int_E^\infty Q(E',t)R(E,E')\frac{dE'}{E'} \quad .. \quad (1)$$

$$\frac{\partial Q(E,t)}{\partial t}=\int_E^\infty P(E',t)R(E,E')\frac{EdE'}{E'^2}-Q(E,t)\int_0^E R(E',E)\frac{dE'}{E} \quad .. \quad (2)$$

Applying Mellin transform in E, the above equations are transformed into

$$\frac{\partial p(s,t)}{\partial t}=-A_s p(s,t)+B_s q(s,t)-\beta(s-1)p(s-1,t) \quad .. \quad .. \quad .. \quad (3a)$$

$$\frac{\partial q(s,t)}{\partial t}=C_s p(s,t)-Dq(s,t) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3b)$$

where the A_s, B_s, C_s, D are functions of s and have been defined in B.

Equations ($3a$, b) are the same as in the case of electron-initiated shower. But for a cascade shower initiated by a photon of energy E_0, the boundary conditions are different; they are given by

$$P(E,0)=0;\quad Q(E,0)=\delta(E_0-E) \quad .. \quad .. \quad .. \quad (4)$$

giving

$$\left.\begin{array}{l} p(s,0)=0;\quad q(s,0)=E_0^{s-1}. \\ \dfrac{\partial}{\partial t}[p(s,t)]_{t=0}=B_sE_0^{s-1} \end{array}\right\} \quad .. \quad .. \quad .. \quad (5)$$

Following B, we define a set of functions $\psi_n(s,t)$ satisfying the following differential equations

$$\left.\begin{array}{l} \left\{\dfrac{\partial^2}{\partial t^2}+(A_s+D)\dfrac{\partial}{\partial t}+A_sD-B_sC_s\right\}\psi_0(s,t)=0. \\ \left\{\dfrac{\partial^2}{\partial t^2}+(A_{s+n}D-B_{s+n}C_{s+n})\dfrac{\partial}{\partial t}+A_{s+n}D-B_{s+n}C_{s+n}\right\}\psi_n(s,t)=\left(\dfrac{\partial}{\partial t}+D\right)\psi_{n-1}(s,t) \end{array}\right\} \quad (6)$$

(where n is a +ve integer) together with the boundary conditions

$$\left.\begin{array}{l} \psi(s,0)=0;\quad \left.\dfrac{\partial}{\partial t}\psi_0(s,t)\right]_{t=0}=B_s. \\ \psi_n(s,0)=0;\quad \left.\dfrac{\partial}{\partial t}\psi_n(s,t)\right]_{t=0}=0 \end{array}\right\} \quad .. \quad .. \quad .. \quad (7)$$

From (6) and (7) we find

$$\psi_0(s,t)=\frac{B_s}{\mu_s-\lambda_s}(e^{-\lambda_s t}-e^{-\mu_s t}) \quad .. \quad .. \quad .. \quad (8)$$

The differential electron spectrum is then given by

$$P(E,t)=\frac{1}{2\pi\iota\beta}\int_{C_s}ds\left(\frac{E_0}{\beta}\right)^{s-1}\left(\frac{\beta}{E}\right)^{s}\sum_{n=0}^{\infty}\left(-\frac{\beta}{E}\right)^{n}\frac{\Gamma(s+n)}{\Gamma(s)}\psi_n(s,t) \quad .. \quad (9)$$

as may be seen directly by eliminating $Q\ (E, t)$ from (1) and (2) and substituting for $P\ (E, t)$ in the resulting equation. The boundary conditions (7) imposed on ψ's correspond to (4).

Now, let (Chakrabarty, 1946)

$$\psi_n(s,r)=\int_0^{\infty}e^{-rt}\,\psi_n(s,t)dt$$

Then

$$\psi_n(s,t)=\frac{1}{2\pi\iota}\int_{\rho-\iota\alpha}^{\rho+\iota\alpha}e^{rt}\,\psi_n(s,r)dr \quad .. \quad .. \quad .. \quad (10)$$

where $n=0$ or any integer. Evidently for $n=0$ we get from (8)

$$\psi_0(s,r)=\frac{B_s}{D+r}\phi_0(s,r) \quad .. \quad .. \quad .. \quad .. \quad (11)$$

where

$$\phi_0(s,r)=\frac{D+r}{(r+\lambda_s)\,(r+\mu_s)}.$$

Combining (10) with the second equation of (6) we get for integral values of n, the following recurrence relation

$$\psi_n(s,r)=\phi_0(s+n,r)\psi_{n-1}(s,r)$$

$$=\frac{Bs}{D+r}\prod_{i=0}^{n}\phi_0(s+i,r) \quad .. \quad .. \quad .. \quad (12)$$

