ON THE REFLECTION AND REFRACTION OF MAGNETO-HYDRODYNAMIC WAVES

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I. INTRODUCTION

The reflection and refraction of magneto-hydrodynamic waves at the interface of discontinuity of two fluids of different densities has been discussed briefly by Walén (1944, 1946), Alfvén (1950) and Lindquist (1952). Recently Ferraro (1954) and Roberts (1955) have discussed this problem, in considerable details, by taking the same uniform permanent magnetic field in the two media.

In this paper we discuss Ferraro's case of two infinitely extended media but start by assuming different permanent magnetic fields in the two media. It is shown that reflection and refraction are possible, provided the discontinuity in the permanent magnetic fields is perpendicular to the plane of incidence, and provided, as in Ferraro's case, the incident waves are polarized perpendicular to the plane of incidence. The amplitude relations come out to be the same as in Ferraro's case, but the laws of reflection and refraction are slightly modified. The later reduce to Ferraro's result in his special case. It is also found that the interface remains undisturbed by the magneto-hydrodynamic waves.

II. FUNDAMENTAL EQUATIONS OF MAGNETO-HYDRODYNAMICS

The well-known equations governing magneto-hydrodynamic phenomena in an infinitely conducting, inviscid, and incompressible fluid of density ρ and permeability unity reduce in the presence of an external homogeneous field $\vec{H}_0$ to the form (Walén, 1944)

$$\frac{\partial \vec{h}}{\partial t} - \left(\vec{H}_0 \cdot \vec{\nabla}\right)\vec{v} = \left(\vec{h} \cdot \vec{\nabla}\right)\vec{v} - \left(\vec{v} \cdot \vec{\nabla}\right)\vec{h} \qquad \ldots \quad \ldots \quad (1a)$$

$$\frac{\partial \vec{v}}{\partial t} - \frac{1}{4\pi\rho}\left(\vec{H}_0 \cdot \vec{\nabla}\right)\vec{h} = \vec{v}\times\left(\vec{\nabla}\times\vec{v}\right) - \frac{1}{4\pi\rho}\vec{h}\times\left(\vec{\nabla}\times\vec{h}\right)$$
$$- \vec{\nabla} \cdot \left\{\frac{1}{2}\vec{v}^2 - \frac{\vec{h}^2}{4\pi\rho}\right\} - \vec{\nabla}\,\omega. \qquad \ldots \quad \ldots \quad \ldots \quad (1b)$$

Here

$$\omega = \frac{1}{\rho}\left(p + \frac{1}{8\pi}\vec{H}^2\right), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$\vec{h}$ denotes the induced magnetic field, $\vec{v}$ the fluid velocity and p the hydrostatic pressure at any point in the fluid, while $\vec{H} = \vec{H}_0 + \vec{h}$ is the total magnetic field.

Gaussian units are used for electro-magnetic quantities. The equations (1a) and (1b) simplify to

$$\frac{\partial \vec{h}}{\partial t} = \left(\vec{H}_0 \cdot \vec{\nabla}\right) \vec{v}, \qquad (3a)$$

and

$$\frac{\partial \vec{v}}{\partial t} = \frac{1}{4\pi\rho} \left(\vec{H}_0 \cdot \vec{\nabla}\right) \vec{h}, \qquad (3b)$$

respectively, provided Walén's conditions

$$\vec{v} = \mp \frac{\vec{h}}{(4\pi\rho)^{\frac{1}{2}}}, \qquad (4)$$

and

$$\omega = \text{constant} \qquad (5)$$

are satisfied. $\vec{h}$ then satisfies the equation

$$\frac{\partial^2 \vec{h}}{\partial t^2} = \frac{1}{4\pi\rho} \left(\vec{H}_0 \cdot \vec{\nabla}\right)^2 \vec{h}. \qquad (6)$$

The same equation is also satisfied by $\vec{v}$.

For the case of the plane harmonic waves condition (4) is identically satisfied. Also, as shown by Walén (1944), equation (5) holds for an infinitely extended medium, but as pointed out by Roberts (1955), it does not hold for semi-infinite regions. Walén's conditions will be assumed here.

