THERMAL STRESSES IN A SEMI-INFINITE ELASTIC SOLID DUE TO PERIODIC TEMPERATURE DISTRIBUTION OVER A PORTION OF ITS PLANE SURFACE

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INTRODUCTION

In this paper the problem of thermal stresses due to periodic temperature distribution on the surface of a semi-infinite solid has been solved following the general method of the author's previous paper (Sharma, 1956). The particular case in which $T = T_0 \sin pt$ on the surface inside a circular area and zero outside is considered. This problem arises in the case of a periodic supply of heat by a blow jet on a circular area of the plane surface. The results have been given in a closed form.

NOMENCLATURE

The following nomenclature is used in this paper:—

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$ = Cartesian components of stress,

$\sigma_r, \sigma_\theta, \sigma_z, \tau_{r\theta}, \tau_{\theta z}, \tau_{rz}$ = cylindrical components of stress,

u, v, w = Cartesian components of displacement,

u_r, u_θ, w = cylindrical components of displacement,

$\odot = \sigma_x+\sigma_y+\sigma_z = \sigma_r+\sigma_\theta+\sigma_z$,

ν = Poisson's ratio,

E = modulus of elasticity,

$\mu = G$ = Lame's constant,

α = coefficient of linear expansion,

$\epsilon_x, \epsilon_y, \epsilon_z$ = unit elongation in x, y, z-directions,

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ = shearing strain components in rectangular co-ordinates,

∇^2 = Laplacian operator,

k = coefficient of conductivity.

METHOD OF SOLUTION

If T is the temperature at any point in the solid, the stress-strain relations are (Timoshenko and Goodier, 1951)

$$\left.\begin{aligned} \epsilon_x-\alpha T &= \frac{\partial u}{\partial x}-\alpha T = \frac{1}{E}\left[\sigma_x-\nu(\sigma_y+\sigma_z)\right], \\ \epsilon_y-\alpha T &= \frac{\partial v}{\partial y}-\alpha T = \frac{1}{E}\left[\sigma_y-\nu(\sigma_z+\sigma_x)\right], \\ \epsilon_z-\alpha T &= \frac{\partial w}{\partial z}-\alpha T = \frac{1}{E}\left[\sigma_z-\nu(\sigma_x+\sigma_y)\right], \end{aligned}\right\} \quad \cdots \quad \cdots \quad (1)$$

and

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = \frac{2(1+\nu)}{E}\tau_{xy}, \\ \gamma_{yz} &= \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} = \frac{2(1+\nu)}{E}\tau_{yz}, \\ \gamma_{zx} &= \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} = \frac{2(1+\nu)}{E}\tau_{zx}. \end{aligned}\right\} \quad \ldots \quad (2)$$

Also the compatibility equations are

$$\frac{\partial^2 \epsilon_y}{\partial z^2}+\frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \ldots \quad (3)$$

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and

$$2\,\frac{\partial^2 \epsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right], \quad \ldots \quad (4)$$

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Substituting the values of ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{yz}, γ_{zx} in equations (3) and with the help of the equations of equilibrium

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z} = 0 \quad \ldots \quad (5)$$

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we get

$$(1+\nu)\left[\nabla^2 \odot - \nabla^2 \sigma_x - \frac{\partial^2 \odot}{\partial x^2}\right] - \nu\left[\nabla^2 \odot - \frac{\partial^2 \odot}{\partial x^2}\right] + \alpha E\left[\nabla^2 T - \frac{\partial^2 T}{\partial x^2}\right] = 0 \quad (6)$$

and two similar equations. Adding all these equations we have

$$(1-\nu)\nabla^2 \odot + 2\alpha E \nabla^2 T = 0. \quad \ldots \quad (7)$$

In the unsteady state

$$k\nabla^2 T = \frac{\partial T}{\partial t}. \quad \ldots \quad (8)$$

With the help of equations (7) and (8), equations (6) reduce to

$$\nabla^2\left[\sigma_x + \frac{1}{1-\nu}\,\alpha E T\right] + \frac{1}{1+\nu}\,\frac{\partial^2}{\partial x^2}\left[\odot + \alpha E T\right] = 0 \quad \ldots \quad (9)$$