As a generalisation of (12) for any p real or complex, we now define as in A

$$\psi_p(s,r)=\frac{B_s}{D+r}\lim_{N\to\infty}\,[\phi_0(s+N+1,r)]^{p+1}\prod_{i=0}^{N}\frac{\phi_0(s+i,r)}{\phi_0(s+p+i+1,r)} \quad .. \quad (13)$$

satisfying a similar recurrence relation

$$\psi_p(s,r)=\phi_0(s+p,r)\psi_{p-1}(s,r)$$

By applying the transform (10) in t, $P(E,t)$ is reduced to

$$P(E,t)=\frac{1}{2\pi\iota\beta}\int_{C_s}ds\left(\frac{E_0}{\beta}\right)^{s-1}\left(\frac{\beta}{E}\right)^{s}\sum_{n=0}^{\infty}\left(-\frac{\beta}{E}\right)^{n}\frac{\Gamma(s+n)}{\Gamma(s)}\cdot\frac{1}{2\pi\iota}\int_{\rho-\iota\alpha}^{\rho+\iota\alpha}e^{rt}\,\psi_n(s,r)dr$$

the above infinite series can be transformed into an integral over p taken along a contour C_p running parallel to the imaginary axis and with $-1<Re(p)<0$, viz.

$$P(E,t)=\frac{1}{(2\pi\iota)^3\beta}\int_{C_s}ds\left(\frac{E_0}{\beta}\right)^{s-1}\int_{C_p}dp\left(\frac{\beta}{E}\right)^{s+p}\frac{\Gamma(s+p)\Gamma(-p)}{\Gamma(s)}\times$$

$$\times\Gamma(p+1)\int_{\rho-\iota\alpha}^{\rho+\iota\alpha}dr e^{rt}\psi_p(s,r) \quad .. \quad (14)$$

Evaluating the integral over r in terms of the residues at $r = -\lambda_{s+m}$ $r = -\mu_{s+m}$, we get

$$P(E, t) = \sum_{m=0}^{\infty} P_m(E, t) = \sum_{m=0}^{\infty} \frac{-1}{4\pi^2\beta} \int_{C_s} ds \left(\frac{E_0}{\beta}\right)^{s-1} B_s \int dp \left(\frac{\beta}{E}\right)^{s+p} \times$$

$$\times \frac{\Gamma(s+p)\Gamma(-p)}{\Gamma(s)} \times \frac{1}{\mu_{s+m}-\lambda_{s+m}} \left[G_m(s, p) e^{-\lambda_{s+m}t} - F_m(s, p) e^{-\mu_{s+m}t}\right] \quad (15)$$

where

$$G_m(s, p) = \Gamma(p+1) \lim_{N\to\infty} [\phi_0(s+N+1, -\lambda_{s+m})]^{p+1} \prod_{\substack{i=0 \\ \neq m}}^{N} \phi_0(s+i, -\lambda_{s+m}) \times$$

$$\times \prod_{i=0}^{N} \frac{1}{\phi_0(s+p+i+1, -\lambda_{s+m})} \quad .. \quad (16)$$

$$F_m(s, p) = \Gamma(p+1) \lim_{N\to\infty} [\phi_0(s+N+1, -\mu_{s+m})]^{p+1} \prod_{\substack{i=0 \\ \neq m}}^{N} \phi_0(s+i, -\mu_{s+m}) \times$$

$$\times \prod_{i=0}^{N} \frac{1}{\phi_0(s+p+i+1, -\mu_{s+m})} \quad .. \quad (17)$$