III. Reflection and refraction of plane magneto-hydrodynamic waves

An elementary solution of the form

$$\vec{h} = \vec{A} + \vec{h}_0 \exp \left\{ jw \left(t - \frac{\vec{n} \cdot \vec{r}}{W} \right) \right\}, \qquad (7)$$

where $\vec{A}$ and $\vec{h}_0$ are constant vectors, $\vec{n}$ a unit vector and $\vec{r}$ the position vector of a field point, satisfies equation (6) if

$$W = \pm V \left(\vec{n} \cdot \vec{1}_\xi \right), \qquad (8)$$

where

$$V = \frac{H_0}{(4\pi\rho)^{\frac{1}{2}}} \qquad (9)$$

and $\vec{1}_\xi$ denotes a unit vector in the direction of $\vec{H}_0$. This solution represents a plane harmonic magneto-hydrodynamic wave travelling with a phase velocity $|W|$, in the positive or negative direction according as the upper or the lower signs are taken in equations (4) and (8). Also $\vec{n}$ is the normal to the wavefront. We shall further denote by $\vec{1}_x, \vec{1}_y, \vec{1}_z$ unit normals in the directions of positive x, y, and z axes respectively.

The reflection and refraction of such waves travelling from a medium A $(z < 0)$ to a medium B $(z > 0)$ incident, say in the YZ plane, on the plane surface $z = 0$

separating the two media A and B which consist of infinitely extended and infinitely conducting, incompressible and inviscid fluids of different densities, say ρ_A and ρ_B respectively, has been considered by Ferraro (1954), in the presence of a single permanent uniform magnetic field $\vec{H}_0$. We shall here investigate the more general case, corresponding to the presence of unequal permanent fields $\vec{H}_{0A}$ and $\vec{H}_{0B}$ in the media A and B respectively. Let their orientations be given respectively by the unit vectors $\vec{1}_{\xi A}$ and $\vec{1}_{\xi B}$ where

$$\vec{1}_{\xi k} = \left(\sin\alpha_k \sin\beta_k,\ \cos\alpha_k \sin\beta_k,\ \cos\beta_k\right),\ (k = A, B) \quad \ldots \quad (10)$$

As pointed out by Roberts (1955), no surface currents can be generated in these infinitely conducting fluids for the case of harmonic waves. Hence we have the usual boundary conditions, viz. the continuity across the boundary of the magnetic field $\vec{H}$, the electric field $\vec{E}$ and v_n, the normal component of the particle velocity. Recalling that in the case of infinite conductivity $\vec{E} = -\frac{1}{c}\left(\vec{v}\times\vec{H}\right)$, the boundary conditions are, in the obvious notations

$$\vec{H}_A = \vec{H}_B \quad \ldots \quad (11)$$

$$\vec{v}_A \times \vec{H}_A = \vec{v}_B \times \vec{H}_B \quad \ldots \quad (12)$$

$$\vec{1}_Z \cdot \vec{v}_A = \vec{1}_Z \cdot \vec{v}_B = v_n \text{ (say)} \quad \ldots \quad (13)$$

at $z = 0$. Since

$$\operatorname{div}\vec{H} \equiv \operatorname{div}(\vec{H}_0 + \vec{h}) = 0,$$

we have the further conditions

$$\operatorname{div}\vec{h}_A = 0;\ \operatorname{div}\vec{h}_B = 0. \quad \ldots \quad (14)$$

Let the solutions, of the same form as in equation (7), for the incident, the reflected and the refracted waves be represented by

$$\vec{h}_k = \vec{A}_k + \vec{h}_{0k}\exp\Omega_k, \quad \ldots \quad (15)$$

according as we replace the subscript k by the subscript i, r, or R respectively. Here

$$\left.\begin{aligned} \Omega_k &= jw_k\left(t - \frac{\vec{n}_k\cdot\vec{r}}{W_k}\right), \\ W_k &= \pm V\left(\vec{n}_k\cdot\vec{1}_\xi\right), \end{aligned}\right\} \quad \ldots \quad (16)$$

and

$$\left.\begin{aligned} V &= \frac{H_{0A}}{(4\pi\rho_A)^{\frac{1}{2}}}(\equiv V_A) \text{ or } \frac{H_{0B}}{(4\pi\rho_B)^{\frac{1}{2}}}(\equiv V_B), \\ 1_\xi &= 1_{\xi A} \text{ or } 1_{\xi B}. \end{aligned}\right\} \quad \ldots \quad (17)$$

