SOME PROBLEMS RELATING TO STRESSES IN AN INFINITE PLATE CONTAINING CIRCULAR HOLES

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I. TWO NON-INTERSECTING UNEQUAL CIRCLES

1. The problem of the stresses in a perforated plate is of importance in many engineering applications. The simplest case, that of a single circular hole in an infinite plate, has been discussed in great detail by various authors (Timoshenko and Goodier, 1951). The next less simple case is that of a group of two equal circular holes, for which the solution was given by C. B. Ling (1948). More recently B. Karunes (1952) has considered the question of initial dislocations in an infinite plate with two unequal circular holes, but in the numerical illustration he treated only the case of equal circular holes. In the present note, following the method of Ling, the general problem of two unequal circular holes is discussed by using bipolar co-ordinates. The solution is numerically evaluated for a particular example. The cases when the two circles touch or intersect will be treated in subsequent papers.

2. Take the line joining the centres of the two holes as X-axis and their radical axis as Y-axis. The two circles may be thought of as two members of a co-axial system whose limiting points are known, when the radii of the two circles as well as the distance between them is given.

Now consider the bipolar transformation

$$x+iy = -a \coth \frac{1}{2}i(\xi+i\eta), \qquad \ldots \quad (1)$$

where a represents the distance of each limiting point from the radical axis.

The co-axial system of which the two given holes are members is then given by $\eta =$ constant. Positive values of η give those members which are situated on the positive side of the Y-axis, while negative values of η give those which lie on the negative side of it.

Let the rims of the two circular holes be then given by $\eta = \alpha$ and $\eta = -\beta$, where α, β are real numbers, either both positive or both negative.

Let the stresses in the complete plate (without the given holes) arise from a stress function χ_0. The solution for the actual case under discussion will be obtained by superposing upon χ_0 another stress function χ_1 which should satisfy the following conditions

(i) The stresses at infinity, that is, for $\xi = \eta = 0$ arising from χ_1, are all zero (so that the presence of the holes does not alter the state of stress at great distances from the holes).

(ii) The normal and tangential stresses arising from $\chi_0+\chi_1$ vanish identically at $\eta = \alpha$ and at $\eta = -\beta$. This is the condition of no traction at the boundary of the holes. (2)

Now, in the bipolar co-ordinate system, the stresses arising from any stress function χ are given by

$$\left.\begin{aligned} a\widehat{\eta\eta} &= \left[(\cosh\eta-\cos\xi)\frac{\partial^2}{\partial\xi^2}-\sin\xi\frac{\partial}{\partial\xi}-\sinh\eta\frac{\partial}{\partial\eta}+\cosh\eta\right]\frac{\chi}{J}, \\ a\widehat{\xi\xi} &= \left[(\cosh\eta-\cos\xi)\frac{\partial^2}{\partial\eta^2}-\sin\xi\frac{\partial}{\partial\xi}-\sinh\eta\frac{\partial}{\partial\eta}+\cos\xi\right]\frac{\chi}{J}, \\ \text{and}\quad a\widehat{\xi\eta} &= (\cosh-\cos\xi)\frac{\partial^2}{\partial\xi\partial\eta}\frac{\chi}{J}, \end{aligned}\right\} \quad \ldots \quad (3)$$

where $$\frac{a}{J} = \cosh\eta-\cos\xi.$$

From this it is clear that for $\xi = \eta = 0$, $a\widehat{\xi\eta}$ is always zero, while the other stresses will also vanish if $\chi = 0$, for $\xi = \eta = 0$. Hence the first of conditions (2) is equivalent to the condition that

$$\chi_1 = 0, \quad \text{for} \quad \xi = \eta = 0. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Next, we may assume that χ_0 is an even function of ξ. If χ_0 is an odd function of ξ, the argument which follows will have to be modified only slightly, while if χ_0 is neither even nor odd in ξ, it can be split up into two parts, one even and the other odd in ξ, and the solution then obtained by superposition.

The stress $\widehat{\eta\eta}$ arising from the stress function χ_0 will be an even function of ξ and being periodic with period 2π, can be expanded as a Fourier Cosine Series.

If we put $\eta_1 = \eta - \frac{\alpha+\beta}{2}$, the two circles are given by $\eta_1 = \pm\frac{\alpha-\beta}{2}$. Each coefficient in this Fourier's Series can be split up into two parts, one even and the other odd in η_1. We may therefore write

$$a(\widehat{\eta\eta})_0 = \frac{1}{2}(a_0+a_0')+\sum_{n=1}^{\infty}(a_n+a_n')\cos n\xi, \quad \ldots \quad \ldots \quad (5)$$

where the a_n are all even functions of η_1, while all the a_n' are odd functions of η_1. The suffix 0 indicates that the corresponding stress arises from the stress function χ_0.

