A PARTICULAR SOLUTION OF THE FIELD EQUATIONS IN EINSTEIN'S GENERALIZED THEORY OF GRAVITATION

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1. Introduction

Sometime back we (V. V. Narlikar and Ramji Tiwari, 1949*a*) considered a field which up to the first order of approximations, in the absence of a point mass, turns out to be a field of uniform monochromatic radiation given by the electric and magnetic force vectors,

$$\hat{E} = (0, -\phi, 0), \quad \hat{H} = (0, 0, -\phi), \qquad \ldots \qquad \ldots \qquad (1.1)$$

expressed in a rectangular Cartesian frame. Here

$$\phi = A \cos 2\pi (x-t)/\lambda, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1.2)$$

A and λ being constants. The same field in the absence of the electromagnetic field reduces to the field of a point mass m as given by the Riemannian metric,

$$ds^2 = -(1+m/2r)^4(dx^2+dy^2+dz^2)+(1-m/2r)^2(1+m/2r)^{-2}dt^2, \quad \ldots \qquad (1.3)$$

in the usual notation. The field equations were calculated on the basis of Einstein's theory of 1948. Up to the second order of approximations the gravitational terms of interaction appeared in the electromagnetic equations, but no such electromagnetic terms were found in the gravitational equations. So the gravitational equations had to be calculated further up to the third order of approximations. It was not possible then to obtain a solution of the field equations although their consistency and integrability were assured by the fact that the four identities and the strength conditions were satisfied by them. The present paper deals with a particular solution of the gravitational equations. The terms of the field equations have now been calculated according to Einstein's 1953 theory. In this theory the total field is non-symmetric, the antisymmetric parts of which besides the symmetric ones are real. The technique adopted (now) to calculate the field equations is essentially the one used in the previous paper referred to above. The field equations obtained here differ from the field equations of the previous paper only in so far as the sign of a few terms is concerned.

2. Outline of the Method

We take for the non-symmetric tensor

$$g_{ik} = a_{ik} + b_{ik} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.1)$$

where

$$a_{ii} = -\left(1+\frac{2m}{r}+\frac{3}{2}\frac{m^2}{r^2}+\frac{m^3}{2r^3}\right)+h_{ii} \qquad \ldots \qquad \ldots \qquad (2.2)$$

for $i = 1, 2, 3$ and

$$a_{44} = 1-\frac{2m}{r}+\frac{2m^2}{r^2}-\frac{3}{2}\frac{m^3}{r^3}+h_{44},$$

$$a_{ij} = a_{ji}, \quad a_{ij} = h_{ij}, \quad i \neq j, \qquad (2.3)$$

while

$$\left.\begin{array}{lll} b_{12} = \alpha, & b_{13} = \phi+\beta, & b_{14} = \gamma, \\ b_{23} = \delta, & b_{24} = \epsilon, & b_{34} = \phi+\chi, \\ & b_{ij} = -b_{ji}. & \end{array}\right\} \qquad (2.4)$$

The orders of smallness of various terms are as given below:

$$m = 0\,(1), \quad \phi = 0\,(1), \quad h_{ij} = 0\,(3)$$

$$\alpha, \beta, \gamma, \delta, \epsilon, \chi = 0\,(2). \qquad (2.5)$$

The modification of the field of pure radiation as affected by gravitation is measured in terms of α, β, γ, etc. These six unknowns are to be determined from the field equations

$$Q^s_{is} = 0 \text{ or its equivalent } (\sqrt{-g}\,B^{is})_{,s} = 0 \qquad (2.6)$$

and

$$Q_{ik,\,l}+Q_{kl,\,i}+Q_{li,\,k} = 0 \qquad (2.7)$$

with

$$Q_{ik} = Q^a_{ik,\,a}-P^a_{ib}\,Q^b_{ak}-Q^a_{ib}\,P^b_{ak}-Q^a_{ia,\,k}+P^a_{ik}\,Q^b_{ab}+P^b_{ab}\,Q^a_{ik}. \qquad (2.8)$$