In equation (17) the subscript appropriate to the medium of propagation is to be used, and in equation (16) the sign for the reflected waves is to be taken opposite to that for the incident and the refracted waves.

Without loss of generality, we can put $A_i = 0$. The wave in the media A and B are then respectively

$$\vec{h}_A = \vec{A}_r + \vec{h}_{0i} \exp \Omega_i + \vec{h}_{0r} \exp \Omega_r \, .$$

and

$$\vec{h}_B = \vec{A}_R + \vec{h}_{0R} \exp \Omega_R \, . \qquad (18)$$

To satisfy the boundary condition (11), one must have

$$\Delta \vec{H}_0 \equiv \vec{H}_{0B} - \vec{H}_{0A} = \vec{A}_r - \vec{A}_R \, , \qquad (19)$$

$$\vec{h}_{0R} = \vec{h}_{0i} + \vec{h}_{0r} \, , \qquad (20)$$

and the equality of the exponential parts at the boundary $\vec{1}_z \, . \, \vec{r} = 0$.
Using the identity

$$\vec{r} \equiv \left(\vec{1}_z \, . \, \vec{r}\right)\vec{1}_z - \vec{1}_z \times \left(\vec{1}_z \times \vec{r}\right)$$

the latter gives, as usual, the relations

$$w_i = w_r = w_R \, , \qquad (21)$$

and

$$\frac{\vec{n}_i \times \vec{1}_z}{W_i} = \frac{\vec{n}_r \times \vec{1}_z}{W_r} = \frac{\vec{n}_R \times \vec{1}_z}{W_R} \, . \qquad (22)$$

The relations (22) give the laws of reflection and refraction and show the coplanarity of $\vec{1}_z$, $\vec{n}_i$, $\vec{n}_r$, and $\vec{n}_R$. Denoting by i, r, and R respectively the angles of incidence, reflection and refraction, measured positively as shown in Fig. 1, we have explicitly

$$\left.\begin{aligned} W_i &= V_A(\sin i \cos \alpha_A \sin \beta_A + \cos i \cos \beta_A), \\ W_r &= -V_A(\sin r \cos \alpha_A \sin \beta_A - \cos r \cos \beta_A), \\ W_R &= V_B(\sin R \cos \alpha_B \sin \beta_B + \cos R \cos \beta_B). \end{aligned}\right\} \qquad (23)$$

Substitution of these expressions in (22) now gives the laws of reflection and refraction, respectively, in the form

$$\cot r = \cot i + 2 \cos \alpha_A \tan \beta_A \, , \qquad (24)$$

and

$$\cot R = \frac{V_A \cos \beta_A}{V_B \cos \beta_B} \, . \cot i + \left(\frac{V_A \cos \alpha_A \sin \beta_A}{V_B \cos \alpha_B \sin \beta_B} - 1\right) \cos \alpha_B \tan \beta_B \, . \qquad (25)$$

These reduce to the forms obtained by Ferraro (1954) and Roberts (1955) for the case $\vec{H}_{0A} = \vec{H}_{0B}$ if one puts $\alpha_A = \alpha_B = \alpha$; $\beta_A = \beta_B = \beta$.

