POTENTIAL FLOW WITH WAKE PAST A SPHERICAL OBSTACLE

by N. L. GHOSH, *Professor and Head of the Department of Mathematics, Presidency College, Calcutta*

(Communicated by S. N. Bose, F.N.I.)

(*Received February 11, 1956 ; read January 12, 1957*)

1. INTRODUCTION

The fact that the flow of a real fluid like air or water past a solid moving through it is primarily a potential flow except perhaps over a very thin layer on the body and inside the wake—and that it is wholly a potential flow to start with—led the author to suggest his 'Theory of Resistance in Potential Flows' nearly two years ago (Ghosh, 1954). It has been shown in the paper referred to that when the flow is wholly potential without wakes the resistance to the motion of a sphere is $16\pi\mu Ua$, U being the uniform velocity of the sphere or of the stream at infinity, a the radius and μ the coefficient of viscosity of the liquid. This result, as has already been noted, differs from the Stokesian value $6\pi\mu Ua$ but nevertheless resolves D'Alembert's paradox effectively.

On the basis of the same approach, the resistance is likely to vary when the potential flow is attended with a region of 'wake' behind the body (as is usually observed in experimental photographs), the motion in the rest of the fluid being wholly potential. The present investigation seeks to obtain the forms of such potential—wake—flows past a sphere. It has been found that two types of potential flows with a wake are admitted by the boundary conditions at infinity. In the first, there is one continuous flow pattern throughout the liquid, with a wake of a cross-section widening asymptotically at a great distance. In the second the classical flow pattern in front changes along a radial line over to a modified flow which may admit a wake of a similar nature, a discontinuity of velocity appearing on the conical surface separating the two flows.

2. Considering the relative motion, i.e. with the sphere at rest with centre at origin and the fluid at infinity moving with a uniform velocity parallel to the positive x-axis (turning it into an axis of symmetry), the Stoke's stream function in the meridian plane is given by the equation

$$r^2\psi_{rr}+(1-\mu^2)\psi_{\mu\mu} = 0 \quad \ldots \quad (1)$$

where (r, θ) represent the polar co-ordinates, and $\mu = \cos\theta$. The components of the velocity along the radial and transversal directions being

$$q_r = -\frac{1}{r^2}\psi_\mu \quad \ldots \quad (2a)$$

$$q_\theta = -\frac{1}{r\sqrt{1-\mu^2}}\psi_r \quad \ldots \quad (2b)$$

the well-known solution for the *primary* potential flow past the sphere is given by

$$\psi \equiv \psi_0 \equiv \frac{1}{2}Ur^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right) \quad \ldots \quad (3)$$

The stream-line $\psi_0 = 0$ goes round the sphere completely meeting it at the front stagnation point (a, π) and leaving it at the back stagnation point $(a, 0)$ on the x-axis.

We intend to find out solutions, if there be any, of equation (1) in the form

$$\psi = \psi_0\psi_1 \quad \ldots \quad (4)$$

where ψ_1 will remain finite and differentiable everywhere except in the neighbourhood of the back stagnation point or on the real positive x-axis. Besides, ψ must satisfy the same boundary conditions as ψ_0 does. When such a solution is obtained $\psi_1 = 0$ will give us the boundary of the wake.

Now, if (4) be a solution,

$$\psi = \psi_0(1+\psi_1) \quad \ldots \quad (5)$$

is also a solution and we shall seek the wake solution in the form (5) instead of (4). It is obvious that, with ψ in the form (5), it is necessary that $\psi_1 \to 0$ as $r \to \infty$ in order that the condition of uniform flow at infinity may be satisfied. But the wake boundary will be changed to

$$1+\psi_1 = 0 \quad \ldots \quad (6)$$

3. Solution

With $\psi = \psi_0\psi_1$ substituted in (1) we have

$$r^2\{\psi_{1rr}+2p_0\psi_{1r}\}+(1-\mu^2)\{\psi_{1\mu\mu}+2q_0\psi_{1\mu}\} = 0 \quad \ldots \quad (7)$$

where

$$p_0 = \frac{\partial}{\partial r}\log\psi_0 = \frac{2r^3+a^3}{r(r^3-a^3)} \equiv p_0(r) \quad \ldots \quad (7a)$$

$$q_0 = \frac{\partial}{\partial \mu}\log\psi_0 = \frac{-2\mu}{1-\mu^2} \equiv q_0(\mu) \quad \ldots \quad (7b)$$

Trying separable solutions of (7) in the form

$$\psi_1 = R(r)M(\mu) \quad \ldots \quad (8)$$

we obtain

$$R''+\frac{2(2r^3+a^3)}{r(r^3-a^3)}R'-\frac{KR}{r^2} = 0 \quad \ldots \quad (9)$$

$$M''-\frac{4\mu}{1-\mu^2}M'+\frac{KM}{1-\mu^2} = 0 \quad \ldots \quad (10)$$

where K is the separation constant.