The comma behind a suffix denotes as usual a partial differentiation. In what follows when successive differentiations are intended, the corresponding suffixes will be put in proper order after the comma. P^a_{ik} and Q^a_{ik} are symmetric and antisymmetric parts of Γ^a_{ik}. Also $(\sqrt{-g}B^{is})$ is the antisymmetric part of the contravariant tensor density $\mathfrak{g}^{is} = (\sqrt{-g}\,g^{is})$. The ten h_{ij}, correction terms to the gravitational potentials due to the presence of the electromagnetic field are determined from the field equations

$$P_{ik} = 0 \qquad (2.9)$$

where

$$P_{ik} = P^a_{ik,\,a}-P^a_{ib}\,P^b_{ak}-Q^a_{ib}\,Q^b_{ak}-P^a_{ia,\,k}+P^a_{ik}\,P^b_{ab}+Q^a_{ik}\,Q^b_{ab}. \qquad (2.10)$$

It is easy to see that the last term in (2.10) namely $Q^a_{ik}\,Q^b_{ab}$ and the terms $-Q^a_{ia,\,k}$ and $P^a_{ik}\,Q^b_{ab}$ in (2.8) can be dropped because of (2.6). The symbols Γ^a_{ik} and the tensor R_{ik} with P_{ik} and Q_{ik} as its symmetric and antisymmetric parts provide equations which have transposition invariance.

The field strengths of the electromagnetic field and likewise the two components of the current vector density become infinite at all points on the negative side of the x-axis. With the help of a covariant tensor dual to the contravariant tensor associated with the field strengths it has been possible to show that the electric and magnetic fields are perpendicular to one another and that, up to the third order of approximations, they are equal in absolute value. With the determined values of ρ, σ_x, σ_y and σ_z it has now been possible to obtain four partial differential equations of the second order for the electromagnetic vector potentials subject to the condition of continuity. It will not be out of place to recall here that these equations were of the fourth order in our previous paper.

When the values for α, β, γ, etc., are substituted in the ten gravitational equations, they are reduced to those of general relativity for a weak field. The particular solution of the gravitational equations contains singularities all along the negative side of the x-axis. These singularities can be transformed away by means of a singular (and therefore forbidden) transformation of co-ordinates possessing singularities all along the negative side of the x-axis. All this means that, according to this theory, the field of uniform monochromatic radiation does not interact with the gravitational field of a point mass up to the third order of approximations.

3. Preliminary Calculations

The contravariant tensor g^{ls} is given by

$$g_{is}\, g^{ls} = g_{si}\, g^{sl} = \delta^l_i . \qquad (3.1)$$

We take

$$g^{ls} = A^{ls} + B^{ls}$$

with

$$A^{ls} = A^{sl},\ B^{ls} = -B^{sl}. \qquad (3.2)$$

We obtain up to the second order of approximations

$$g^{11} = -\left[1 - \frac{2m}{r} + \frac{5}{2}\frac{m^2}{r^2} - \phi^2\right],$$

$$g^{22} = g^{33} = -\left[1 - \frac{2m}{r} + \frac{5}{2}\frac{m^2}{r^2}\right],$$

$$g^{44} = 1 + \frac{2m}{r} + \frac{2m^2}{r^2} + \phi^2, \qquad (3.3)$$

$$g^{12} = \alpha, \qquad g^{13} = \phi + \beta - \frac{4m}{r}\phi,$$

$$g^{14} = \phi^2 - \gamma, \quad g^{23} = \delta,$$

$$g^{24} = -\epsilon, \qquad g^{34} = -(\phi + \chi).$$

Also

$$g = -\left(1 + \frac{4m}{r} + \frac{13}{2}\frac{m^2}{r^2}\right), \quad g^{-1} = -\left(1 - \frac{4m}{r} + \frac{19}{2}\frac{m^2}{r^2}\right). \qquad (3.4)$$

a^{ij} are determined from (2.2) and (2.3) with the help of the equation

$$a_{ij}\, a^{ik} = \delta^k_j . \qquad (3.5)$$

Up to the second order,

$$a^{11} = a^{22} = a^{33} = -\left(1 - \frac{2m}{r} + \frac{5}{2}\frac{m^2}{r^2}\right),$$

$$a^{44} = \left(1 + \frac{2m}{r} + \frac{2m^2}{r^2}\right), \quad a^{ij} = 0,\ i \neq j. \qquad (3.6)$$

P^a_{ik} and Q^a_{ik} are obtained from the set of equations

$$g_{ik,\,l} = g_{sk}\, \Gamma^s_{il} + g_{is}\, \Gamma^s_{lk} . \qquad (3.7)$$

The actual calculation of Γ-symbols is very tedious. P-symbols and Q-symbols have been determined here by a method of successive approximations (V. V. Narlikar and Ramji Tiwari, 1949*b*).