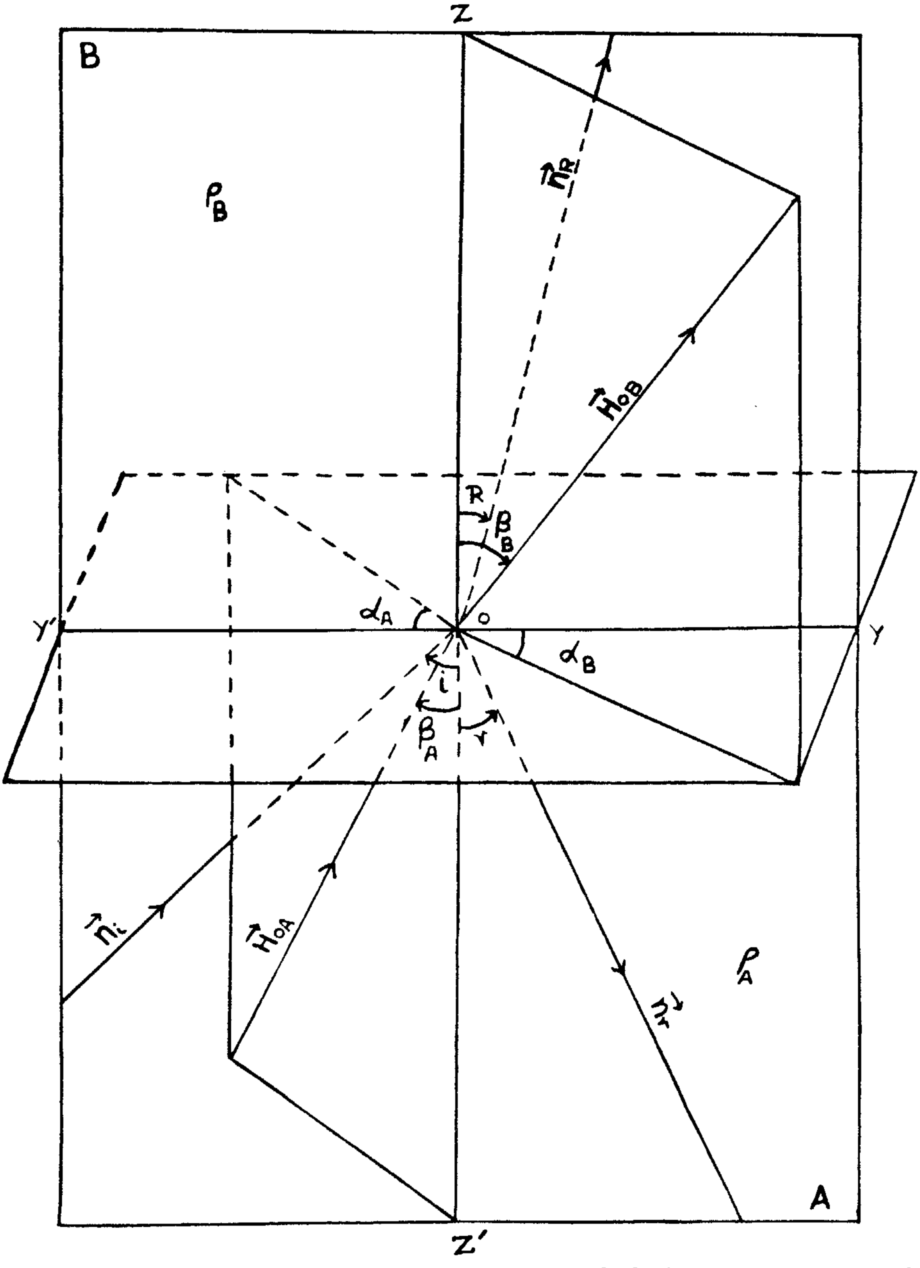

FIG. 1. Illustrating the reflection and refraction of magneto-hydrodynamic waves at the plane surface YY' separating two infinitely conducting and infinitely extended fluid media A and B having different permanent magnetic fields $\vec{H}_{0A}$ and $\vec{H}_{0B}$ respectively. AB is the plane of incidence, i, r and R are respectively the angles of incidence, reflection and refraction. The planes through the z-axis containing the magnetic field vectors $\vec{H}_{0A}$ and $\vec{H}_{0B}$ are inclined at angles α_A and α_B respectively to the plane of incidence, while the angles β_A and β_B give the respective orientations of these vectors with respect to the z-axis.

