INTERNAL BALLISTICS OF A RECOIL-LESS HIGH-LOW PRESSURE GUN

by J. N. KAPUR, *Hindu College, Delhi University, Delhi*

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received September 20, 1956; read January 12, 1957*)

1. INTRODUCTION

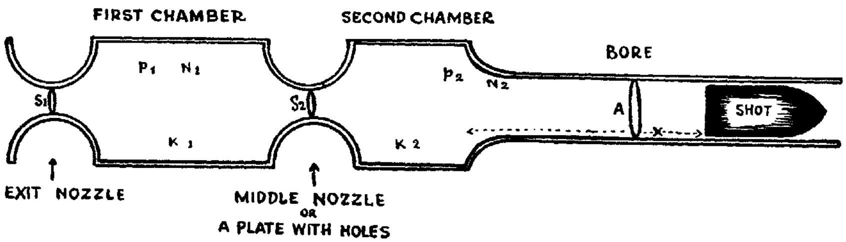
The Internal Ballistics of High-Low pressure guns and of Recoil-less guns have been studied in detail by Corner (1947, 1948, 1950). It is possible, however, to combine the desirable features of both the guns and to develop a Recoil-less High-Low pressure gun; as a matter of fact, the German ballisticians had developed such a gun during the second world war. It is proposed, in the present paper, to develop, for the Internal Ballistics of such a gun, a simple theory of the same order of accuracy as Crow's theory for the orthodox gun and Corner's theory for the High-Low pressure gun.

The theory developed here is, however, of special interest, as it includes, as particular cases, the theories for

(i) Orthodox guns,
(ii) Recoil-less guns,
(iii) High-Low pressure guns,
(iv) Solid-fuel rockets.

2. THE FUNDAMENTAL EQUATIONS

The figure below gives the basic diagram for such a gun.



At time t, let a fraction z of the charge mass C be burnt and let fractions N_1, N_2 be present in the two chambers so that a fraction $z-N_1-N_2$ has escaped from the exit nozzle by this time. Let p_1, p_2 be the pressures in the two chambers whose capacities are K_1 and K_2—the pressure changing continuously through the middle nozzle. Let η be the covolume per unit mass, A the area of the cross-section of the bore and λ the adjusted force constant. Then we have the following equations of Internal Ballistics for the Isothermal Model:

Equation of state for the first chamber

$$p_1\left[K_1-\frac{C(1-z)}{\delta}-CN_1\eta\right]=CN_1\lambda \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Equation of state for the second chamber

$$p_2[K_2+Ax-CN_2\eta] = CN_2\lambda \qquad (2)$$

Equation of continuity for the first chamber

$$C\frac{dz}{dt} = C\frac{dN_1}{dt}+\frac{\psi S_1 p_1}{\sqrt{\lambda}}+\frac{\psi S_2 p_1}{\sqrt{\lambda}} \qquad (3)$$

Here S_1, S_2 are the throat-areas of the two nozzles and

$$\psi = \gamma^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (4)$$

Also we have assumed that

$$\frac{p_2}{p_1} < \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \qquad (5)$$

If $\frac{p_2}{p_1}$ is greater than this critical value, we shall have to multiply the last term in (3) by a suitable back-pressure factor calculated in Corner (1950). This would be equivalent to decreasing S_2 by a suitable factor.

Equation of continuity for the second chamber

$$C\frac{dN_2}{dt} = \frac{\psi S_2 p_1}{\sqrt{\lambda}} \qquad (6)$$

so that (3) can also be written as

$$C\frac{dz}{dt} = C\frac{dN_1}{dt}+C\frac{dN_2}{dt}+\frac{\psi S_1 p_1}{\sqrt{\lambda}} \qquad (7)$$

Equation of motion of the shot

Assuming that conventional Lagrange's corrections for the orthodox gun apply

$$W_2\frac{d^2x}{dt^2} = W_2 v\frac{dv}{dx} = W_2\frac{dv}{dt} = Ap_2 \qquad (8)$$

where

$$W_2 = W_1+\tfrac{1}{3}C, \qquad (9)$$

and W_1 is the effective mass of the shot.