Integrating $P(E, t)$ over the energy from E to infinity we get the integral electron spectrum $N(E, t)$, given by

$$N(E, t) = \sum_{m=0}^{\infty} N_m(E, t) = \frac{-1}{4\pi^2\beta} \sum_{m=0}^{\infty} \int_{C_s} ds \left(\frac{E_0}{\beta}\right)^{s-1} B_s \int_{C'_p} dp \left(\frac{\beta}{E}\right)^{s+p-1} \times$$

$$\times \frac{\Gamma(s+p-1)\Gamma(-p)}{\Gamma(s)} \times \frac{1}{\mu_{s+m}-\lambda_{s+m}} \left[G_m(s, p) e^{-\lambda_{s+m}t} - F_m(s, p) e^{-\mu_{s+m}t}\right] \quad .. \quad (18)$$

where C'_p will as usual run parallel to the imaginary axis with $Re(p) < 0$ and $Re(s+p) > 1$. Except for small thickness, the contribution of the term containing $e^{-\mu_{s+m}} t$ is not significant and further almost the entire contribution to $N(E, t)$ will come from the first term of the series with $m = 0$. Consequently we may write

$$N(E, t) \approx N_0(E, t) = \frac{-1}{4\pi^2\beta} \int_{C_s} ds \left(\frac{E_0}{\beta}\right)^{s-1} B_s \int dp \left(\frac{\beta}{E}\right)^{s+p-1} \times$$

$$\times \frac{\Gamma(s+p-1)\Gamma(-p)}{\Gamma(s)} \frac{e^{-\lambda_s t}}{\mu_s - \lambda_s} G_0(s, p) \quad .. \quad (19)$$

A similar conclusion holds true for the energy spectrum. Except for notations (18) is exactly the same as the solution given by Snyder. He has, however, evaluated the p-integral in terms of the residues at the poles at $p = 1-s-k$ $(k = 0, 1, 2, ..)$. The expansion for $N(E, t)$ obtained by Snyder can be used when $E = 0$, in which case it reduces to the first term of the series obtained by him. For other values of $E(\beta > E > 0)$ the inclusion of the second term in $N_0(E, t)$ gives absurd results as in the case of an electron-initiated shower discussed in A.

The right hand side of (19) can be evaluated by the saddle-point method. The asymptotic value $(E \to 0)$ of the integral spectrum $N_0(E, t)$ can, however, be determined by making $p \to 1-s$.

If we assume that

$$w_0(s,p) = (s-1)y_0-(s+p-1)y-\lambda_s t+\log\frac{B_s}{\mu_s-\lambda_s}+\log\frac{\Gamma(s+p-1)\Gamma(-p)}{\Gamma(s)} \quad .. \quad (20)$$

where

$$y_0 = \log\frac{E_0}{\beta}; \quad y = \log\frac{E}{\beta}$$

then, for any given value of y_0, y and t, the saddle-points s_0, p_0 are given by

$$\frac{\partial w_0}{\partial p_0} = -y+\frac{\partial}{\partial p_0}\log\Gamma(s_0+p_0-1)+\frac{d}{d(-p_0)}\log\Gamma(-p_0) = 0 \quad .. \quad .. \quad .. \quad (21)$$

$$\frac{\partial w_0}{\partial s_0} = y+\frac{d}{d(-p_0)}\log\Gamma(-p_0)-\frac{d}{ds_0}\log\Gamma(s_0)+\frac{d}{ds_0}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}-\lambda'_{s0}t = 0 \quad .. \quad (22)$$

We then have

$$N_0(E,t) = \frac{1}{2\pi}\cdot\frac{\exp\left[(s_0-1)y_0-(s_0+p_0-1)y-\lambda_{s0}t+\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}\right]}{\left[\frac{\partial^2 w_0}{\partial s_0^2}\cdot\frac{\partial^2 w_0}{\partial p_0^2}-\left(\frac{\partial^2 w_0}{\partial s_0\partial p_0}\right)^2\right]^{\frac{1}{2}}}\times$$

$$\times\frac{\Gamma(s_0+p_0-1)\Gamma(-p_0)}{\Gamma(s_0)}G_0(s_0,p_0) \quad .. \quad (23)$$

where

$$\left.\begin{aligned}
\frac{\partial^2 w_0}{\partial p_0^2} &= \frac{\partial^2}{\partial p_0^2}\log\Gamma(s_0+p_0-1)+\frac{d^2}{d(-p)^2}\log\Gamma(-p_0)\\
\frac{\partial^2 w_0}{\partial s_0^2} &= \frac{\partial^2}{\partial s_0^2}\log\Gamma(s_0+p_0-1)+\lambda''_{s0}t+\frac{d^2}{ds_0^2}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}-\frac{d^2}{ds_0^2}\log\Gamma(s_0)\\
\frac{\partial^2 w_0}{\partial s_0\partial p_0} &= \frac{\partial^2}{\partial s_0\partial p_0}\log\Gamma(s_0+p_0-1)
\end{aligned}\right\} \quad .. \quad (24)$$