To derive the amplitude relations we now make use of equations (12) and (13). The latter gives, taking account of the sign in equation (4),

$$\sqrt{\frac{\rho_B}{\rho_A}}\,\vec{1}_z\,.\left\{\vec{h}_{0i}\exp\Omega-\vec{h}_{0r}\exp\Omega-\vec{A}_r\right\}$$
$$=\vec{1}_z\,.\left\{\vec{h}_{0R}\exp\Omega+\vec{A}_R\right\},$$

where Ω denotes the common value of Ω_i, Ω_r and Ω_R at the surface of separation. If this relation is to be satisfied for all times t and for all $\vec{r}$ on the plane $\vec{1}_z\,.\,\vec{r}=0$, one must have

$$-\sqrt{\frac{\rho_B}{\rho_A}}\,\vec{1}_z\,.\,\vec{A}_r=\vec{1}_z\,.\,\vec{A}_R, \qquad \text{(26)}$$

and

$$\sqrt{\frac{\rho_B}{\rho_A}}\left(\vec{1}_z\,.\,\vec{h}_{0i}-\vec{1}_z\,.\,\vec{h}_{0r}\right)=\vec{1}_z\,.\,\vec{h}_{0R} \qquad \text{(27)}$$

Combining the equations (26) and (27) with those resulting from equations (19) and (20) after scalar multiplication of both sides by $\vec{1}_z$ one obtains

$$\left.\begin{aligned}\vec{1}_z\,.\,\vec{A}_r&=\frac{1}{\sqrt{\rho}+1}\,\vec{1}_z\,.\,\Delta\vec{H}_0,\\ \vec{1}_z\,.\,\vec{A}_R&=-\frac{\sqrt{\rho}}{\sqrt{\rho}+1}\,\vec{1}_z\,.\,\Delta\vec{H}_0,\end{aligned}\right\} \qquad \text{(28)}$$

and

$$\left.\begin{aligned}\vec{1}_z\,.\,\vec{h}_{0r}&=\frac{\sqrt{\rho}-1}{\sqrt{\rho}+1}\,\vec{1}_z\,.\,\vec{h}_{0i},\\ \vec{1}_z\,.\,\vec{h}_{0R}&=\frac{2\sqrt{\rho}}{\sqrt{\rho}+1}\,\vec{1}_z\,.\,\vec{h}_{0i},\end{aligned}\right\} \qquad \text{(29)}$$

where

$$\rho=\sqrt{\frac{\rho_B}{\rho_A}}. \qquad \text{(30)}$$

From equation (12) one now obtains

$$\left(\vec{v}_A\times\vec{H}_{0A}\right)+\left(\vec{v}_i+\vec{v}_r\right)\times\left(\vec{h}_i+\vec{h}_r\right)=\left(\vec{v}_B\times\vec{H}_{0B}\right)+\left(\vec{v}_R\times\vec{h}_R\right),$$

or, taking account of equation (4),

$$\left(\vec{v}_A\times\vec{H}_{0A}\right)-\frac{2}{(4\pi\rho_A)^{\frac{1}{2}}}\,\vec{h}_i\times\vec{h}_r=\vec{v}_B\times\vec{H}_{0B}\,. \qquad \text{(31)}$$

Equating the non-exponential terms, the coefficient of exp (Ω) and the coefficient of exp (2Ω) on both sides we obtain

$$\vec{A}_r\times\vec{H}_{0A}=-\frac{1}{\sqrt{\rho}}\,\vec{A}_R\times\vec{H}_{0B}, \qquad \text{(32)}$$

$$\vec{h}_{0i}\times\vec{H}_{0A}-\vec{h}_{0r}\times\vec{H}_{0A}-2\vec{h}_{0i}\times\vec{A}_r=\frac{1}{\sqrt{\rho}}\,\vec{h}_{0R}\times\vec{H}_{0B}, \qquad \text{(33)}$$