Rate of burning equation

$$D\frac{df}{dt} = -\beta p_1 \qquad (10)$$

Form-function

We use

$$z = \phi(f) \text{ (general form-function)} \qquad (11a)$$

or

$$z = (1-f)(1+\theta f) \text{ (quadratic form-function)} \qquad (11b)$$

or

$$z = 1-f \text{ (tubular charge)} \qquad (11c)$$

3. Integration of the equations. Maximum pressure in the first chamber

From (3) and (10)

$$\frac{dz}{dt}=\frac{dN_1}{dt}-\Psi_1\frac{df}{dt}-\Psi_2\frac{df}{dt},$$

where

$$\Psi_1=\frac{\psi S_1 D}{C\beta\sqrt{\lambda}},\quad \Psi_2=\frac{\psi S_2 D}{C\beta\sqrt{\lambda}} \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

are the two dimensionless leakage parameters.

Integrating and remembering that initially $z=0$, $N_1=0$, $f=1$.

$$N_1=z-(\Psi_1+\Psi_2)(1-f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Similarly from (6) and (10)

$$N_2=\Psi_2(1-f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

From (7) and (10) or from (13) and (14)

$$z=N_1+N_2+\Psi_1(1-f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

From (1), (11*a*) and (13)

$$p_1=\frac{C\lambda[\phi(f)-(\Psi_1+\Psi_2)(1-f)]}{K_1-\frac{C}{\delta}+\frac{C}{\delta}\phi(f)-C\eta\phi(f)+C\eta(\Psi_1+\Psi_2)(1-f)} \qquad \ldots \quad \ldots \quad (16)$$

$$\frac{1}{\lambda C}\frac{dp_1}{df}=\frac{\left[K_1-\frac{C}{\delta}\right][\phi'(f)+\Psi_1+\Psi_2]+\frac{C}{\delta}[\Psi_1+\Psi_2][\phi(f)+(1-f)\phi'(f)]}{\left[K_1-\frac{C}{\delta}+\frac{C}{\delta}\phi(f)-C\eta\phi(f)+C\eta(\Psi_1+\Psi_2)(1-f)\right]^2} \qquad (17)$$

For the form $z=(1-f)(1+\theta f)$

$$\frac{1}{\lambda C}\left(\frac{dp_1}{df}\right)_{\substack{z=0\\f=1}}=\frac{\left[K_1-\frac{C}{\delta}\right][-1-\theta+\Psi_1+\Psi_2]+\frac{C}{\delta}[\Psi_1+\Psi_2][-1-\theta]}{\left[K_1-\frac{C}{\delta}\right]^2} \qquad (18)$$

$$\frac{1}{\lambda C}\left(\frac{dp_1}{df}\right)_{\substack{z=1\\f=0}}=\frac{\left[K_1-\frac{C}{\delta}\right][-1+\theta+\Psi_1+\Psi_2]+\frac{C}{\delta}[\Psi_1+\Psi_2]\theta}{[K_1-C\eta+C\eta(\Psi_1+\Psi_2)]^2} \qquad \ldots \quad (19)$$

For a tubular charge ($\theta=0$)

At all-burnt $\frac{dp_1}{df}$ has same sign as $-1+\Psi_1+\Psi_2$

Thus if

$$\Psi_1+\Psi_2>1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

the maximum pressure occurs before all-burnt, and if

$$\Psi_1+\Psi_2\leqslant 1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

the maximum pressure occurs at all-burnt.

However, from (13), for a tubular charge

$$N_1 = (1-f)(1-\Psi_1-\Psi_2)$$

Thus for our equations to remain valid (21) must be satisfied and for a tubular charge in a Recoil-less High-Low pressure gun, the maximum pressure would occur at all-burnt.

For an ordinary H/L gun, $\Psi_1 = 0$, Ψ_2 is of the order of 0·5 and (21) is satisfied. Thus there also the maximum pressure in the first chamber occurs at all-burnt.

For a cord charge ($\theta = 1$)

In this case $\frac{dp_1}{df}$ is essentially positive at all-burnt and therefore maximum pressure occurs definitely before all-burnt. This result is also the same as for orthodox and for ordinary H/L guns.