Cutting short all details, it may be stated that the assumption of a series $\sum_0^\infty a_p r^{-p}$ for R leads to the result that the only solution, which converges to zero at $r \to \infty$, is given by

$$R(r) = \frac{Ar^2+Br}{r^3-a^3} \quad \ldots \quad (9a)$$

where A and B are arbitrary constants and, what is more important, the solution (9a) is furnished only by the value

$$K = -2 \quad \ldots \quad (9b)$$

The interesting point is that equation (10) at once leads to the solution

$$M(\mu) = \frac{C\mu+D}{1-\mu^2} \qquad \text{.. .. .. .. (10}a\text{)}$$

where C and D are again arbitrary constants. Thus ultimately we have, in general,

$$\psi_1 = \frac{(Ar^2+Br)\,(C\mu+D)}{(r^3-a^3)\,(1-\mu^2)} \qquad \text{.. .. .. .. (11)}$$

which by (5) leads to

$$\psi = \frac{1}{2}\,Ur^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right)\left\{1+\frac{(Ar^2+Br)(C\mu+D)}{(1-\mu^2)(r^3-a^3)}\right\} \qquad \text{.. .. (12)}$$

One may verify that (12) satisfies the characteristics mentioned in Sec. 2; it will lead to a solution of the problem when the constants are adjusted suitably. These adjustments lead to two different types (mentioned in Sec. 1) as detailed below.

4. *Type* (1). With $A = -B/a$, $C = D$, $BC = a^2\lambda$, etc., we obtain from (12)

$$\psi = \frac{1}{2}\,Ur^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right)\left\{1-\frac{\lambda ar}{(1-\mu)(r^2+ar+a^2)}\right\},\ \mu \neq 1, \qquad \text{.. (12}a\text{)}$$

where λ is an arbitrary numerical constant.

Characteristic features

(i) ψ given by (12a) vanishes over $\mu+1 = 0$ and over $r = a$; and the wake boundary is given by

$$(1-\mu) = \frac{\lambda ar}{r^2+ar+a^2},\ \lambda \geqslant 0 \qquad \text{.. .. .. (12}b\text{)}$$

(ii) If the wake starts from a point on the sphere, as it is naturally expected to do, one must have

$$0 \leqslant \lambda \leqslant 6 \qquad \text{.. .. .. (12}c\text{)}$$

For $\lambda = 0$, corresponding to the primary potential flow, the wake collapses about the positive x-axis; the other extreme value 6 corresponds to the case $\mu = -1$, the separation for which takes place at the front stagnation point.

(iii) The wake is not closed, the circular cross-section of the wake boundary ultimately $\rightarrow \infty$.

(iv) If $\mu = \mu_0$ corresponds to the point of separation (on $r = a$, obviously), we have $\mu_0 = 1-\dfrac{\lambda}{3}$ so that λ may be regarded as the parameter that fixes the point of separation on the sphere.

5. *Type* (2). ***Flow with a discontinuous velocity—layer and refraction of the flow lines.***

Going back to equation (12) let us consider the radial line given by

$$C\mu+D = 0 \qquad \text{.. .. .. .. (13)}$$

or

$$C\cos\alpha+D = 0,\ 0 \leqslant \alpha \leqslant \frac{\pi}{2} \qquad \text{.. .. .. (13}a\text{)}$$

Now let the stream-function ψ be defined by

$$\left.\begin{aligned}\psi = \psi_0 &\equiv \frac{1}{2} Ur^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right), \pi \geqslant \theta \geqslant \alpha \\ = \psi_1 &\equiv \frac{1}{2} U\left[r^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right)+(A+Br)(C\mu+D)\right], \alpha \geqslant \theta \geqslant 0\end{aligned}\right\} \quad .. \quad (14)$$