and

$$\vec{h}_{0i} \times \vec{h}_{0r} = 0. \qquad \ldots \quad (34)$$

Vector multiplication of both sides of equation (31) by $\vec{1}_z$ gives

$$\vec{v}_B\left(\vec{1}_z \cdot \vec{H}_{0B}\right) - \vec{v}_A\left(\vec{1}_z \cdot \vec{H}_{0A}\right) = \Delta\, \vec{H}_0 v_n - \frac{2}{(4\pi\rho_A)^{\frac{1}{2}}}\, \vec{1}_z \times \left(\vec{h}_i \times \vec{h}_r\right). \qquad \ldots \quad (35)$$

Now

$$v_n = \vec{v}_A \cdot \vec{1}_z = \frac{1}{(4\pi\rho_A)^{\frac{1}{2}}}\left\{-\vec{h}_{0i} \exp \Omega_i + \vec{h}_{0r} \exp \Omega_r + \vec{A}_r\right\} . \vec{1}_z .$$

Using equations (28) and (29) this gives at the boundary

$$v_n = -\frac{1}{(4\pi\rho_A)^{\frac{1}{2}}}\left[\frac{2}{\sqrt{\rho}+1}\left(\vec{h}_{0i} \cdot \vec{1}_z\right) \exp \Omega_i - \frac{1}{\sqrt{\rho}+1}\left(\vec{1}_z \cdot \Delta\, \vec{H}_0\right)\right]. \quad (36)$$

Further, equations (32) and (33) on vector multiplication by $\vec{1}_z$ and on simplification with the help of equations (28) and (29) reduce to

$$\frac{\vec{1}_z \cdot \vec{H}_{0B}}{\sqrt{\rho}}\, \vec{A}_R + \left(\vec{1}_z \cdot \vec{H}_{0A}\right)\vec{A}_r = -\frac{\vec{1}_z \cdot \Delta\, \vec{H}_0}{\sqrt{\rho}+1}\, \Delta\, \vec{H}_0, \qquad \ldots \quad (37)$$

$$\frac{\vec{1}_z \cdot \vec{H}_{0B}}{\sqrt{\rho}}\, \vec{h}_{0R} + \left(\vec{1}_z \cdot \vec{H}_{0A}\right)\vec{h}_{0r} = \frac{2\left(\vec{h}_{0i} \cdot \vec{1}_z\right)}{\sqrt{\rho}+1}\, \Delta \vec{H}_0 + 2\vec{1}_z \times \left(\vec{h}_{0i} \times \vec{A}_r\right) + \left(\vec{1}_z \cdot \vec{H}_{0A}\right)\vec{h}_{0i} . \quad (38)$$

Equations (19) and (37) give

$$\left.\begin{aligned} A_r &= \frac{\Delta\, \vec{H}_0}{\sqrt{\rho}+1}\, \frac{(\sqrt{\rho}+1)\left(\vec{1}_z \cdot \vec{H}_{0B}\right) - \sqrt{\rho}\left(\vec{1}_z \cdot \Delta\, \vec{H}_0\right)}{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)}, \\ A_R &= -\frac{\sqrt{\rho}\, \Delta\, \vec{H}_0}{\sqrt{\rho}+1}\, \frac{(\sqrt{\rho}+1)\left(\vec{1}_z \cdot \vec{H}_{0A}\right) + \left(\vec{1}_z \cdot \Delta\, \vec{H}_0\right)}{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)} . \end{aligned}\right\} \quad \ldots \quad (39)$$