For quadratic form-function

The condition that maximum pressure occurs before all-burnt is

$$\theta > \frac{\left[K_1-\frac{C}{\delta}\right][1-\Psi_1-\Psi_2]}{K_1-\frac{C}{\delta}+\frac{C}{\delta}(\Psi_1+\Psi_2)} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

For general form-function

The corresponding condition in this case, from (17), is

$$K_1[\Psi_1+\Psi_2]+\left[K_1-\frac{C}{\delta}+\frac{C}{\delta}(\Psi_1+\Psi_2)\right]\phi'(0) > 0 \quad \ldots \quad \ldots \quad (23)$$

If $\Psi_1 = 0$, $\Psi_2 = \Psi$, (22) and (23) reduce to the corresponding conditions for ordinary H/L guns (Kapur, 1956*a*).

If (23) is satisfied, maximum pressure occurs when

$$\left[K_1-\frac{C}{\delta}\right][\phi'(f)+\Psi_1+\Psi_2]+\frac{C}{\delta}[\Psi_1+\Psi_2][\phi(f)+(1-f)\phi'(f)] = 0 \quad \ldots \quad (24)$$

The root between 0 and 1 determines the value of f at which maximum pressure occurs in the first chamber and then substituting this value in (16), we get the maximum value.

4. Second chamber. Fundamental differential equation

From (8), (9) and (10)

$$\frac{d}{df}\left[p_1\frac{dx}{df}\right] = \frac{AD^2}{W_2\beta^2}\frac{p_2}{p_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Substituting for p_1, p_2 from (2), (14) and (16), we get

$$\frac{d}{df}\left[\frac{\phi(f)-(1-f)(\Psi_1+\Psi_2)}{K_1-\frac{C}{\delta}-C\left(\eta-\frac{1}{\delta}\right)\phi(f)+C\eta(\Psi_1+\Psi_2)(1-f)}\frac{dx}{df}\right]$$

$$= \frac{AD^2}{W_2\beta^2}\frac{1}{C\lambda}\frac{\Psi_2(1-f)}{K_2+Ax-C\eta\Psi_2(1-f)}\frac{K_1-\frac{C}{\delta}-C\left(\eta-\frac{1}{\delta}\right)\phi(f)+C\eta(\Psi_1+\Psi_2)(1-f)}{\phi(f)-(1-f)(\Psi_1+\Psi_2)} \quad (26)$$

For a tubular charge $\phi(f) = 1-f = z$, this becomes

$$\frac{d}{dz}\left\{\frac{z}{1+bz}\frac{dX}{dz}\right\} = \frac{1+bz}{X-\nu z}, \quad \ldots \quad (27)$$

where

$$b = \frac{C\left[\frac{1}{\delta}-\eta(1-\Psi_1-\Psi_2)\right]}{K_1-\frac{C}{\delta}} \quad \ldots \quad (28)$$

$$\mu X = K_2+Ax \quad \ldots \quad (29)$$

$$\mu\nu = C\eta\Psi_2 \quad \ldots \quad (30)$$

$$\mu^2 = \frac{A^2D^2}{\beta^2}\frac{\left(K_1-\frac{C}{\delta}\right)^2}{C\lambda W_2}\frac{\Psi_2}{[1-\Psi_1-\Psi_2]^2} \quad \ldots \quad (31)$$

(27) is the same as the corresponding equation of Corner for H/L gun, only b and v have to be adjusted. Corner's series solution for (27), as well as his numerical solution in powers of b and v, can be used here.

For a cord charge $\phi(f) = 1-f^2$ and (26) can be integrated in series and solutions given by Aggarwal (1955) as modified by Kapur (1956*a*) can be easily adapted to this case.

Once we know X (or x) and $\frac{dX}{dz}\left(\text{or } \frac{dx}{df}\right)$ as functions of f, we can find p_1 from (16), N_1 from (13), N_2 from (14), p_2 from (3) and

$$v = \frac{dx}{df}\frac{df}{dt} = -\frac{\beta}{D}p_1\frac{dx}{df}$$

from the knowledge of p_1 and $\frac{dx}{df}$.