In deriving (21) and (22) we have neglected the variation of G_0 with s and p, since it is a slowly varying function of s and p at least in the domain of the saddle-points. For $p\to 1-s$, the variation of the function $\frac{D-\lambda_s}{\mu_s-\lambda_s}G_0(s, 1-s)$ with s is even smaller than $G_0(s, 1-s)$ as is evident from Table I.

TABLE I

s	1·5	1·8	2·0	2·2	2·5	2·8
$G_0(s, 1-s)$	1·8089	2·1091	2·2865	2·4752	2·8100	3·2787
$\frac{D-\lambda_s}{\mu_s-\lambda_s}G_0(s, 1-s)$	·9523	1·0071	·9904	·9492	·8811	·7720

From (21) it is easily seen that $\lim_{p_0\to 1-s_0}(s_0+p_0-1)y = -1$ so that from (23), the value of $N_0(0,t)$ can be obtained by making $s_0+p_0-1\to 0$. As discussed above, in this particular case it will be more justified to neglect the variation of $\frac{D-\lambda_s}{\mu_s-\lambda_s}$ also in equations (22) and (24). Consequently,

$$N_0(0,t) = \frac{1}{2\pi} \frac{\exp\left[(s_0-1)y_0-\lambda_s t_A+\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}\right]}{\left[\frac{1}{(s_0-1)^2}-\lambda''_{s0}t_A+\frac{d^2}{ds_0^2}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}\right]^{\frac{1}{2}}} \cdot \frac{1}{s_0-1} \cdot G_0(s_0, 1-s_0) \quad .. \quad (25)$$

where

$$t_A = \frac{y_0-\frac{1}{s_0-1}+\frac{d}{ds_0}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}}{\lambda'_{s0}}.$$

III. Discussions

The values of $N_0(E, t)$ as determined from (23) and (25) for different values of y_0 and y are given in Table II. In the numerical computation the values of the function $G_0(s, p)$ for different s and p are taken from A. For $y_0 = 6$, $\log_{10}N_0(E, t)$ has been plotted against t for $y = -\infty$, -4 and -2 in Fig. 1. The curve for

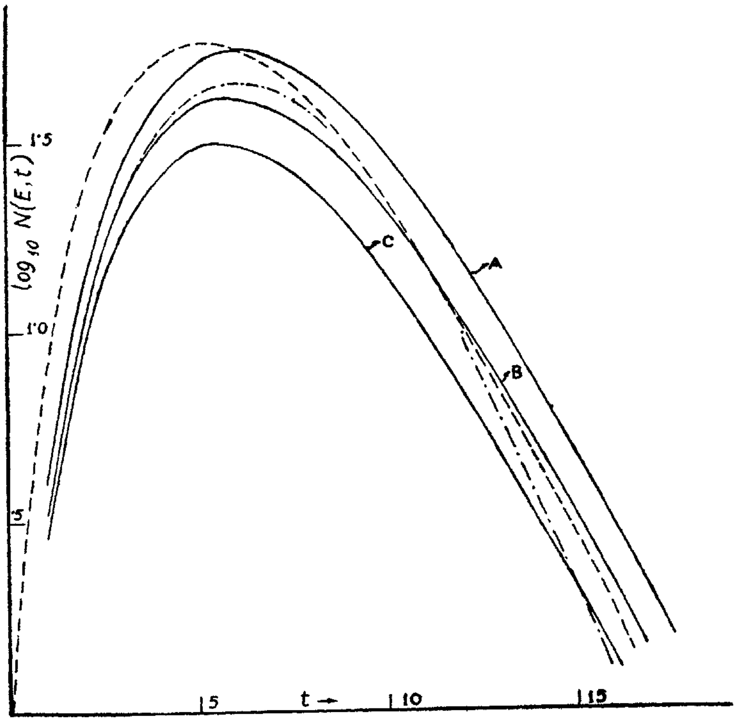

Fig. 1. Plot of $\log_{10}N(E,t)$ against t for $y_0 = 6$; $(A) \to y = -\infty$ $(B) \to y = -4$; $C \to y = -2$. Dashed curve corresponds to electron-initiated shower with $y_0 = 6$. Dash-dot one is Arley's curve for photon-induced shower with same y_0.