Similarly combining equations (20) and (38) we obtain

$$\left.\begin{aligned} \vec{h}_{0r} &= \frac{(\sqrt{\rho}+1)\left\{\sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right) - \vec{1}_z \cdot \vec{H}_{0B}\right\}\vec{h}_{0i} + 2\sqrt{\rho}\left(\vec{h}_{0i} \cdot \vec{1}_z\right)\Delta \vec{H}_0}{(\sqrt{\rho}+1)\left\{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)\right\}} \\ &\quad + \frac{2\sqrt{\rho}\left\{(\sqrt{\rho}+1)\left(\vec{1}_z \cdot \vec{H}_{0B}\right) - \sqrt{\rho}\left(\vec{1}_z \cdot \Delta \vec{H}_0\right)\right\}\left\{\vec{1}_z \times \left(\vec{h}_{0i} \times \Delta \vec{H}_0\right)\right\}}{\left\{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)\right\}^2} . \\ \vec{h}_{0R} &= \frac{2\sqrt{\rho}\left[(\sqrt{\rho}+1)\left(\vec{1}_z \cdot \vec{H}_{0A}\right)\vec{h}_{0i} + \left(\vec{1}_z \cdot \vec{h}_{0i}\right)\Delta \vec{H}_0\right]}{(\sqrt{\rho}+1)\left\{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)\right\}} \\ &\quad + \frac{2\sqrt{\rho}\left\{(\sqrt{\rho}+1)\left(\vec{1}_z \cdot \vec{H}_{0B}\right) - \sqrt{\rho}\left(\vec{1}_z \cdot \Delta \vec{H}_0\right)\right\}\left\{\vec{1}_z \times \left(\vec{h}_{0i} \times \Delta \vec{H}_0\right)\right\}}{\left\{\left(\vec{1}_z \cdot \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \cdot \vec{H}_{0A}\right)\right\}^2} . \end{aligned}\right\} \quad \ldots \quad (40)$$

From equations (39) and (40) we observe that condition (34) is satisfied in general only if

$$\vec{h}_{0i} \times \Delta \vec{H}_0 = 0, \qquad \ldots \quad (41)$$

and we can write

$$\left.\begin{aligned} \vec{h}_{0r} &= \frac{(\sqrt{\rho}+1)\left\{\sqrt{\rho}\left(\vec{1}_z \,.\, \vec{H}_{0A}\right) - \vec{1}_z \,.\, \vec{H}_{0B}\right\}\vec{h}_{0i} + 2\sqrt{\rho}\left(\vec{h}_{0i} \,.\, \vec{1}_z\right)\Delta\vec{H}_0}{(\sqrt{\rho}+1)\left\{\left(\vec{1}_z \,.\, \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \,.\, \vec{H}_{0A}\right)\right\}}, \\ \vec{h}_{0R} &= \frac{2\sqrt{\rho}\left[(\sqrt{\rho}+1)\left(\vec{1}_z \,.\, \vec{H}_{0A}\right)\vec{h}_{0i} + \left(\vec{1}_z \,.\, \vec{h}_{0i}\right)\Delta\vec{H}_0\right]}{(\sqrt{\rho}+1)\left\{\left(\vec{1}_z \,.\, \vec{H}_{0B}\right) + \sqrt{\rho}\left(\vec{1}_z \,.\, \vec{H}_{0A}\right)\right\}} \end{aligned}\right\} \quad \ldots \quad (42)$$

We shall now consider the restrictions imposed by equations (14). These give the relations

$$\frac{\vec{h}_{0i} \,.\, \vec{n}_i}{W_i} + \frac{\vec{h}_{0r} \,.\, \vec{n}_r}{W_r} = 0,$$

$$\frac{\vec{h}_{0R} \,.\, \vec{n}_R}{W_R} = 0,$$

or

$$\left.\begin{aligned} \frac{\left(\vec{1}_y \,.\, \vec{h}_{0i}\right)\sin i + \left(\vec{1}_z \,.\, \vec{h}_{0i}\right)\cos i}{W_i} + \frac{\left(\vec{1}_y \,.\, \vec{h}_{0r}\right)\sin r - \left(\vec{1}_z \,.\, \vec{h}_{0r}\right)\cos r}{W_r} &= 0, \\ \frac{\left(\vec{1}_y \,.\, \vec{h}_{0R}\right)\sin r + \left(\vec{1}_z \,.\, \vec{h}_{0R}\right)\cos R}{W_R} &= 0. \end{aligned}\right\} \quad (43)$$