Also from (10) and (16)

$$-\frac{\beta}{D}t = \int_f^1 \frac{K_1-\frac{C}{\delta}+\frac{C}{\delta}\phi(f)-C\eta\phi(f)+C\eta(\Psi_1+\Psi_2)(1-f)}{C\lambda[\phi(f)-(\Psi_1+\Psi_2)(1-f)]}df \quad \ldots \quad (32)$$

When $\phi(f) = 1-f$ or $(1-f)(1+\theta f)$, we easily get explicit relations between f and t, on evaluating the integral on the R.H.S. of (32).

5. Internal ballistics after burnt

After all-burnt $z = 1$, $f = 0$. On neglecting η, equations (1), (2), (3), (6) become

$$p_1K_1 = CN_1\lambda \quad \ldots \quad (33)$$

$$p_2[K_2+Ax] = CN_2\lambda \quad \ldots \quad (34)$$

$$0 = C\frac{dN_1}{dt}+\frac{\psi S_1p_1}{\sqrt{\lambda}}+\frac{\psi S_2p_1}{\sqrt{\lambda}} \quad \ldots \quad (35)$$

$$C\frac{dN_2}{dt} = \frac{\psi S_2p_1}{\sqrt{\lambda}} \quad \ldots \quad (36)$$

From (33) and (35)

$$\frac{dN_1}{dt} = -\left[\frac{\psi S_1}{C\sqrt{\lambda}}+\frac{\psi S_2}{C\sqrt{\lambda}}\right]\frac{CN_1\lambda}{K_1}$$

Integrating and using (12)

$$N_1 = [1-\Psi_1-\Psi_2]e^{-\frac{\beta}{D}\frac{C\lambda}{K_1}[\Psi_1+\Psi_2][t-t_B]}, \quad \ldots \quad \ldots \quad (37)$$

since, from (13), at all-burnt ($z = 1$, $f = 0$).

From (33), (36) and (37)

$$C\frac{dN_2}{dt} = \frac{\psi S_2}{\sqrt{\lambda}}\frac{C\lambda}{K_1}[1-\Psi_1-\Psi_2]e^{-\frac{\beta}{D}\frac{C\lambda}{K_1}[\Psi_1+\Psi_2][t-t_B]}$$

Integrating

$$N_2 = \Psi_2+\frac{\Psi_2}{\Psi_1+\Psi_2}[1-\Psi_1-\Psi_2]\left[1-e^{-\frac{\beta C\lambda}{DK_1}[\Psi_1+\Psi_2][t-t_B]}\right] \quad (38)$$

Again

$$W_2\frac{d^2x}{dt^2} = Ap_2 = \frac{ACN_2\lambda}{K_2+Ax} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

Since N_2 is from (38) a known function of t, integration of (39) determines x and v as functions of t and then (34) determines p_2 as a function of t. We can also determine the instant of shot-ejection and muzzle velocity.

6. Internal ballistics after burnt, variation of temperature being taken into account

In the last section, we discussed the Internal Ballistics after burnt on the Isothermal assumption, but this assumption is very poor after burnt, as gases cool considerably by expansion. At any time, let the temperatures of the two chambers be T_1 and T_2. This introduces two new variables and the two additional equations we have to write are the energy equations for the two chambers.

Energy equation of the first chamber

A mass $-CdN_1$ escapes through the nozzles. Since the expansion is adiabatic, we have

$$\frac{dT_1}{T_1} = (\gamma-1)\frac{d\rho}{\rho}$$

Also

$$\frac{d\rho}{\rho} = \frac{dN_1}{N_1}$$

$$\therefore \quad \frac{dT_1}{T_1} = (\gamma-1)\frac{dN_1}{N_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

Energy equation of the second chamber

(i) Net change in the internal energy of the gas in the second chamber $= Cc_v d(N_2T_2) = \frac{CR}{\gamma-1}d(N_2T_2)$