Thus, we assume the primary flow pattern ψ_0 to remain unchanged up till the radial line, over which it changes continuously into the pattern ψ_1, which holds on the right of the line $\theta = \alpha$. With $\mu_0 = \cos\alpha$ and adjusting the constants, we put ψ_1 in the form [$A = -aB$, $BC = -\lambda a$, $D = -C\mu_0$]

$$\psi_1 = \frac{1}{2} U\left[r^2(1-\mu^2)\left(1-\frac{a^3}{r^3}\right)-\lambda a(r-a)(\mu-\mu_0)\right] \quad .. \quad .. \quad (15)$$

$$= \frac{1}{2} U.\left(1-\frac{a}{r}\right)\left[(1-\mu^2)(a^2+ar+r^2)-\lambda ar(\mu-\mu_0)\right] \quad .. \quad (15a)$$

(so that λ becomes merely a numerical constant).

This ensures that ψ_1 satisfies the boundary condition on $r = a$. Besides, (15*a*) shows that the inner boundary of the flow region for ψ_1 can be taken as

$$(1-\mu^2)(a^2+ar+r^2)-\lambda ar(\mu-\mu_0) = 0,\ 1 > \mu \geqslant \mu_0, \quad .. \quad .. \quad (16)$$

where the constant λ still remains to be specified. Obviously when $\lambda \to 0$, ψ_1 reduces to ψ_0 and ψ gives back the primary potential flow.

Equation (16) shows, again, that for a finite non-zero λ when $r \to \infty$, $\mu \to 1$.

The point of separation $\mu = \mu_1$ is related to λ by the equation

$$3(1-\mu_1^2) = \lambda(\mu_1-\mu_0) \quad .. \quad .. \quad .. \quad .. \quad (17)$$

i.e.

$$\lambda = \frac{3(1-\mu_1^2)}{\mu_1-\mu_0} \quad .. \quad .. \quad .. \quad .. \quad (17a)$$

Equation (17*a*) shows that the point of separation of the wake can never reach the line of discontinuity, for when $\mu_1 \to \mu_0$ $\lambda \to \infty$. Besides, the discriminant of the quadratic in μ_1 being $\lambda^2+12(3+\lambda\mu_0)$ is always positive for all real positive values of λ, whence we obtain that for every positive value of λ, there is a real point of separation, between $\mu = 1$ and $\mu = \mu_0$.

Further analysis of equation (15*a*) shows that the wake boundary leaving the spherical surface extends continuously towards its asymptote.

The discontinuous velocity surface

Using q^0, q^1 for the velocity in the two parts of the motion we have, generally, from (14)

$$q_r^1 = q_r^0 - \frac{1}{2} UC\frac{A+Br}{r^2} \quad .. \quad .. \quad .. \quad .. \quad (18)$$

and

$$q_\theta^1 = q_\theta^0 - \frac{1}{2}\frac{UB}{\bar{\omega}}.(C\mu+D) \quad .. \quad .. \quad .. \quad (19)$$

where

$$\bar{\omega} = r\sin\theta = r\sqrt{1-\mu^2}.$$

It is obvious from (18) that the radial velocities of the two motions tally only at a very large distance. They also tally on the boundary $r = a$.
As $C\mu + D = 0$ everywhere on the radial cone, $q_\theta^1 = q_\theta^0$ everywhere so that the normal velocity is continuous. This leads to a refraction of the stream lines.

The discontinuity in the velocity vanishes on the sphere and at infinity.

6. Conclusion

As the existence of the potential flow outside the wake and the boundary layer is admitted to be a fact the above solution specially the continuous one is seen to be the only separable type admitted by the boundary conditions. The wake boundary having reduced to only a one parameter family of surfaces, one is naturally tempted to trace the resistance curve for variations in the parameter λ, if the same can be calculated from the boundary of the potential flow alone. In the case of a steady potential flow with a finite closed wake the author has demonstrated such a possibility in the paper referred to. We reserve the attempt in the present case for a future communication.

Reference

Ghosh, N. L. (1954). A Theory of Resistance in Potential Flows, Parts I–IV. *Proc. Nat. Inst. Sci. Ind.*, **20**, 74–103.

Issued October 14, 1957.