In a similar manner, we may write

$$a(\widehat{\xi\eta})_0 = \sum_{n=1}^{\infty}(b_n+b_n')\sin n\xi, \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where the b_n are all even and the b_n' are all odd functions of η_1.

We shall have occasion to consider the values of the stresses (5) and (6) only at $\eta = \alpha$ and $\eta = -\beta$. Hence we may economise in notation by understanding that the a's and the b's occurring in (5) and (6) represent the values at $\eta=\alpha$ of the corresponding coefficients. With this understanding stresses at $\eta = -\beta$ will be given by

$$\left.\begin{aligned} a(\widehat{\eta\eta})_0 &= \frac{1}{2}(a_0-a_0')+\sum_{n=1}^{\infty}(a_n-a_n')\cos n\xi, \\ \text{and}\quad a(\widehat{\xi\eta})_0 &= \sum_{n=1}^{\infty}(b_n-b_n')\sin n\xi. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

3. Let us now consider a possible form for the stress function χ_1.
The biharmonic equation $\nabla^4\chi = 0$ becomes in bipolar co-ordinates

$$\left(\frac{\partial^4}{\partial\xi^4}+2\frac{\partial^4}{\partial\xi^2\partial\eta^2}+\frac{\partial^4}{\partial\eta^4}+2\frac{\partial^2}{\partial\xi^2}-2\frac{\partial^2}{\partial\eta^2}+1\right)\frac{\chi}{J}=0. \quad \ldots \quad \ldots \quad (8)$$

Consider a solution of this equation of the form

$$\frac{\chi}{J}=\phi_n(\eta)\cos n\xi. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

On actual substitution, we easily obtain

$$\left.\begin{aligned}\phi_n(\eta) = A_n\cosh\overline{n+1}\,\eta+B_n\cosh\overline{n-1}\,\eta\\ +C_n\sinh\overline{n+1}\,\eta+D_n\sinh\overline{n-1}\,\eta,\end{aligned}\right\} \quad \ldots \quad \ldots \quad (10)$$

where A_n, B_n, C_n, D_n are constants to be suitably determined.

This is true for all positive integral values of n except $n = 1$, when the last term on the right hand side of (10) has to be replaced by $D_1\eta$, i.e.

$$\phi_1(\eta) = A_1\cosh 2\eta+B_1+C_1\sinh 2\eta+D_1\eta. \quad (11)$$

Moreover, it is seen that

$$\frac{\chi}{J}=K(\cosh\eta-\cos\xi)\log(\cosh\eta-\cos\xi), \quad \ldots \quad \ldots \quad (12)$$

where k is another constant, is also a solution of (8).

Combining all these a possible form for the stress function χ_1 is

$$\frac{\chi_1}{A}=K(\cosh\eta-\cos\xi)\log(\cosh\eta-\cos\xi)+\sum_{n=1}^{\infty}\phi_n(\eta)\cos n\xi, \quad \ldots \quad (13)$$

the condition (4) applied to χ_1 leads to

$$\sum_{n=1}^{\infty}(A_n+B_n)=0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

We now split $\phi_n(\eta)$ into two parts

$$\phi_n(\eta)=\phi_{1,n}(\eta)+\phi_{2,n}(\eta), \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where ϕ_1, (η) is odd and ϕ_2, (η) is even in η_1.

The following notation will be found convenient:—

$$\left.\begin{aligned}(n-1)\,n\,(n+1)\,\phi_{1,n}(\alpha) &= \Psi_{1,n},\\ (n-1)\,n\,(n+1)\,\phi_{2,n}(\alpha) &= \Psi_{2,n},\\ n\,\phi'_{1,n}(\alpha) &= \Psi'_{1,n},\\ \text{and}\quad n\,\phi'_{2,n}(\alpha) &= \Psi'_{2,n}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

where, in the last two equations, the dashes denote differentiation with respect to η.