4. Electromagnetic Equations

The four equations (2.6) can be expressed as

$$\alpha,_2+\beta,_3-\gamma,_4-2\phi\left(\frac{m}{r}\right),_3=0, \qquad (4.1a)$$

$$\alpha,_1-\delta,_3+\epsilon,_4=0, \qquad (4.1b)$$

$$-\beta,_1-\delta,_2-\chi,_4+2\left(\frac{m}{r}\right),_1\phi+\frac{4m}{r}\phi,_1=0, \qquad (4.1c)$$

$$\gamma,_1+\epsilon,_2+\chi,_3+2\left(\frac{m}{r}\right),_3\phi=0. \qquad (4.1d)$$

With the help of the definition

$$\sqrt{-g}\,B^{is}=(\sqrt{-g}\,g^{isl}),_l \qquad (4.2)$$

where $\sqrt{-g}\,g^{ikl}\ (\equiv\mathfrak{g}^{ikl})$ is an antisymmetric tensor density in i, k and l, or even otherwise we can interpret

$$\begin{aligned}
\sqrt{-g}\,B^{12}&=\alpha\\
&=-\Phi,_3-H,_4=E_z,\\
\sqrt{-g}\,B^{13}&=\beta+\phi-\frac{2m}{r}\phi\\
&=\Phi,_2+G,_4=-E_y,\\
\sqrt{-g}\,B^{14}&=-\gamma\\
&=H,_2-G,_3=H_x,\\
\sqrt{-g}\,B^{23}&=\delta\\
&=-\Phi,_1-F,_4=E_x,\\
\sqrt{-g}\,B^{24}&=-\epsilon\\
&=F,_3-H,_1=H_y,\\
\sqrt{-g}\,B^{34}&=-\chi-\phi-\frac{2m}{r}\phi\\
&=G,_1-F,_2=H_z,
\end{aligned} \qquad (4.3)$$

where F, G, H and Φ are the components of the electromagnetic vector potentials. $\mathfrak{g}^{ikl}$ plays the rôle of the electromagnetic vector potentials:

$$\mathfrak{g}^{123}=-\Phi,\quad \mathfrak{g}^{314}=-G,$$
$$\mathfrak{g}^{234}=-F,\quad \mathfrak{g}^{124}=-H. \qquad (4.4)$$

The equations (4.1) turn out to be Maxwell's equations

$$\operatorname{curl}\hat{E}+\frac{\partial\hat{H}}{\partial t}=0,\quad \operatorname{div}\hat{H}=0. \qquad (4.5)$$

Up to the second order of approximations the condition

$$\mathfrak{g}^{ijk}{}_{;\lambda}\, g^{\lambda l} + \mathfrak{g}^{jlk}{}_{;\lambda}\, g^{\lambda i} + \mathfrak{g}^{lik}{}_{;\lambda}\, g^{\lambda j} + \mathfrak{g}^{ilj}{}_{;\lambda}\, g^{\lambda k} = 0 \quad \text{(4.6)}$$

expresses the equation of continuity in the form

$$\left(1-\frac{2m}{r}\right)\left(\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}\right)+\left(1+\frac{2m}{r}\right)\frac{\partial \Phi}{\partial t}=0. \quad \text{(4.7)}$$

Here the semicolon refers to a covariant differentiation as defined by Einstein (1948) in connection with the covariant differentiation of a tensor density. The current density is given by

$$b_{ik,\,l} + b_{kl,\,i} + b_{li,\,k} \quad \text{(4.8)}$$

from which we get

$$\begin{aligned} \rho &= \alpha,_3 - \beta,_2 + \delta,_1, \\ \sigma_x &= -\delta,_4 + \epsilon,_3 - \chi,_2, \\ \sigma_y &= \chi,_1 + \beta,_4 - \gamma,_3, \\ \sigma_z &= \gamma,_2 - \epsilon,_1 - \alpha,_4, \end{aligned} \quad \text{(4.9)}$$

where ρ is the charge density and σ_x, σ_y, σ_z are the components of current vector density at x, y, z at time t. The set of four equations (2.7) takes the form