(ii) Change in energy due to the kinetic energy imparted to the shot $= -Ap_2dx$

(iii) Internal energy entering from the first chamber into the second $= CdN_2c_vT_1 = CdN_2\dfrac{R}{\gamma-1}T_1$

(iv) Energy taken from the first chamber in the gas forcing its way through the nozzle $= CdN_2RT_1$

$$\therefore \qquad d(N_2T_2) = \gamma T_1 dN_2 - \frac{Ap_2}{CR}(\gamma-1)dx$$

or

$$\frac{d}{dt}(N_2T_2) = -(\bar{\gamma}-1)\frac{Ap_2}{CR}\frac{dx}{dt} + \gamma T_1\frac{dN_2}{dt}, \qquad \ldots \qquad \ldots (41)$$

if we take heat losses into account. (41) differs from the corresponding equation of Corner (1950) deduced by him for the H/L gun in as much as Corner's equation does not contain the term $\gamma T_1\dfrac{dN_2}{dt}$. It appears that the addition of energy from the middle nozzle was not considered by him*.

Moreover if we neglect heat losses and put γ, $\bar{\gamma}$ equal to unity in (41), we get the Isothermal Model $T_1 = T_2 =$ constant; but if this substitution is made in Corner's equation, it would give $N_2T_2 =$ constant and since N_2 is apparently varying, it would imply the variation in T_2.

Equation of state for the two chambers

Neglecting co-volume η, we have after all-burnt

$$p_1K_1 = CN_1RT_1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (42)$$

$$p_2[K_2+Ax] = CN_2RT_2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (43)$$

Equation of continuity for the two chambers

$$0 = C\frac{dN_1}{dt} + \frac{\psi S_1p_1}{\sqrt{RT_1}} + \frac{\psi S_2p_1}{\sqrt{RT_1}} \qquad \ldots \quad \ldots \quad \ldots (44)$$

$$C\frac{dN_2}{dt} = \frac{\psi S_2p_1}{\sqrt{RT_1}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (45)$$

Integration of the equations

From (40), we get

$$\frac{T_1}{T_{1,B}} = \left(\frac{N_1}{N_{1,B}}\right)^{\gamma-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (46)$$

(42) then gives

$$\frac{p_1}{p_{1,B}} = \left(\frac{N_1}{N_{1,B}}\right)^{\gamma} = \left(\frac{T_1}{T_{1,B}}\right)^{\frac{\gamma}{\gamma-1}} \qquad \ldots \quad \ldots \quad \ldots (47)$$

---

* Dr. Corner, in a private communication, has expressed his agreement with the author and has kindly agreed to make the necessary correction in the next edition of his book. He has evaluated the error due to the neglect of the energy passing from first chamber into second and he has found that it may make a difference of as much as 30% in the square of the muzzle velocity in the H/L gun.

From (44) and (47)

$$\frac{dN_1}{dt} = m_1(N_1)^{\frac{\gamma+1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

where

$$m_1 = \frac{\psi}{\sqrt{R}}(S_1+S_2)\frac{p_{1,\,B}(T_{1,\,B})^{-\frac{1}{2}}}{(N_{1,\,B})^{\frac{\gamma+1}{2}}} \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

Integrating (48)

$$\frac{N_1}{N_{1,\,B}} = \left[1+\frac{t-t_B}{\theta_1}\right]^{-\frac{2}{\gamma-1}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

where

$$\theta_1 = \frac{2N_{1,\,B}^{-\frac{\gamma-1}{2}}}{(\gamma-1)m_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

From (47) and (50)

$$\frac{p_1}{p_{1,\,B}} = \left[1+\frac{t-t_B}{\theta_1}\right]^{-\frac{2\gamma}{\gamma-1}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

$$\frac{T_1}{T_{1,\,B}} = \left[1+\frac{t-t_B}{\theta_1}\right]^{-2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

From (45)

$$\frac{dN_2}{dt} = \frac{\psi S_2}{C\sqrt{R}}\frac{p_{1,\,B}}{\sqrt{T_{1,\,B}}}\left[1+\frac{t-t_B}{\theta_1}\right]^{-\frac{\gamma+1}{\gamma-1}}$$