According to this notation,

$$\Psi_{1,1}=\Psi_{2,1}=0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Now calculating the stresses $a\,\widehat{\eta\eta}$ and $a\,\widehat{\xi\eta}$ arising from $\chi_0+\chi_1$, splitting them into two parts each, one odd and the other even in η_1, and putting each part at $\eta=\alpha$ equal to zero, we get the following results

$$
\left.\begin{aligned}
&\text{(i)}\quad \phi_{2,1}(\alpha) = \tfrac{1}{2}\left(-a_0+K\cosh\overline{\alpha+\beta}\cosh\overline{\alpha-\beta}\right),\\
&\text{(ii)}\quad \phi_{1,1}(\alpha) = \tfrac{1}{2}\left(-a_0'+K\sinh\overline{\alpha+\beta}\sinh\overline{\alpha-\beta}\right),\\
&\text{(iii)}\quad \psi_{1,1}'+\psi_{2,1}' = 2Ke^{-\alpha}\sinh\alpha-2\sum_{n=1}^{\infty}(b_n+b_n')e^{-n\alpha},\\
&\text{(iv)}\quad \psi_{1,1}'-\psi_{2,1}' = -2Ke^{-\beta}\sinh\beta-2\sum_{n=1}^{\infty}(b_n-b_n')e^{-n\beta},\\
&\text{(v)}\quad \psi_{1,n}'+\psi_{2,n}' = (\psi_{1,1}'+\psi_{2,1}')\frac{\sinh n\alpha}{\sinh\alpha}-2K\sinh\overline{n-1}\alpha\\
&\qquad\qquad +2\sum_{m=1}^{n-1}(b_m+b_m')\frac{\sinh\overline{n-m}\alpha}{\sinh\alpha},\\
&\text{(vi)}\quad \psi_{1,n}'-\psi_{2,n}' = (\psi_{1,1}'-\psi_{2,1}')\frac{\sinh n\beta}{\sinh\beta}+2K\sinh\overline{n-1}\beta\\
&\qquad\qquad +2\sum_{m=1}^{n-1}(b_m-b_m')\frac{\sinh\overline{n-m}\beta}{\sinh\beta},\\
&\text{(vii)}\quad (\psi_{2,n}+\psi_{1,n})\sinh\alpha = (\psi_{2,1}'+\psi_{1,1}')(n\cosh n\alpha-\sinh n\alpha\coth\alpha)\\
&\qquad\qquad +K\left[(n+1)\sinh\overline{n-2}\alpha-(n-1)\sinh n\alpha\right]\\
&\qquad +2\sum_{m=1}^{n-1}\Big[(n-m)(b_m+b_m')\cosh\overline{n-m}\alpha-(m.\overline{a_m+a_m'}+\overline{b_m+b_m'}\coth\alpha)\\
&\qquad\qquad \times\sinh\overline{n-m}\alpha\Big]\\
&\text{(viii)}\quad (\psi_{2,n}-\psi_{1,n})\sinh\beta = (\psi_{2,1}'-\psi_{1,1}')(n\cos s\,n\beta-\sinh n\beta\coth\beta,\\
&\qquad\qquad +K\left[(n+1)\sinh\overline{n-2}\beta-(n-1)\sinh n\beta\right]\\
&\qquad -2\sum_{m=1}^{n-1}\Big[(n-m)(b_m-b_m')\cosh\overline{n-m}\beta+(m.\overline{a_m-a_m'}-\overline{b_m-b_m'}\coth\beta)\\
&\qquad\qquad \times\sinh\overline{n-m}\beta\Big]
\end{aligned}\right\}\quad (18)
$$

By taking the last six relations in (18) in pairs separate values of $\psi_{1,1}'$, $\psi_{2,1}'$, $\psi_{1,n}'$, $\psi_{2,n}'$, $\psi_{1,n}$, $\psi_{2,n}$ can be obtained.

The eight relations in (18) together with the condition (14) are enough for the determination of the constants K, A_1, B_1, C_1, D_1 and A_n, B_n, C_n, D_n for $n \geqslant 2$. The problem is thus completely solved.

We now proceed to illustrate the method by means of an example.

Example

4. *An infinite plate with two circular holes under an all round tension T.*

If the holes were not present, the state of stress would be given by the stress function χ_0, where

$$\chi_0 = \frac{1}{2}T(n^2+y^2),\text{ i.e. }\frac{\chi_0}{J} = \frac{1}{2}aT(\cosh\eta+\cos\xi)\ldots \qquad \ldots \quad (19)$$

The stresses arising from χ_0 are actually given by

$$(\widehat{\eta\eta})_0 = (\widehat{\xi\xi})_0 = T \text{ and } (\widehat{\xi\eta})_0 = 0. \qquad (20)$$

From this it is easy to see that in this case

$$\frac{1}{2}a_0 = aT,\ a_0' = a_n = b_n = a_n' = b_n' = 0, \text{ for } n \geqslant 1. \qquad (21)$$