TABLE II

y_0	y \ t	·5	1	2	3	4	5	6	8	10	12	16	20	24
2	$-\infty$	1·012	1·413	1·652	1·429	1·035	6·998							
	-2	1·023	1·905	4·217	5·408	5·559	4·819	3·733	1·905	0·871				
4	-4	1·109	2·042	4·467	6·918	7·328	6·637	5·370	2·951	1·445				
	$-\infty$	1·216	2·570	6·095	8·318	8·913	8·414	6·998	3·758	1·778				
	-2		2·570	9·120	18·20	26·00	30·90	31·33	23·99	14·29	6·998	1·349		
6	-4		3·162	11·22	22·65	33·11	40·27	41·69	34·04	20·89	10·59	2·089		
	$-\infty$		3·980	13·96	27·54	41·69	51·29	54·33	46·77	28·97	15·14	3·055		
	-2		3·548	17·58	45·19	88·10	133·4	172·2	192·8	154·9	100·0	32·36	9·120	
8	-4		4·217	20·89	57·54	110·9	167·9	217·8	263·0	218·8	147·9	45·71	11·35	
	$-\infty$		6·310	28·40	79·43	154·9	229·1	288·4	350·8	311·2	218·8	65·92	14·79	
	-2		4·571	21·88	69·18	223·9	436·5	691·8	1122	1202	1047	457·1	128·8	28·84
10	-4		6·026	28·84	112·2	288·4	562·3	891·3	1445	1660	1446	645·7	190·5	47·86
	$-\infty$		8·318	43·65	166·0	426·6	794·3	1216	1905	2113	1995	1000	281·8	66·07

Values of $N_0(E, t)$ for different values of E_0 E, t

$y_0 = \log \frac{E_0}{\beta}$; $y = \log \frac{E}{\beta}$.

$\log_{10}N_{el}(0, t)$ giving the total integral spectrum for an electron-initiated shower with $y_0 = 6$ is also shown in the same figure from which it is clear that the nature of a photon-initiated shower is nearly similar to that of an electron-initiated one. There is however the following difference. In each case the maximum of the photon-initiated shower as given by $N_0(0, t)$ is shifted towards larger depths by nearly one unit as compared to that of an electron-initiated shower. This characteristic holds good for other values of y_0 as well and can be seen analytically from equation (26) which shows that for the same y_0 and s, the value of t_A for a photon-initiated shower differs from that of an electron-initiated one by the amount $\frac{1}{\lambda'_{s0}}\frac{d}{ds_0}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}$. At the maximum of the shower which occurs near $s_0 = 2$, independently of the value of y_0, $\left(\frac{1}{\lambda'_{s0}}\frac{d}{ds_0}\log\frac{B_{s0}}{\mu_{s0}-\lambda_{s0}}\right)_{s_0=2} \sim \cdot 8$ and this is what we actually find in our figure.

For comparison we have plotted (Fig. 2) $N_0(0, t)$ against t for a shower induced