$$\begin{aligned} \Box\, \rho &= -4\left(\frac{m}{r}\right)_{,12} \phi,_1, \\ \Box\, \sigma_x &= -4\left(\frac{m}{r}\right)_{,12} \phi,_1, \\ \Box\, \sigma_y &= 4\left(\frac{m}{r}\right)_{,11} \phi,_1 + 8\left(\frac{m}{r}\right)_{,33} \phi,_1, \\ \Box\, \sigma_z &= -8\left(\frac{m}{r}\right)_{,32} \phi,_1, \end{aligned} \quad \text{(4.10)}$$

where

$$\Box = \frac{\partial^2}{\partial_x^2}+\frac{\partial^2}{\partial_y^2}+\frac{\partial^2}{\partial_z^2}-\frac{\partial^2}{\partial t^2}. \quad \text{(4.11)}$$

It has been now possible to obtain from (4.10)

$$\begin{aligned} \rho &= -2\left(\frac{m}{r}\right)_{,2} \phi + f_1(x, y, z, t), \\ \sigma_x &= -2\left(\frac{m}{r}\right)_{,2} \phi + f_2(x, y, z, t), \\ \sigma_y &= 2\left(\frac{m}{r}\right)_{,1} \phi + 4m\phi\, \{\log(x+r)\}_{,33} + f_3(x, y, z, t), \\ \sigma_z &= -4\, m\, \phi\, \{\log(x+r)\}_{,32} + f_4(x, y, z, t), \end{aligned} \quad \text{(4.12)}$$

where f_1, f_2, f_3 and f_4 are functions satisfying the wave equations. By virtue of the boundary condition that when m is zero the field reduces to the field of a uniform monochromatic radiation wherein the charge-and-current vector density does not exist we observe that these functions are all zero. Hence the complete solution can be written as

$$\begin{aligned}
\rho &= -2\left(\frac{m}{r}\right)_{,2}\phi, \\
\sigma_x &= -2\left(\frac{m}{r}\right)_{,2}\phi, \\
\sigma_y &= 2\left(\frac{m}{r}\right)_{,1}\phi+4m\phi\,\{\log(x+r)\}_{,33}, \\
\sigma_z &= -4m\phi\,\{\log(x+r)\}_{,32}.
\end{aligned} \qquad (4.13)$$

Here we see that ρ and σ_x are zero when y is zero. σ_y is zero for all points on the positive side of the x-axis and also for all points on the line in the $y-z$ plane inclined at an angle of 45° with the z-axis. σ_z too is zero all along the positive side of the x-axis. Both σ_y and σ_z become infinite for all points on the negative side of the x-axis. (4.9) now takes the form

$$\alpha_{,3}-\beta_{,2}+\delta_{,1} = -2\left(\frac{m}{r}\right)_{,2}\phi, \qquad (4.14a)$$

$$-\delta_{,4}+\epsilon_{,3}-\chi_{,2} = -2\left(\frac{m}{r}\right)_{,2}\phi, \qquad (4.14b)$$

$$\chi_{,1}+\beta_{,4}-\gamma_{,3} = 2\left(\frac{m}{r}\right)_{,1}\phi+4m\phi\,\{\log(x+r)\}_{,33}, \qquad (4.14c)$$

$$\gamma_{,2}-\epsilon_{,1}-\alpha_{,4} = -4m\phi\,\{\log(x+r)\}_{,32}. \qquad (4.14d)$$

Differentiating (4.1b), (4.1a), (4.14a), (4.14d) with respect to x, y, z, t and adding the results we obtain

$$\Box\,\alpha = 4m\phi_{,1}\,\{\log(x+r)\}_{,32} \qquad (4.15)$$

from which we have

$$\alpha = 2m\phi\,\{x\log(x+r)-r\}_{,32}. \qquad (4.16)$$

Proceeding in the same manner we find

$$\begin{aligned}
\Box\,\beta &= 8\left(\frac{m}{r}\right)_{,1}\phi_{,1}+4\frac{m}{r}\phi_{,11}+4m\phi_{,1}\{\log(x+r)\}_{,33}, \\
\Box\,\gamma &= -4\left(\frac{m}{r}\right)_{,3}\phi_{,1}, \\
\Box\,\delta &= 4\left(\frac{m}{r}\right)_{,2}\phi_{,1}, \\
\Box\,\epsilon &= 4m\phi_{,1}\{\log(x+r)\}_{,32}, \\
\Box\,\chi &= 4\left(\frac{m}{r}\right)_{,1}\phi_{,1}+4\frac{m}{r}\phi_{,11}+4m\phi_{,1}\{\log(x+r)\}_{,33},
\end{aligned} \qquad (4.17)$$