Integrating

$$N_2 = N_{2,\,B} - \frac{\gamma-1}{2}\frac{\psi S_2}{C\sqrt{R}}\frac{p_{1,\,B}\theta_1}{\sqrt{T_{1,\,B}}}\left[\left(1+\frac{t-t_B}{\theta_1}\right)^{-\frac{2}{\gamma-1}}-1\right]$$

or

$$N_2 = \Psi_2+\frac{\gamma-1}{2}\frac{\psi S_2}{C\sqrt{R}}\frac{p_{1,\,B}}{\sqrt{T_{1,\,B}}}\theta_1\left[1-\left(1+\frac{t-t_B}{\theta_1}\right)^{-\frac{2}{\gamma-1}}\right] \quad \ldots \quad (54)$$

since from (13) and (14)

$$N_{1,\,B} = 1-\Psi_1-\Psi_2\,;\; N_{2,\,B} = \Psi_2$$

From (18) and (41)

$$N_2T_2-N_{2,\,B}T_{2,\,B} = -\frac{\bar{\gamma}-1}{2cR}W_2\left\{\left(\frac{dx}{dt}\right)^2-\left(\frac{dx}{dt}\right)_B^2\right\}+\gamma\int_{t_B}^{t}T_1\frac{dN_2}{dt}\,dt$$

or

$$N_2T_2-N_{2,\,B}T_{2,\,B} = -\frac{\bar{\gamma}-1}{2cR}W_2\left\{\left(\frac{dx}{dt}\right)^2-\left(\frac{dx}{dt}\right)_B^2\right\}$$

$$-\frac{\psi S_2}{C\sqrt{R}}\sqrt{T_{1,\,B}}\,p_{1,\,B}\frac{\theta_1(\gamma-1)}{2}\left[1+\left(\frac{t-t_B}{\theta_1}\right)\right]^{\frac{-2\gamma}{\gamma-1}} \quad (55)$$

or

$$\frac{C\lambda}{\gamma-1}[\Psi_2-N_2T_2'] = \frac{1}{2}W_2\left\{\left(\frac{dx}{dt}\right)^2-\left(\frac{dx}{dt}\right)_B^2\right\}\left(\frac{\bar{\gamma}-1}{\gamma-1}\right)$$
$$+\frac{1}{2}\psi S_2\sqrt{\lambda}\,p_{1,B}\theta_1\left[1+\frac{t-t_B}{\theta_1}\right]^{\frac{-2\gamma}{\gamma-1}}, \quad .. \quad (56)$$

where T_2' is the ratio of the gas temperature in the second chamber to the mean value it had during burning.

From (8), (41), (43) and (55)

$$\frac{1}{ACR}W_2\frac{d^2x}{dt^2}[K_2+Ax]-N_{2,B}T_{2,B}$$
$$= -\frac{\bar{\gamma}-1}{2CR}W_2\left(\frac{dx}{dt}\right)^2+\frac{\bar{\gamma}-1}{2CR}W_2\left(\frac{dx}{dt}\right)_B^2$$
$$-\frac{1}{2}\theta_1(\gamma-1)\frac{\psi S_2}{C\sqrt{R}}\sqrt{T_{1,B}}\,p_{1,B}\left[1+\left(\frac{t-t_B}{\theta_1}\right)^{\frac{-2\gamma}{\gamma-1}}\right] \quad .. \quad (57)$$

which is an ordinary differential equation to obtain x and $\frac{dx}{dt}$ as functions of t.