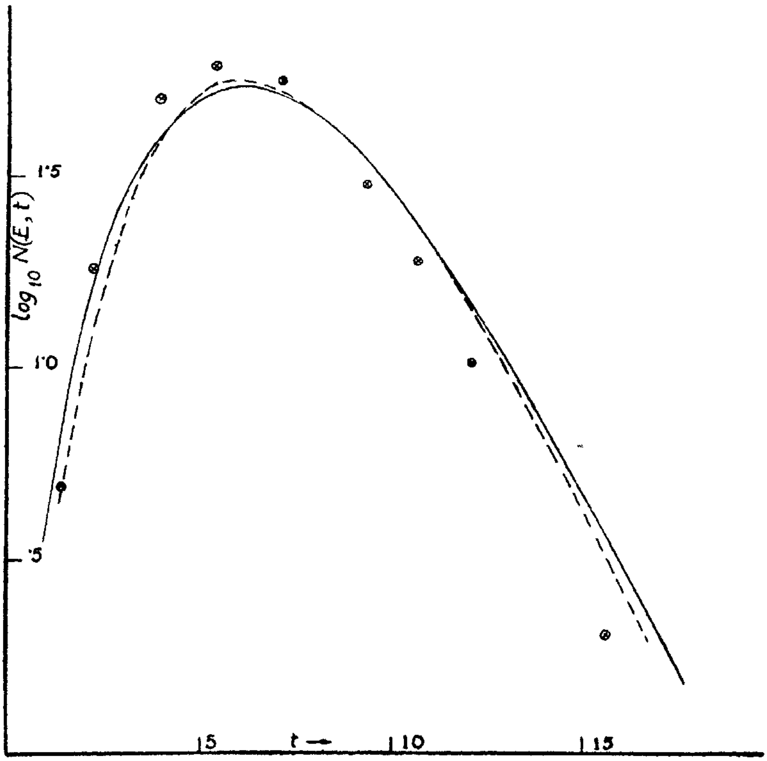

FIG. 2. Complete curve → $\log_{10}N_0(0, t)$ against t for a photon-induced shower with $E_0 = \beta e^6$. Dashed curve → sum of the contributions from two electron-initiated showers with $E_0 = \beta e^{5\cdot 95}$ and $\beta e^{2\cdot 98}$. Points ⊗ → sum of contributions from two electron-initiated showers with $E_0 = \frac{1}{2}\beta e^6$.

by a photon of energy $E_0 = \beta e^6$ and the combined contribution of two electron-initiated showers with energies $\beta e^{5\cdot 95}$ and $\beta e^{2\cdot 98}$ respectively, the latter curve being shifted one unit to the right. The two curves agree satisfactorily except at the early beginning and at the tail end of the shower. The discrepancy will, however, be reduced to some extent, if we remember that the contribution to $N_0(0, t)$ from the second term involving $e^{-\mu st}$(Equ^n 18) is negative for a photon-induced shower, while the same is positive for a shower started by an electron. Thus it appears that within one radiation length, the incident photon creates a pair of electrons one of which has a very low energy, while the energy of the other is nearly equal to that of the initial photon. The contribution from the low energy electron will come only at the early beginning of the shower, while from and after the maximum, it is only the high energy electron which will be important. This is also suggested by the fact that for the same y_0, the maximum value of $N_0(0, t)$ is slightly less than that of $N_{el}(0, t)$.

We have also compared the integral spectrum $N_0(0, t)$ of a shower started by a photon and that due to a pair of electrons of equal energy $E_0/2$. It is found that the agreement in the previous case is everywhere much better than here. This is what is expected from the expression for the cross-section for pair creation given by Bethe and Heitler and assumed in the present analysis.

The results derived in the present paper have been compared to the experimental results of Bender (1955) where he has analysed the cascade showers started by two photons which result from the decay of a π^0 meson in flight. Bender has plotted N_{max}, the number of particles at the maximum of the shower against E_0, the energy of the shower producing photon and according to him there is good agreement between his experimental results and the theoretical curve $E_0 = 87{\cdot}4\ N_{max}$ *mev.*

