A RELATIVISTIC ANALOGUE OF A SIMPLE NEWTONIAN RESULT

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1. INTRODUCTION

A liquid sphere of mass m and radius R satisfies the relation

$$m = \frac{4}{3}\pi(\alpha-\beta)r_0^3 + \frac{4}{3}\pi\beta R^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots (1.1)$$

if the density is α within an inner core of radius r_0 and β outside. In a relativistic treatment of the problem such as this one has to take note of three basic facts. Firstly, due to the curvature of space, the co-ordinates that are used are not Euclidean and hence the radius of the sphere in (1.1) is quite different from the corresponding quantity in relativity which is also referred to as the radius of the sphere. This distinction is important and it is often slurred over when comparisons are made between the classical and the relativistic theories. Secondly, in the non-relativistic theory the operational meaning of the word density is quite clear but in general relativity one is not sure as to which function of the stress-energy tensor components is to be identified as the density. We do not know for certain whether in our present treatment the density should be identified with the component T_4^4 of the stress-energy tensor T_ν^μ, or with the invariant T formed from this tensor. Thirdly, in the classical theory the concept of mass does not present any difficulty. It is regarded as an absolutely inherent property of matter which is uninfluenced by the presence of other matter. In general relativity, however, mass is a co-operative phenomenon. Its measure is a more complicated function of matter and motion and is determined by the total amount of matter surrounding the object of interest.

In the relativistic formulation of the problem we are confronted with nine equations involving nine constants from which eight constants arising out of the metrics have to be eliminated to obtain the analogue of (1.1). As the equations are not all linear we have resorted to a method of approximation. As a first step towards an understanding of a complicated problem like this, the approximate result that we have obtained here will be found to be of some interest.

2. THE NINE EQUATIONS

The distribution of pressure and density in the relativistic treatment is given by T_μ^ν, where

$$T_\mu^\nu = (\rho+p)v_\mu v^\nu - pg_\mu^\nu$$

in the usual notation. The line-element will be supposed to be static and of the following form:—

$$ds^2 = -e^\mu(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2)+e^\nu dt^2,\ \mu = \mu(r),\ \nu = \nu(r). \quad \ldots (2.1)$$

The particles of liquid have no motion relative to the co-ordinate system. The field equations are

$$R^{\nu}_{\mu} - \tfrac{1}{2}Rg^{\nu}_{\mu} = -8\pi T^{\nu}_{\mu}. \quad \ldots \quad (2.2)$$

The pressure p and the density ρ are given by

$$T^1_1 = T^2_2 = T^3_3 = -p,\ T^4_4 = \rho. \quad \ldots \quad (2.3)$$

All other components of T^{ν}_{μ} vanish.

Since $T^4_4 = \alpha$ in the core and $T^4_4 = \beta$ outside the core in the sphere from the field equations we get separate metrics for the three zones, viz. the core, the shell and the free space. Thus

$$e^{\mu} = (A_0+B_0r^2)^{-2},\ e^{\nu} = \left(L_0 - \frac{M_0}{A_0+B_0r^2}\right)^2 \text{ for } 0 < r < r_0 \quad \ldots \quad (2.4a)$$

$$e^{\mu} = (A+Br^2)^{-2},\ e^{\nu} = \left(L - \frac{M}{A+Br^2}\right)^2 \text{ for } r_0 < r < R \quad \ldots \quad (2.4b)$$

$$e^{\mu} = \left(1+\frac{m}{2r}\right)^4,\ e^{\nu} = \left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 \text{ for } r > R. \quad \ldots \quad (2.4c)$$

In the above A_0, B_0, L_0, M_0, A, B, L, M and m are the unknown constants to be determined from the boundary conditions. The continuity of $g_{\mu\nu}$ at $r = r_0$ and at $r = R$ gives

$$\left.\begin{aligned}
&\text{(i)} \quad A_0+B_0r_0^2 = A+Br_0^2\\
&\text{(ii)} \quad L_0 - \frac{M_0}{A_0+B_0r_0^2} = L - \frac{M}{A+Br_0^2}\\
&\text{(iii)} \quad A+BR^2 = \left(1+\frac{m}{2R}\right)^{-2}\\
&\text{(iv)} \quad L - \frac{M}{A+BR^2} = \left(1-\frac{m}{2R}\right)\left(1+\frac{m}{2R}\right)^{-1}.
\end{aligned}\right\} \quad \ldots \quad (2.5)$$