7. Composite charges in R.C.L. H/L gun. Uniqueness of maximum pressure in the first chamber

If F_i, C_i, D_i, β_i, θ_i denote the parameters for the ith component charge, the form-function for the rth stage of burning is (Kapur, 1956*b*)

$$z = \phi(f) = A_r+B_r(1-f)-E_r(1-f)^2 \quad .. \quad .. \quad .. \quad (58)$$

where

$$A_r = \sum_{i=1}^{r-1}\lambda_i \quad .. \quad .. \quad .. \quad .. \quad .. \quad (59a)$$

$$B_r = \sum_{i=r}^{n}\lambda_i k_i(1+\theta_i) \quad .. \quad .. \quad .. \quad .. \quad (59b)$$

$$E_r = \sum_{i=r}^{n}\lambda_i k_i^2\theta_i \quad .. \quad .. \quad .. \quad .. \quad (59c)$$

$$\lambda_i = \frac{F_iC_i}{\sum_{i=1}^{n}F_iC_i} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (60a)$$

$$k_i = \frac{D_i/B_i}{D/B} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (60b)$$

(17) gives for the rth stage,

$$\frac{1}{\lambda C}\frac{dp_1}{df} = \frac{P_r+B_r(1-f)+R_r(1-f)^2}{\left[K_1-\frac{C}{\delta}-C\left(\eta-\frac{1}{\delta}\right)(A_r+B_r(1-f)-E_r(1-f)^2)+C\eta(\Psi_1+\Psi_2)(1-f)\right]^2} \quad (61)$$

where

$$P_r = A_r \frac{C}{\delta}(\Psi_1+\Psi_2) - \left(K_1 - \frac{C}{\delta}\right)(B_r - \Psi_1 - \Psi_2) \quad \ldots \quad \ldots (62a)$$

$$Q_r = 2E_r\left[K_1 - \frac{C}{\delta}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (62b)$$

$$R_r = E_r \frac{C}{\delta}(\Psi_1+\Psi_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots (62c)$$

We assume all the propellants have either constant-burning or degressive burning surfaces so that $E_r \geqslant 0$. In this case throughout the rth stage, the numerator on the R.H.S. of (61) continues to increase and accordingly in the rth stage $\frac{dp_1}{df}$ can change from negative to positive, but not vice versa. Thus a pressure maximum can arise in the rth stage, but a pressure minimum cannot arise there.

At the end of the rth stage, $\phi'(f)$ is discontinuous unless $\theta_r = 1$ and receives a non-negative increment. Since coefficient of $\phi'(f)$ in the numerator of (17) is positive; $\frac{dp_1}{df}$ can change at the end of a stage, from negative to positive and not vice versa.

Thus once $\frac{dp_1}{df}$ has become positive in or at the end of any stage, it cannot become negative again and thus the maximum pressure is unique.

The condition that the maximum pressure occurs in the rth stage is

$$P_r + \frac{Q_r}{k_{r-1}} + \frac{R_r}{k^2_{r-1}} < 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (63a)$$

and

$$P_r + \frac{Q_r}{k_r} + \frac{R_r}{k^2_r} > 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (63b)$$

The condition that the maximum pressure occurs at the end of the rth stage is either (i)

$$P_r + \frac{Q_r}{k_r} + \frac{R_r}{k^2_r} < 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (64a)$$

and

$$P_{r+1} + \frac{Q_{r+1}}{k_r} + \frac{R_{r+1}}{k^2_r} > 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (64b)$$

or (ii)

$$P_r + \frac{Q_r}{k_r} + \frac{R_r}{k^2_r} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (64c)$$

or (iii)

$$P_{r+1} + \frac{Q_{r+1}}{k_r} + \frac{R_{r+1}}{k^2_r} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (64d)$$

The condition that the maximum pressure occurs at all-burnt is

$$P_n + Q_n + R_n \leqslant 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (65)$$

If $\Psi_1 = 0$, $\Psi_2 = \Psi$, these conditions reduce to the corresponding conditions for H/L gun.

8. Composite charges. Fundamental differential equation for the rth stage

If in (26), we put

$$\phi(f) = A_r + B_r(1-f) - E_r(1-f)^2,$$

we get the fundamental differential equation for the rth stage. The initial conditions for its integration at the beginning of the first stage are

$$x = 0,\ f = 1,\ \frac{dx}{df} = -\frac{AD^2}{W_2\beta^2 C\lambda}\frac{\Psi_2}{K_2}\frac{\left(K_1 - \frac{C}{\delta}\right)^2}{[B_1 - \Psi_1 - \Psi_2]^2} \qquad \ldots \quad \ldots \quad (66)$$