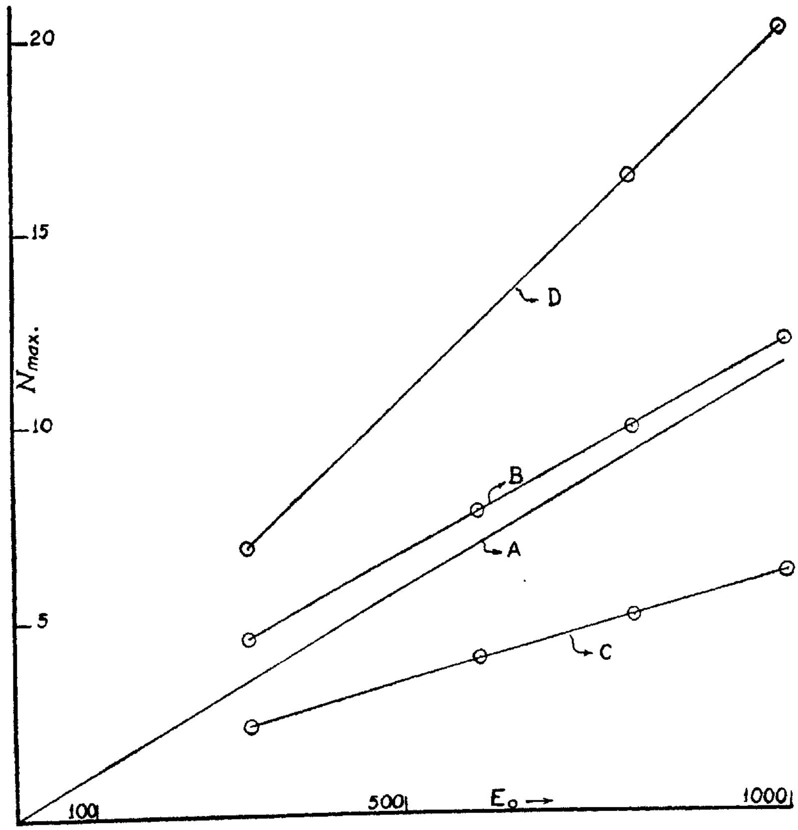

Fig. 3. N_{max} against E_0. $(A) \rightarrow$ Bender's observational curve. $B \rightarrow y = -2$; $C \rightarrow y = 0$; $D \rightarrow y = -\infty$.

given by the Monte Carlo calculations (carried out by R. R. Wilson) when cut off at about 8 *mev.* while the theoretical expressions for N_{max} as given by Rossi and Greisen give values which are considerably higher than the data indicate. In Fig. 3 we have plotted $N_{max}(E, t)$ against E for different values of E, viz. $E = 0$, βe^{-2}, β. Since the showers are observed in lead chambers ($y = -2$ corresponds to $E \approx 2mc^2$ for lead), it is not possible to observe particles with energies less than a certain minimum which is of the order $2mc^2$ as pointed out previously in A. This at once explains why the results of Rossi and Greisen as well as ours with $E = 0$ give values much higher than that observed. Further since particles with energies less than 8 *mev.* (8 *mev.* is greater than the critical energy in lead) can be observed, it appears that at the above cut off the Monte Carlo calculations give values which are rather high. Thus for comparison with experimental results it seems to be more justified to use theoretical values of N_{max} (E, t) when cut off at about $E \approx 2mc^2$ ($\approx \beta e^{-2}$ for lead) and from Fig. 3 it is found that the best agreement with the observational curve derived by Bender is obtained when $E = \beta e^{-2}$ as it should be.

As regards the dependence of the frequency of burst production by mesons on the size of showers initiated by photons it is seen, following the analysis of Chakrabarty (1942*a*), that the same is decreased to about two-thirds its value, if for N_m the maximum of the average number of particles in the showers, we consider the values at $y = -2$ instead of at $y = -\infty$. Apart from the question of fluctuation, this modification alone will influence the results of Christy and Kusaka in a manner which supports the $\frac{1}{2}$ spin theory of the μ-meson instead of zero spin which they suggested.

IV. Energy Spectrum

The electron energy spectrum $P(E, t)$ is obtained in a similar manner from (15) by applying the saddle-point method. We get

$$\beta P(E,t) \approx \beta P_0(E.t) = \frac{1}{2\pi}\,\frac{\exp\left[(s_0'-1)y_0-(s_0'+p_0')y-\lambda s_0' t\right]}{\left[\dfrac{\partial^2\omega_0'}{\partial s_0'^2}\cdot\dfrac{\partial^2\omega_0'}{\partial p_0'^2}-\left(\dfrac{\partial^2\omega_0'}{\partial s_0'\partial p_0'}\right)^2\right]^{\frac{1}{2}}}\times$$

$$\times\,\frac{B(s_0')}{\mu s_0'-\lambda s_0'}\cdot\frac{\Gamma(s_0'+p_0')\,\Gamma(-p_0')}{\Gamma(s_0')}\,G_0(s_0',p_0') \quad \ldots \quad (27)$$