The continuity of pressure at $r = r_0$ and at $r = R$ gives

$$\left.\begin{aligned}
&\text{(v)} \quad B_0[A_0(2M_0-L_0) - B_0r_0^2(M_0+L_0A_0)]\\
&\qquad = B[A(2M-LA) - Br_0^2(M+LA)]\\
&\text{(vi)} \quad A(2M-LA) - BR^2(M+LA) = 0.
\end{aligned}\right\} \quad \ldots \quad (2.6)$$

Calculations for T^4_4 for (2.4a) and (2.4b) give

$$\left.\begin{aligned}
&\text{(vii)} \quad 8\pi\alpha = 12A_0B_0\\
&\text{(viii)} \quad 8\pi\beta = 12AB.
\end{aligned}\right\} \quad \ldots \quad (2.7)$$

These are eight equations in nine unknowns. To obtain the ninth equation we note that m, the total energy of the system, is the sum of the energies of the three fields of our system. Using Tolman's (1934) formula the expression for the energy content is found to be

$$U = \left[\tfrac{1}{2}\nu' e^{(\mu+\nu)/2}\, r^2\right]_0^r \quad \ldots \quad (2.8)$$

The formula when applied to the three zones gives

$$\text{(ix)} \quad \frac{2M_0B_0r_0^3}{(A_0+B_0r_0^2)^3}+\frac{2MBR_0^3}{(A+BR^2)^3}-\frac{2MBr_0^3}{(A+Br_0^2)^3}=m. \quad .. \ (2.9)$$

The analogue of (1.1) is to be found by eliminating A_0, B_0, L_0, M_0, A, B, L, and M from the nine equations i–ix. As the equations are non-linear it is sufficient for our purpose if we adopt a method of approximation ignoring $\left(\frac{m}{R}\right)^3$ and quantities of that order.

3. Approximate Solution

From the equations (iii) and (viii) we obtain

$$A+\frac{2\pi}{3A}\beta R^2=\left(1+\frac{m}{2R}\right)^{-2}=1-\frac{m}{R}+\frac{3}{4}\frac{m^2}{R^2}. \quad .. \quad .. \ (3.1)$$

Let

$$A=1-\frac{m}{R}+\frac{3}{4}\frac{m^2}{R^2}+x$$

where x contains terms up to the second order. Substituting this value of A in (3.1) we get

$$x=-\frac{2\pi}{3}\beta R^2\left(1+\frac{m}{R}\right)-\left(\frac{2\pi}{3}\beta R^2\right)^2.$$

Thus

$$A=1-\frac{m}{R}+\frac{3}{4}\frac{m^2}{R^2}-\frac{2\pi}{3}\beta R^2\left(1+\frac{m}{R}\right)-\left(\frac{2\pi}{3}\beta R^2\right)^2. \quad .. \quad .. \ (3.2)$$

From (iii) on substitution of the value of A we get

$$B=\frac{2\pi}{3}\beta\left(1+\frac{m}{R}\right)+\left(\frac{2\pi}{3}\beta R\right)^2. \quad .. \quad .. \quad .. \ (3.3)$$

Equations (vii) and (i) give

$$B_0=\frac{2\pi\alpha}{3}\left[\left(1+\frac{m}{R}\right)+(R^2-r_0^2)\frac{2\pi}{3}\beta\right]+\frac{4\pi^2}{9}r_0^2\alpha^2 \quad .. \quad .. \ (3.4)$$

and

$$A_0=1-\frac{m}{R}+\frac{3}{4}\frac{m^2}{R^2}-\frac{2\pi}{3}\beta R^2\left(1+\frac{m}{R}\right)-\left(\frac{2\pi}{3}\beta R^2\right)^2$$

$$+r_0^2\left[\frac{2\pi}{3}\beta\left(1+\frac{m}{R}\right)+\frac{4\pi^2}{9}\beta^2R^2-\frac{2\pi}{3}\alpha\left(1+\frac{m}{R}\right)-\frac{4\pi^2}{9}\alpha\beta\,(R^2-r_0^2)-\frac{4\pi^2}{9}\alpha^2r_0^2\right]. \ (3.5)$$