the solutions of which can be given as

$$\begin{aligned}
\beta &= 4\frac{m}{r}\phi+2m\phi,_1 \log(x+r)+2m\phi\{x\log(x+r)-r\},_{33}, \\
\gamma &= -2m\phi\{\log(x+r)\},_3, \\
\delta &= 2m\phi\{\log(x+r)\},_2, \qquad (4.18) \\
\epsilon &= 2m\phi\{x\log(x+r)-r\},_{32}, \\
\chi &= 2\frac{m}{r}\phi+2m\phi,_1 \log(x+r)+2m\phi\{x\log(x+r)-r\},_{33}.
\end{aligned}$$

It is worth noticing that the values of α, β, γ, etc., as determined in (4.16) and (4.18) are exact in the sense that the complementary functions are all zero because of the boundary condition.

(4.14) by virtue of (4.3) and (4.7) can be expressed as

$$\begin{aligned}
&\Box\Phi-4\frac{m}{r}\Phi,_{44}=0, \\
&\Box F+4\left(\frac{m}{r}\Phi\right),_{14}=4\left(\frac{m}{r}\right),_2\phi, \qquad (4.19) \\
&\Box G+4\left(\frac{m}{r}\Phi\right),_{24}=4m\phi\left\{\log(x+r)\right\},_{22}-4\frac{m}{r}\phi,_1, \\
&\Box H+4\left(\frac{m}{r}\Phi\right),_{34}=4m\phi\left\{\log(x+r)\right\},_{32}.
\end{aligned}$$

It can easily be verified that (4.19) satisfy identically

$$\Box\left\{F,_1+G,_2+H,_3+\left(1+4\frac{m}{r}\right)\Phi,_4\right\}=0 \qquad (4.20)$$

as is expected from (4.7) which may as well be expressed as

$$F,_1+G,_2+H,_3+\left(1+4\frac{m}{r}\right)\Phi,_4=0. \qquad (4.21)$$

5. Determination of the Nature of the Electromagnetic Field

With the help of (4.3) we now get from (4.18)

$$\begin{aligned}
E_x &= \frac{2my}{r(r+x)}\phi, \\
E_y &= -\phi-\frac{2m}{x+r}\phi+\frac{2my^2}{r(r+x)^2}\phi-2m\phi,_1\log(x+r), \\
E_z &= \frac{2myz}{r(r+x)^2}\phi, \qquad (5.1) \\
H_x &= \frac{2mz}{r(r+x)}\phi, \\
H_y &= \frac{-2myz}{r(r+x)^2}\phi, \\
H_z &= -\phi-\frac{2m}{r}\phi-\frac{2m}{x+r}\phi+\frac{2my^2}{r(r+x)^2}\phi-2m\phi,_1\log(x+r).
\end{aligned}$$

As indicated in (4.3) the components of $(\sqrt{-g}\,B^{ik})$ represent the electric and magnetic field strengths. The covariant components of the dual tensor corresponding to B^{ik} are given by

$$F_{lm} = \tfrac{1}{2}\,\delta_{lmik}\left(\sqrt{-g}\,B^{ik}\right) \qquad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

where δ_{lmik} is the Levi-Civita symbol. Maxwell's four equations (4.5) can now be recast in the usual tensor form

$$F_{\mu\nu,\,\sigma}+F_{\nu\sigma,\,\mu}+F_{\sigma\mu,\,\nu} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

Also the second set of Maxwell's equations, namely

$$\operatorname{curl}\hat{H}-\frac{\partial\hat{E}}{\partial t} = \hat{I}, \quad \operatorname{div}\hat{E} = \rho \qquad \ldots \quad \ldots \quad \ldots \quad (5.4)$$

can likewise be expressed as

$$F^{\mu\nu};\,\nu = I^{\mu} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.5)$$

with

$$F^{\mu\nu} = a^{\mu\alpha}a^{\nu\beta}F_{\alpha\beta}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.6)$$

$\hat{I}$ in (5.4) represents only current vector density, while I^{μ} in (5.5) represents charge-and-current vector density. The semicolon refers to the covariant differentiation with respect to Christoffel's symbols. (5.3) and (5.5) can alternatively be expressed as

$$B^{ik};\,k = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.7)$$

and

$$b_{ik,\,l}+b_{kl,\,i}+b_{li,\,k} = I_{ikl} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

where I_{ikl} is an antisymmetric tensor in the suffixes i, k, l.