The saddle-points s_0', p_0' being determined from the equation $\dfrac{\partial w_0'}{\partial s_0'} = 0$ and $\dfrac{\partial w_0'}{\partial p_0'} = 0$,

where $w_0'(s, p)$ is exactly the same as $\omega(s, p)$ except that the two terms $(s+p-1)y$ and $\Gamma(s+p-1)$ are to be replaced by $(s+p)y$ and $\Gamma(s+p)$. The expression $\beta P(E, t)$ is valid for all energies and goes to infinity logarithmically as $E \to 0 (s+p \to 0)$. For large values of E which can be obtained by making $-p \to 0$, so that $y(-p) \to 1$ (27) merges to the expression for $P(E, t)$ neglecting collision loss (Chakrabarty, 1942*b*) as shown below

$$P(E,t) \approx P_0(E,t) = \frac{1}{2\pi\beta}\,\frac{\exp\left[(s_0'-1)y_0-s_0'y-\lambda_{s_0'}'t+1\right]}{\left[-\lambda_{s_0}''t+\dfrac{d^2}{ds_0'^2}\log\dfrac{B_{s_0}'}{\mu s_0'-\lambda s_0'}\right]^{\frac{1}{2}}}\cdot\frac{B(s_0')}{\mu s_0'-\lambda s_0'}$$

$$= \frac{e}{\sqrt{2\pi}}\cdot\frac{1}{2\pi\iota E_0}\int\left(\frac{E_0}{E}\right)^s\frac{B(s)}{\mu s-\lambda s}\cdot e^{-\lambda s t}ds.$$

For $y_0 = 6$ and $t = 6$ and 11, we have plotted in Fig. 4, the values of $\log_{10}\beta P(E, t)$

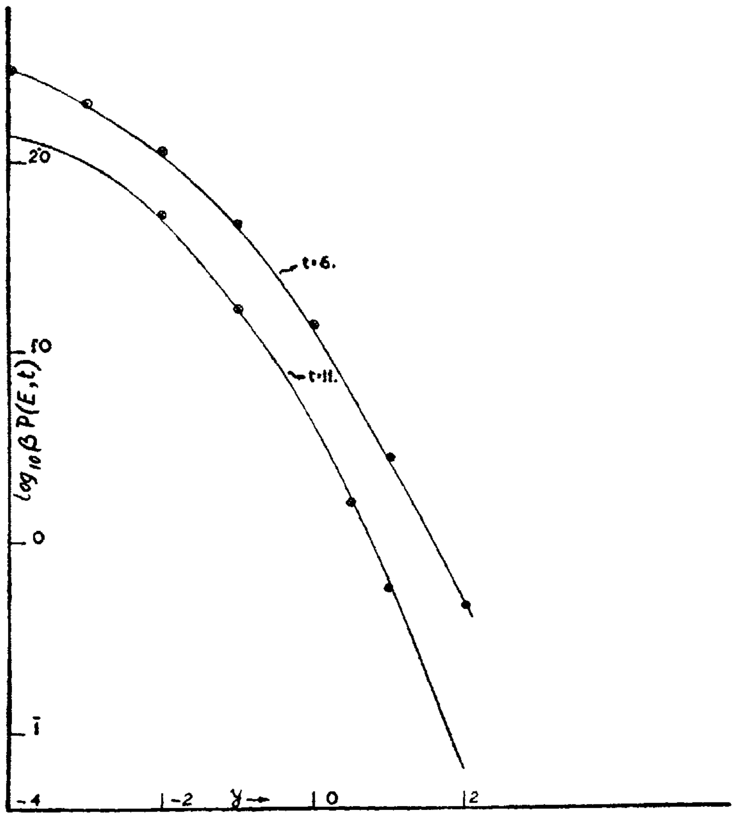

FIG. 4. $\log_{10}\beta P(E, t)$ against y for $t = 6$ and $t = 11$. Points ⊗ correspond to values of $\log\beta P_{el}(E, t)$ for $t = 5$ and $t = 10$ respectively.

against y. These curves are similar to those of the energy spectra for an electron-initiated shower and almost coincides with the corresponding curves for βP_{el} (E, t) with $t = 5$ and 10 respectively, a fact which supports our previous contention as regards the connection between an electron-initiated and a photon-initiated shower.

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SUMMARY

In the present paper the solution of the diffusion equations for a cascade shower initiated by a photon has been obtained and numerical values for both the differential and integral electron spectra have been calculated through saddle-point integration. On the basis of the results obtained, the connection between a photon and an electron-initiated shower has been investigated and comparison with experimental results is also made.

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