Again (iv) and (vi) give

$$L=2\left(1-\frac{m}{R}+\frac{m^2}{2R^2}\right)+\frac{2\pi}{3}\beta R^2\left(1+\frac{m}{R}\right)+\frac{4\pi^2}{3}\beta^2R^4 \quad .. \quad .. \ (3.6)$$

and

$$M=1-\frac{2m}{R}+\frac{9}{4}\frac{m^2}{R^2}+\frac{2\pi}{3}\beta R^2+\frac{4\pi^2}{3}\beta^2R^4. \quad .. \quad .. \ (3.7)$$

Equations (i), (ii), (vii) and (viii) give

$$M_0B_0(A_0-B_0r_0^2) = \frac{2\pi}{3}(\alpha-\beta)\left(A+Br_0^2\right)L+M\left(2AB-B^2r_0^2-A_0B_0\right).$$

Using the values of L and M from (3.6) and (3.7) we obtain

$$M_0B_0 = \frac{2\pi}{3}\left[\alpha\left(1-\frac{m}{R}\right)-\frac{2\pi}{3}\left(R^2-r_0^2\right)\alpha\beta+2\pi\left(R^2-r_0^2\right)\beta^2+\frac{4\pi}{3}\alpha^2r_0^2\right]. \quad (3.8)$$

Also

$$MB = \frac{2\pi}{3}\beta\left(1-\frac{m}{R}\right)+\frac{8\pi^2}{9}\beta^2R^2.$$

We now substitute these values in equation (ix) which can be written as

$$\frac{2r_0^3(M_0B_0-MB)}{(A+Br_0^2)^3}+\frac{2R^3MB}{(A+BR^2)^3} = m.$$

We thus obtain the following equation

$$m\left\{1-\frac{8\pi}{3}(\alpha-\beta)\frac{r_0^3}{R}-\frac{8\pi}{3}\beta R^2\right\} = \frac{4\pi}{3}(\alpha-\beta)\,r_0^3+\frac{4\pi}{3}\beta R^3$$
$$+\frac{16\pi^2}{9}\left(\alpha^2r_0^5+\alpha\beta R^2r_0^3-\alpha\beta r_0^5-\beta^2R^2r_0^3+\beta^2R^5\right).$$

Hence

$$m = \frac{4\pi}{3}(\alpha-\beta)r_0^3+\frac{4\pi}{3}\beta R^3+\frac{16\pi^2}{9}(\alpha-\beta)r_0^3\left\{\left(r_0^2+\frac{2r_0^3}{R}\right)\alpha\right.$$
$$\left.-\left(\frac{2r_0^3}{R}-5R^2\right)\beta\right\}+\frac{16}{3}\pi^2\beta^2R^5. \quad \ldots \quad \ldots \quad (3.9)$$

In the case of a homogeneous sphere of density β (3.9) reduces to

$$m = \frac{4}{3}\pi\beta R^3+\frac{16}{3}\pi^2\beta^2R^5. \quad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

4. Discussions

The equation (3.9) is the approximate relativistic analogue of (1.1). As already remarked above in the introduction the radii r_0 and R in (1.1) are not the same as those in (3.9). The densities also do not convey the same meaning in both cases. It is therefore not a simple matter to determine the relativistic correction to the classical result. We may, however, regard the density as being the same in simple cases and attempt to find the required correction. We may for instance take the case of the homogeneous sphere of radius R and density β. If R_1 be the proper radius the relation between R and R_1 will be

$$R = R_1-\frac{4}{15}\pi R_1^3\beta-m, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

the terms of order higher than the first being ignored. Substitution of this value of R in (3.10) gives

$$m = \frac{4\pi}{3}\beta R_1^3 - \frac{16}{15}\pi^2\beta^2 R_1^5 \quad \ldots \quad (4.2)$$

The second term in (4.2) is the usual expression for the gravitational potential energy. This result, already obtained by Tolman by a different procedure, is interesting because it gives an insight into the approximate relativistic correction and its physical significance.

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Abstract

A liquid sphere of radius R is considered with a density α for $0 < r < r_0$ and a density β for $r_0 < r < R$. The relativistic relation defining the mass m in terms of α, β, r_0 and R is obtained, in an approximate form.

Reference

Tolman, R. C. (1934). Relativity, Thermodynamics and Cosmology, 235.

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