Substituting for $\vec{h}_{0r}$ and $\vec{h}_{0R}$ from equation (42) the equations (43) reduce to a pair of homogeneous equations in $\left(\vec{1}_y \,.\, \vec{h}_{0i}\right)$ and $\left(\vec{1}_z \,.\, \vec{h}_{0i}\right)$ and hence, in general, lead to the conditions

$$\vec{1}_y \,.\, \vec{h}_{0i} = 0; \quad \vec{1}_z \,.\, \vec{h}_{0i} = 0. \qquad \ldots \quad (44)$$

Hence we must have, in general, the incident wave vector $\vec{h}_{0i}$ perpendicular to the plane of incidence in order to have the possibility of reflection and refraction. This restriction on the polarization of incident wave is the same as obtained by Ferraro (1954). Equation (41) now gives

$$\vec{1}_y \,.\, \Delta\vec{H}_0 = 0; \quad \vec{1}_z \,.\, \Delta\vec{H}_0 = 0, \qquad \ldots \quad (45)$$

that is the vector $\Delta\vec{H}_0$ should also be perpendicular to the plane of incidence. The normal component of the permanent magnetic field must be continuous, in conformity with general boundary conditions. As a consequence of the conditions (44) and (45) we obtain from (36)

$$v_n = 0, \qquad \ldots \quad (46)$$

showing that the interface $z = 0$ remains undisturbed by the magneto-hydrodynamic waves. Also since $\vec{h}_{0i} \times \vec{A}_r = 0$, $\vec{h}_i \times \vec{h}_r = 0$, the equation (32) simplifies to

$$\vec{v}_B\left(\vec{1}_z \cdot \vec{H}_{0B}\right) - \vec{v}_A\left(\vec{1}_z \cdot \vec{H}_{0A}\right) = 0,$$

giving for

$$\vec{1}_z \cdot \vec{H}_{0B} = \vec{1}_z \cdot \vec{H}_{0A} \neq 0,$$

$$\vec{v}_A = \vec{v}_B . \qquad \qquad (47)$$

These results obtained by Ferraro (1954) are thus found to be true also in a more general case.

Equations (37) and (40) further simplify with the help of equations (42) and (43) and we get

$$\left.\begin{aligned} \vec{A}_r &= \frac{1}{\sqrt{\rho}+1} \, \Delta \vec{H}_0 , \\ \vec{A}_R &= -\frac{\sqrt{\rho}}{\sqrt{\rho}+1} \, \Delta \vec{H}_0 , \end{aligned}\right\} \qquad (48)$$

and

$$\left.\begin{aligned} \vec{h}_{0r} &= \frac{\sqrt{\rho}-1}{\sqrt{\rho}+1} \vec{h}_{0i} , \\ \vec{h}_{0R} &= \frac{2\sqrt{\rho}}{\sqrt{\rho}+1} \vec{h}_{0i} . \end{aligned}\right\} \qquad (49)$$

The latter result is the same as obtained by Ferraro (1954).

The laws of reflection and refraction also get simplified with the help of equations (45) giving

$$\left.\begin{aligned} \cot r &= \cot i + 2 \cos \alpha_A \tan \beta_A , \\ \cot R &= \sqrt{\rho} \cot i + (\sqrt{\rho}-1) \cos \alpha_B \tan \beta_B . \end{aligned}\right\} \qquad (50)$$

These relations show that the laws of reflection and refraction, apart from density ratio and angle of incidence, will be dependent on the orientation of the magnetic field of the respective media in which the waves are being propagated. Further, there is no possibility of total internal reflection because $\cot R$ and hence R is always real. When $\cot R$ becomes negative R is to be measured in the opposite sense.

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Summary

The laws of reflection and refraction of magneto-hydrodynamic waves from the surface of discontinuity of two infinitely extended and infinitely conducting fluid media of different densities and having different homogeneous permanent magnetic fields have been derived. The laws are quite simple and depend on the orientations of the magnetic field of the respective media. It is also shown that reflection and refraction are possible only provided the incident wave is polarized perpendicular to the plane of incidence and the discontinuity of the magnetic field is perpendicular to the plane of incidence.

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