Also

$$\frac{dx}{df} = \frac{dx}{dt}\Big/\frac{df}{dt} = -\frac{vD}{\beta p}$$

and since x, f, v, p are continuous in crossing from one stage to the other, the initial conditions for the rth stage are

$$x = x_{r-1},\ f = f_{r-1} = 1 - \frac{1}{k_{r-1}},\ \frac{dx}{df} = \left(\frac{dx}{df}\right)_{r-1} \qquad \ldots \quad \ldots \quad (67)$$

If all the component charges are tubular, $E_r = 0$ and

$$\phi(f) = A_r + B_r(1-f)$$

In this case (26) can be transformed to

$$\frac{d}{dz}\left[\frac{e_r + Z}{1 + b_r Z}\frac{dX}{dZ}\right] = \frac{Z}{X - v_r Z}\frac{1 + b_r Z}{e_r + Z}, \qquad \ldots \quad \ldots \quad \ldots \quad (68)$$

where

$$Z = 1 - f \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (69)$$

$$K_2 + Ax = \mu_r X \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (70a)$$

$$C\eta\Psi_2 = \mu_r v_r \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (70b)$$

$$b_r = \frac{C\eta\left[\Psi_1 + \Psi_2 - CB_r\left(\eta - \frac{1}{\delta}\right)\right]}{K_1 - \frac{C}{\delta} - C\left(\eta - \frac{1}{\delta}\right)A_r} \qquad \ldots \quad \ldots \quad (71)$$

$$e_r = \frac{A_r}{B_r - \Psi_1 - \Psi_2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

$$\mu_r^2 = \frac{A^2 D^2}{W_2\beta^2 C\lambda}\Psi_2\frac{\left[K_1 - \frac{C}{\delta} - C\left(\eta - \frac{1}{\delta}\right)A_r\right]^2}{[B_r - \Psi_1 - \Psi_2]^2} \quad \ldots \quad (73)$$

The initial conditions for the integration of (68) are

$$X = \frac{K_2}{\mu_1},\ Z = 0,\ \frac{dX}{dZ} = -\frac{A}{\mu_1}\frac{dx}{df} = -\mu \qquad \ldots \quad \ldots \quad (74)$$

In the rth stage

$$\frac{dX}{dZ} = -\frac{A}{\mu_r}\frac{dx}{df} = \frac{A}{\mu_r}\frac{D}{\beta}\frac{v}{p}$$

Accordingly initial conditions for the rth stage are

$$X = \frac{\mu_{r-1}}{\mu_r} X_{r-1}, Z = Z_{r-1} = \frac{1}{k_{r-1}}, \frac{dX}{dZ} = \frac{\mu_{r-1}}{\mu_r} \left(\frac{dX}{dZ}\right)_{r-1} \quad .. \quad (75)$$

ACKNOWLEDGEMENTS

The author is extremely grateful to Prof. P. L. Bhatnagar for his interest and encouragement, and to Dr. J. Corner, Ph.D., for his valuable advice.

SUMMARY

In the present paper, we have developed a simple theory for the Isothermal Model for a R.C.L. H/L gun up to burnt. After burnt we have discussed the general model taking the variation of temperature into account. For the Isothermal Model, we are able to establish uniqueness of maximum pressure for composite charges when the component charges are constant-burning or degressive. The results for an ordinary H/L gun follow as particular cases.

REFERENCES

Aggarwal, S. P. (1955). Internal ballistics of H/L gun using propellant of any shape. *PNISIPS*, **21**, 350–362.
Corner, J. (1947). The internal ballistics of a leaking gun. *Proc. Roy. Soc.*, **188**A, 237.
——— (1948). Internal ballistics of H/L gun. *Jour. Frank. Inst.*, **246**, 233.
——— (1950). Theory of Interior Ballistics of Guns. John Wiley & Sons, New York.
Kapur, J. N. (1956*a*). Internal ballistics of H/L gun. *PNISIPS* (*in press*).
——— (1956*b*). Equivalent-charge method in the general theory of composite charges. *PNISIPS*, **22A**, 63–81.

Issued October 9, 1957.