Now the equations (18) give

$$\left.\begin{aligned}
A_1 &= \frac{1}{2}K\frac{\cosh\overline{\alpha-\beta}}{\sinh\overline{\alpha+\beta}}(e^{-\alpha}\sinh\alpha+e^{-\beta}\sinh\beta)\\
&\quad -\frac{1}{2}K\sinh^2\overline{\alpha-\beta}\,\frac{(\alpha+\beta)(\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta})\sinh\overline{\alpha+\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}},\\
B_1 &= -aT+\tfrac{1}{2}K\coth\overline{\alpha+\beta}\,(\sinh\overline{\alpha+\beta}\cosh\overline{\alpha+\beta}-e^{-\alpha}\sinh\alpha\\
&\qquad -e^{-\beta}\sinh\beta)\\
&\quad -\frac{1}{2}K(\alpha-\beta)\frac{\sinh^2\overline{\alpha-\beta}\sinh\overline{\alpha-\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}},\\
C_1 &= -\frac{1}{2}K\frac{\sinh\overline{\alpha-\beta}}{\sinh\overline{\alpha+\beta}}(e^{-\alpha}\sinh\alpha+e^{-\beta}\sinh\beta)\\
&\quad +\frac{1}{2}K\sinh\overline{\alpha-\beta}\cosh\overline{\alpha-\beta}\,\frac{(\alpha+\beta)(\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta})-\sinh\overline{\alpha+\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}},\\
D_1 &= K\frac{\sinh^2\overline{\alpha+\beta}\sinh\overline{\alpha-\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}},
\end{aligned}\right\} \qquad (22)$$

while for $n \geqslant 2$,

$$\left.\begin{aligned}
A_n' &= -K\big[(\sinh 2n\alpha-n\sinh 2\alpha)(\sinh^2 n\beta+n\sinh^2\beta)\\
&\quad +(\sinh 2n\beta-n\sinh 2\beta)(\sinh^2 n\alpha+n\sinh^2\alpha)\big]\\
&\qquad\times\frac{1}{\sinh^2 n\overline{\alpha+\beta}-n^2\sinh^2\overline{x+\beta}},\\
B_n' &= K\big[(\sinh 2n\alpha-n\sinh 2\alpha)(\sinh^2 n\beta-n\sinh^2\beta)\\
&\quad +(\sinh 2n\beta-n\sinh 2\beta)(\sinh^2 n\alpha-n\sinh^2\alpha)\big]\\
&\qquad\times\frac{1}{\sinh^2 n\overline{\alpha+\beta}-n^2\sinh^2\overline{\alpha+\beta}},\\
\text{and}\quad C_n' &= K\bigg[\frac{1}{2}(\sinh 2n\alpha+n\sinh 2\alpha)(\sinh 2n\beta-n\sinh 2\beta)\\
&\quad +2(\sinh^2 n\alpha-n\sinh^2\alpha)(\sinh^2 n\beta+n\sinh^2\beta)\\
&\quad +2(\sinh^2 n\beta-n^2\sinh^2\beta)\bigg]\frac{1}{\sinh^2 n\overline{\alpha+\beta}-n^2\sinh^2\overline{\alpha+\beta}},
\end{aligned}\right\} \qquad (23)$$

where $A_n' = n(n+1)A_n-K$, $B_n' = n(n-1)B_n+K$, $C_n' = n(n+1)C_n+K$.

The condition (14) now gives the value of K and we find that

$$\left.\begin{aligned}
a\frac{T}{K} &= -\frac{1}{2}+\frac{1}{2}\cosh\overline{\alpha+\beta}\cosh\overline{\alpha-\beta}-\frac{\sinh\alpha\sinh\beta}{\sinh\overline{\alpha+\beta}}\\
&\quad+(e^{-\alpha}\sinh\alpha+e^{-\beta}\sinh\beta)-\frac{1}{2}\sinh\overline{\alpha-\beta}\\
&\times\frac{(\alpha-\beta)\sinh^2\overline{\alpha+\beta}+(\alpha+\beta)\sinh\overline{\alpha-\beta}\,e^{-\overline{\alpha+\beta}}-\sinh\overline{\alpha-\beta}\sinh\overline{\alpha+\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}}\\
&\quad+2\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}\big[(\sinh 2n\alpha-n\sinh 2\alpha)(\sinh^2 n\beta-n^2\sinh^2\beta)\\
&\quad+(\sinh 2n\beta-n\sinh 2\beta)(\sinh^2 n\alpha-n^2\sinh^2\alpha)\big]\\
&\quad\times\frac{1}{\sinh^2 n\overline{\alpha+\beta}-n^2\sinh^2\overline{\alpha+\beta}}
\end{aligned}\right\} \tag{24}$$