The covariant components of a vector dual to the antisymmetric tensor g^{ikl} are given by

$$A_{i} = \frac{1}{6}\,\delta_{ilmn}\,\mathfrak{g}^{\,lmn}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.9)$$

The equation of continuity (4.7) can simply be expressed as

$$A^{\mu};\,\mu = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.10)$$

where

$$A^{\mu} = a^{\mu s}A_{s}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.11)$$

It has been found that

$$F_{ik}B^{ik} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.12)$$

(5.12) clearly indicates that the electric and magnetic fields are at right angles to one another. Also up to the third order of approximations

$$F_{ik}F^{ik} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.13)$$

which shows that the electric and magnetic field strengths are equal to one another in absolute value. Thus it is clear that the electromagnetic field is the field of a plane wave. Having once obtained the electromagnetic field tensor it is easy to calculate the electromagnetic energy tensor E^{μ}_{ν}:

$$E^{\mu}_{\nu} = -F^{\mu\alpha}F_{\nu\alpha}+\tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}. \qquad \ldots \quad \ldots \quad \ldots \quad (5.14)$$

Up to the third order of approximations we have

$$\begin{aligned}
&\underset{2}{E}^{3} = \underset{2}{E}^{2} = \underset{3}{E}^{3} = 0, \\
&\underset{1}{E}^{1} = -\underset{4}{E}^{4} = -\phi^2 - 4m\phi\phi_{,1} \log(x+r) + 4m\phi^2 \{x \log(x+r) - r\}_{,22}, \\
&\underset{1}{E}^{2} = \underset{2}{E}^{4} = -\underset{4}{E}^{2} = -2m\phi^2 \{\log(x+r)\}_{,2}, \\
&\underset{1}{E}^{3} = \underset{3}{E}^{4} = -\underset{4}{E}^{3} = -2m\phi^2 \{\log(x+r)\}_{,3}, \\
&\underset{1}{E}^{4} = -\underset{4}{E}^{1} = -\phi^2 - 2\frac{m}{r}\phi^2 - 4m\phi\phi_{,1} \log(x+r) \\
&\qquad + 4m\phi^2 \{x \log(x+r) - r\}_{,22}.
\end{aligned} \tag{5.15}$$

6. The Gravitational Equations

The ten gravitational equations are:

$$\begin{aligned}
P_{11} &= \tfrac{1}{2}(h_{11,22} + h_{11,33} - h_{11,44}) - h_{12,12} - h_{13,13} + h_{14,14} \\
&\quad + \tfrac{1}{2}(h_{22} + h_{33} - h_{44})_{,11} - \phi\Box\beta - \phi(\beta - \chi)_{,11} + 16\left(\frac{m}{r}\right)_{,1}\phi\phi_{,1} \\
&\quad - \phi_{,11}(\beta - \chi) - 2\phi_{,1}(\beta_{,1} - \chi_{,1} + \gamma_{,3}) + 8\frac{m}{r}\phi\phi_{,11} \\
&\quad + 4\frac{m}{r}(\phi_{,1})^2 + 2\phi^2\left(\frac{m}{r}\right)_{,11} = 0, \\
P_{12} &= \tfrac{1}{2}(h_{12,33} - h_{23,13} - h_{31,23}) + \tfrac{1}{2}(h_{14,24} + h_{42,14} - h_{21,44}) \\
&\quad + \tfrac{1}{2}(h_{33} - h_{44})_{,12} - \tfrac{1}{2}\phi\Box\delta - \phi(\beta - \chi)_{,12} + 6\left(\frac{m}{r}\right)_{,2}\phi\phi_{,1} \\
&\quad + \phi_{,1}(\alpha_{,3} - \beta_{,2} + \chi_{,2} - \epsilon_{,3}) + 2\phi^2\left(\frac{m}{r}\right)_{,12} = 0, \\
P_{13} &= \tfrac{1}{2}(h_{13,22} - h_{32,12} - h_{21,32}) + \tfrac{1}{2}(h_{22} - h_{44})_{,13} \\
&\quad + \tfrac{1}{2}(h_{14,34} + h_{43,14} - h_{13,44}) + \tfrac{1}{2}\phi\Box\gamma - \phi(\beta - \chi)_{,13} \\
&\quad + \phi_{,1}(\gamma_{,1} + \gamma_{,4}) + 6\phi\phi_{,1}\left(\frac{m}{r}\right)_{,3} + 2\phi^2\left(\frac{m}{r}\right)_{,13} = 0, \\
P_{14} &= \tfrac{1}{2}(h_{14,22} - h_{42,12} - h_{21,42}) + \tfrac{1}{2}(h_{14,33} - h_{43,13} - h_{31,43}) \\
&\quad + \tfrac{1}{2}(h_{22} + h_{33})_{,14} + \tfrac{1}{2}\phi\Box(\beta + \chi) - \phi(\beta - \chi)_{,14} - 10\left(\frac{m}{r}\right)_{,1}\phi\phi_{,1} \\
&\quad - \phi_{,1}(\chi_{,1} - \beta_{,1} + \beta_{,4} - \chi_{,4} - 2\gamma_{,3}) - 4\frac{m}{r}(\phi_{,1})^2 + 8\frac{m}{r}\phi\phi_{,14} \\
&\quad - \phi_{,14}(\beta - \chi) = 0, \\
P_{22} &= \tfrac{1}{2}(h_{22,11} + h_{22,33} - h_{22,44}) + \tfrac{1}{2}(h_{11} + h_{33} - h_{44})_{,22} - h_{21,21} \\
&\quad - h_{23,23} + h_{24,24} - \phi(\beta - \chi)_{,22} + 2\phi^2\left(\frac{m}{r}\right)_{,22} = 0,
\end{aligned} \tag{6.1}$$

$$P_{23} = \tfrac{1}{2}(h_{23,11}-h_{31,21}-h_{12,31})+\tfrac{1}{2}(h_{24,34}+h_{43,24}-h_{23,44})$$
$$+\tfrac{1}{2}(h_{11}-h_{44})_{,23}-\tfrac{1}{2}\phi\Box(\alpha-\epsilon)-\phi(\beta-\chi)_{,23}+2\phi^2\left(\frac{m}{r}\right)_{,23}$$
$$-\phi_{,1}(\alpha_{,1}+\alpha_{,4}-\epsilon_{,1}-\epsilon_{,4}) = 0,$$

$$P_{24} = \tfrac{1}{2}(h_{24,11}-h_{41,21}-h_{12,41})+\tfrac{1}{2}(h_{24,33}-h_{43,23}-h_{32,43})$$
$$+\tfrac{1}{2}(h_{11}+h_{33})_{,24}+\tfrac{1}{2}\phi\Box\delta-\phi(\beta-\chi)_{,24}-6\left(\frac{m}{r}\right)_{,2}\phi\phi_{,1}$$
$$-\phi_{,1}(\alpha_{,3}-\epsilon_{,3}-\beta_{,2}+\chi_{,2}) = 0,$$

$$P_{33} = \tfrac{1}{2}(h_{33,11}+h_{33,22}-h_{33,44})-h_{31,31}-h_{32,32}+h_{34,34}$$
$$+\tfrac{1}{2}(h_{11}+h_{22}-h_{44})_{,33}-\phi\Box(\beta-\chi)+2\phi^2\left(\frac{m}{r}\right)_{,33}-\phi(\beta-\chi)_{,33}$$
$$-2\phi_{,1}(\beta_{,1}+\beta_{,4}-\chi_{,1}-\chi_{,4})+8\left(\frac{m}{r}\right)_{,1}\phi\phi_{,1} = 0,$$

$$P_{34} = \tfrac{1}{2}(h_{34,11}-h_{41,31}-h_{13,41})+\tfrac{1}{2}(h_{34,22}-h_{42,32}-h_{32,42})$$
$$+\tfrac{1}{2}(h_{11}+h_{22})_{,34}-\tfrac{1}{2}\phi\Box\gamma-\phi(\beta-\chi)_{,34}-\phi_{,1}(\gamma_{,1}+\gamma_{,4})$$
$$-6\phi\phi_{,1}\left(\frac{m}{r}\right)_{,3} = 0,$$