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And equations (4) by virtue of relations (1), (2) and (5) give

$$\nabla^2 \tau_{xz} + \frac{1}{1+\nu}\,\frac{\partial^2}{\partial x \partial z}\left[\odot + \alpha E T\right] = 0 \quad \ldots \quad (10)$$

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Solutions of equations containing τ_{xz}, τ_{yz} and σ_z in (9) and (10) can be put in the form

$$\left.\begin{aligned}\tau_{xz} &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial x}(\odot+\alpha ET)+\psi_1\\ \tau_{yz} &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial y}(\odot+\alpha ET)+\psi_2\\ \sigma_z &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial z}(\odot+\alpha ET)-\frac{1}{1-\nu}\alpha ET+\psi_3\end{aligned}\right\} \quad \ldots \quad (11)$$

where ψ_1, ψ_2, and ψ_3 are some functions of r, z, and t. Operating equations (11) with ∇^2 and comparing with the original equations we get after a little reduction

$$\nabla^2(\psi_1, \psi_2, \psi_3) = -\frac{\alpha E}{2(1-\nu)k}z\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{\partial T}{\partial t} \quad \ldots \quad \ldots \quad (12)$$

Solving these equations

$$(\psi_1, \psi_2, \psi_3) = -\frac{\alpha E}{2(1-\nu)}\left[z\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)T -2k\int\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}\frac{\partial T}{\partial z}dt\right]+(\phi_1, \phi_2, \phi_3) \quad \ldots \quad \ldots \quad (13)$$

where ϕ_1, ϕ_2, ϕ_3 are as yet undetermined harmonic functions. Substituting in equations (11):

$$\left.\begin{aligned}\tau_{xz} &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial x}(\odot+\alpha ET)-\frac{\alpha E}{2(1-\nu)}\left[z\frac{\partial T}{\partial x}-2k\int\frac{\partial^2 T}{\partial x\partial z}dt\right]+\phi_1\\ \tau_{yz} &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial y}(\odot+\alpha ET)-\frac{\alpha E}{2(1-\nu)}\left[z\frac{\partial T}{\partial y}-2k\int\frac{\partial^2 T}{\partial y\partial z}dt\right]+\phi_2\\ \sigma_z &= -\frac{1}{2(1+\nu)}z\frac{\partial}{\partial z}(\odot+\alpha ET)-\frac{\alpha E}{2(1-\nu)}\left[z\frac{\partial T}{\partial z}-2k\int\frac{\partial^2 T}{\partial z^2}dt\right]\\ &\quad -\frac{1}{1-\nu}\alpha ET+\phi_3\end{aligned}\right\} \quad (14)$$

Substituting these values in the equation of equilibrium:

$$\frac{\partial}{\partial z}[(1-\nu)\odot+2\alpha ET] = 2(1-\nu^2)\left[\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial z}\right] \quad \ldots \quad \ldots \quad (15)$$

whence $\odot$ is obtained by integration and the constant of integration is determined by the conditions of the problem. Thus we know completely the stress components τ_{xz}, τ_{yz}, σ_z. Substituting these values in equations (1) and (2) on integration we find u, v, and w which in turn determine other stress components.

Solution for a Periodic Surface Temperature.—A particular solution of equation (8) is given by

$$T = \int_0^\infty A\sin(pt+\gamma z)e^{-\beta z}J_0(mr)dm \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

where

$$\left.\begin{aligned} \beta^2 &= \frac{1}{2}\left[\sqrt{m^4+\frac{p^2}{k^2}}+m^2\right] \\ \gamma^2 &= \frac{1}{2}\left[\sqrt{m^4+\frac{p^2}{k^2}}-m^2\right] \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

and A is some function of m to be determined subsequently. Substituting the value of T in equations (14) and imposing the condition

$$\tau_{xz} = \tau_{yz} = 0 \qquad \text{when } z = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

we get

$$\left.\begin{aligned} [\phi_1]_{z=0} &= \int_0^\infty C\frac{\partial}{\partial x}J_0(mr)dm, \\ [\phi_2]_{z=0} &= \int_0^\infty C\frac{\partial}{\partial y}J_0(mr)dm, \\ [\phi_3]_{z=0} &= \int_0^\infty mDJ_0(mr)dm, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where

$$\left.\begin{aligned} C &= -\frac{\alpha EAk}{(1-\nu)p}[\gamma \sin pt+\beta \cos pt] \\ mD &= \frac{\alpha EA}{(1-\nu)p}[(2\beta\gamma k+p)\sin pt+k(\beta^2-\gamma^2)\cos pt] \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (20)$$