All the constants having been evaluated, the stress at any point can now be calculated. Most important naturally are the stresses at the edges of the two holes. On calculation we find the following two results:

(i) At the edge of the hole $\eta=\alpha$,

$$\left.\begin{aligned}
a\widehat{\xi\xi} &= 2K(\cosh\alpha-\cos\xi)\Bigg[\sinh\alpha+\cos\xi\Bigg\{\sinh 2\alpha\\
&\quad-\coth\overline{\alpha+\beta}(\sinh^2\alpha+\sinh^2\beta)-\frac{(\alpha+\beta)\sinh\overline{\alpha-\beta}\sinh^2\overline{\alpha+\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}}\Bigg\}\\
&\quad+4\sum_{n=2}^{\infty}\frac{\sinh\alpha\sinh n\beta\sinh n\overline{\alpha+\beta}-n\sinh n\alpha\sinh\beta\sinh\overline{\alpha+\beta}}{\sinh^2\overline{\alpha+\beta}-n^2\sinh^2\overline{\alpha+\beta}}\cos n\xi\Bigg]
\end{aligned}\right\} \tag{25}$$

and (ii) at the edge of the hole $\eta=-\beta$.

$$\left.\begin{aligned}
a\widehat{\xi\xi} &= 2K(\cosh\beta-\cos\xi)\Bigg[\sinh\beta+\cos\xi\Bigg\{\sinh 2\beta\\
&\quad-\coth\overline{\alpha+\beta}(\sinh^2\alpha+\sinh^2\beta)+\frac{(\alpha+\beta)\sinh\overline{\alpha-\beta}\sinh^2\overline{\alpha+\beta}}{(\alpha+\beta)\cosh\overline{\alpha+\beta}-\sinh\overline{\alpha+\beta}}\Bigg\}\\
&\quad+4\sum_{n=2}^{\infty}\frac{\sinh\beta\sinh n\alpha\sinh n\overline{\alpha+\beta}-n\sinh\alpha\sinh n\beta\sinh\overline{\alpha+\beta}}{\sinh^2 n\overline{\alpha+\beta}-n^2\sinh^2\overline{\alpha+\beta}}\cos n\xi\Bigg]
\end{aligned}\right\} \tag{26}$$

It is easily verified that in case $\beta=\alpha$, i.e. in the case of two equal circular holes, all these results reduce to those obtained by C. B. Ling (*loc. cit.*).

5. *Maximum stress; numerical calculations.*

For numerical calculations, it is better to change the parameters α, β to λ_1 and λ_2 given by the relations $\cosh\alpha=\lambda_1$, $\cosh\beta=\lambda_2$. It is easily seen that λ_1 represents the ratio between the distance of the centre of the hole; $\eta=\alpha$ from the radical axis to the radius of that hole and λ_2 represents the corresponding ratio for the hole $\eta=-\beta$.

It is found that extreme stresses occur at $\xi = 0$ and at $\xi = \pi$ in the case of each hole, i.e. at the ends of the common diameter.

Table I gives the stresses at $\eta = \alpha$, $\xi = 0$ for different values of λ_1 and λ_2 (for $T = 1$). The corresponding stresses for the hole $\eta = -\beta$ can be read from the same table by interchanging the values of λ_1 and λ_2.

Table II gives the stresses at $\eta = \alpha$, $\xi = \pi$. As before the stresses at $\eta = -\beta$, $\xi = \pi$ can be read off from the same table by an interchange of the values of λ_1 and λ_2.

TABLE I

Values of $\widehat{\xi\xi}$ at $\eta = \alpha$, $\xi = 0$ for $T = 1$

λ_1 \ λ_2	1·5	2	3	5	8
1·5	2·258	2·134	2·106	1·954	1·909
2	1·919	2·157	2·250	2·299	2·320
3	0·329	1·402	2·080	2·912	2·834
5	−4·812	−1·585	−0·227	2·033	3·046
8	−14·209	−7·515	−3·274	−0·184	2·014

TABLE II

Values of $\widehat{\xi\xi}$ at $\eta = \alpha$, $\xi = \pi$ for $T = 1$

λ_1 \ λ_2	1·5	2	3	5	8
1·5	2·893	−1·305	−4·846	−6·933	−8·136
2	7·535	2·410	−1·370	−4·135	−5·564
3	14·890	7·206	2·155	−2·369	−3·847
5	25·579	14·787	8·672	2·049	−1·601
8	31·866	24·354	14·619	7·272	2·018

Note:—The corresponding values of $\widehat{\xi\xi}$ for the hole $\eta = -\beta$ can be read from these tables by interchanging the values of λ_1 and λ_2.

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