But ϕ_1, ϕ_2, ϕ_3 are harmonic functions, so we take

$$\left.\begin{aligned} \phi_1 &= \int_0^\infty Ce^{-mz}\frac{\partial}{\partial x}J_0(mr)dm, \\ \phi_2 &= \int_0^\infty Ce^{-mz}\frac{\partial}{\partial y}J_0(mr)dm, \\ \phi_3 &= \int_0^\infty mDe^{-mz}J_0(mr)dm. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

Substituting the values of ϕ_1, ϕ_2, ϕ_3 in (15) and integrating

$$(1-\nu)\odot+2\alpha ET = 2(1-\nu^2)\int_0^\infty m(C+D)e^{-mz}J_0(mr)dm \qquad \ldots \quad (22)$$

Thus using equations (1) and (2) we get:

$$\left.\begin{aligned} u_r &= -\frac{1+\nu}{E}\int_0^\infty \Big[\{(C+D)(mz-1+2\nu)-C\}e^{-mz} \\ &\qquad -\frac{\alpha EAkm}{(1-\nu)p}\cos(pt+\gamma z)e^{-\beta z}\Big]J_1(mr)dm, \\ w &= -\frac{1+\nu}{E}\int_0^\infty \Big[\{(C+D)(mz+1-2\nu)+D\}e^{-mz} \\ &\qquad -\frac{\alpha EAk}{(1-\nu)p}\{\gamma\sin(pt+\gamma z)+\beta\cos(pt+\gamma z)\}e^{-\beta z}\Big]J_0(mr)dm \end{aligned}\right\} \quad \ldots \quad (23)$$

Also we know

$$\left.\begin{aligned} \sigma_r+\sigma_\theta &= \odot-\sigma_z \\ \text{and} \qquad \sigma_r-\sigma_\theta &= \frac{E}{1+\nu}\left[\frac{\partial u_r}{\partial r}-\frac{u_r}{r}\right] \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

Hence from equations (14), (23) and (24):

$$\left.\begin{aligned} \sigma_r+\sigma_\theta &= \int_0^\infty \Big[\{m(C+D)(2+2\nu-mz)-mD\}e^{-mz} \\ &\quad -\frac{\alpha EAk}{(1-\nu)p}\left\{(\gamma^2-\beta^2)\cos(pt+\gamma z)+\left(\frac{p}{k}-2\beta\gamma\right)\sin(pt+\gamma z)\right\}e^{-\beta z}\Big]J_0(mr)dm, \\ \sigma_r-\sigma_\theta &= \int_0^\infty \Big[\{(C+D)(mz-1+2\nu)-C\}e^{-mz} \\ &\quad -\frac{\alpha EAkm}{(1-\nu)p}\cos(pt+\gamma z)e^{-\beta z}\Big]mJ_2(mr)dm. \end{aligned}\right\} \quad (25)$$

Now, when $z=0$, if

$$\left.\begin{aligned} T &= T_0\sin pt \quad r \leqslant a \\ &= 0 \qquad\quad\;\; r > a \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

by Fourier-Bessel representation we can put

$$T = T_0 a\int_0^\infty \sin pt\, J_1(ma)J_0(mr)dm \quad \ldots \quad \ldots \quad (27)$$

Comparing equation (27) with equation (16) we have

$$A = T_0 a J_1(ma) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

A being determined, we can find the stresses and displacements from equations established. Thus the problem is completely solved.

In the case of an infinite disk, we can introduce in equations (21) the harmonic function $\sinh mz\, J_0(mr)$ which vanishes on the plane $z=0$. In the axisymmetric case we can thus introduce two more arbitrary constants which can be determined from the conditions at the base involving displacements and/or stresses.

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Abstract

A new method of solving three-dimensional problems of thermal stresses in an isotropic elastic solid has been described in this paper, and the method has been applied to the solution of the problem of a semi-infinite elastic solid having periodic supply of heat on a portion of its plane surface.

References

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