$$P_{44} = \tfrac{1}{2}(h_{44,11}+h_{44,22}+h_{44,33})-h_{41,41}-h_{42,42}-h_{43,43}$$
$$+\tfrac{1}{2}(h_{11}+h_{22}+h_{33})_{,44}-\phi\Box\chi-\phi(\beta-\chi)_{,44}+4\frac{m}{r}(\phi_{,1})^2$$
$$+4\left(\frac{m}{r}\right)_{,1}\phi\phi_{,1}+2\phi_{,1}(\beta_{,4}-\chi_{,4}-\gamma_{,3})-8\frac{m}{r}\phi\phi_{,14}+\phi_{,14}(\beta-\chi) = 0.$$

It has been observed that when α, β, γ, etc., as determined in (4.16) and (4.18) are substituted in (6.1) the electromagnetic terms cancel out in each of the field equations which are reduced to

$$N_{ik} = 0 \quad \ldots \quad (6.2)$$

where

$$N_{ik} = -\tfrac{1}{2}\delta^{\sigma\rho}\left(\frac{\partial^2 h_{ik}}{\partial x^\sigma \partial x^\rho}+\frac{\partial^2 h_{\sigma\rho}}{\partial x^i \partial x^k}-\frac{\partial^2 h_{i\sigma}}{\partial x^k \partial x^\rho}-\frac{\partial^2 h_{k\rho}}{\partial x^i \partial x^\sigma}\right) \quad \ldots \quad (6.3)$$

with

$$\delta^{11} = \delta^{22} = \delta^{33} = -\delta^{44} = -1, \qquad \delta^{ij} = 0, \qquad i \neq j. \quad \ldots \quad (6.4)$$

It is indeed remarkable that the solutions of (2.6) and (2.7) when substituted in (2.9) should lead to the cancellation of all the electromagnetic terms from the gravitational equations. Whatever electromagnetic effects remain can be found only in the h-terms.

From the structure of the gravitational equations (6.1) a particular solution is suggested by the assumption

$$h_{12} = \phi\delta = -h_{24}, \qquad h_{13} = -\phi\gamma = -h_{34}, \qquad h_{23} = \phi(\alpha-\epsilon). \quad \ldots \quad (6.5)$$

With these values of h's the field equations (6.1) are simplified to

$$\tfrac{1}{2}(h_{11\,,\,22}+h_{11\,,\,33}-h_{11\,,\,44})+\tfrac{1}{2}(h_{22}+h_{33}-h_{44})_{\,,\,11}+h_{14\,,\,14}$$

$$-2\left(\frac{m}{r}\right)_{,\,22}\phi^2-2\left(\frac{m}{r}\right)_{,\,33}\phi^2+4\left(\frac{m}{r}\right)_{,\,1}\phi\phi_{\,,\,1}=0\,,$$

$$h_{14\,,\,24}+(h_{33}-h_{44})_{\,,\,12}+4\left(\frac{m}{r}\right)_{,\,2}\phi\phi_{\,,\,1}=0\,,$$

$$h_{14\,,\,34}+(h_{22}-h_{44})_{\,,\,13}+4\left(\frac{m}{r}\right)_{,\,3}\phi\phi_{\,,\,1}=0\,,$$

$$h_{14\,,\,22}+h_{14\,,\,33}+(h_{22}+h_{33})_{\,,\,14}+2\left(\frac{m}{r}\right)_{,\,22}\phi^2+2\left(\frac{m}{r}\right)_{,\,33}\phi^2$$

$$+8m\phi\phi_{\,,\,1}\{\log(x+r)\}_{\,,\,22}+8m\phi\phi_{\,,\,1}\{\log(x+r)\}_{\,,\,33}=0\,, \qquad (6.6)$$

$$\tfrac{1}{2}(h_{22\,,\,11}+h_{22\,,\,33}-h_{22\,,\,44})+\tfrac{1}{2}(h_{11}+h_{33}-h_{44})_{,\,22}-2\phi^2\left(\frac{m}{r}\right)_{,\,22}=0,$$

$$\tfrac{1}{2}(h_{11}-h_{44})_{\,,\,23}-2\phi^2\left(\frac{m}{r}\right)_{,\,23}=0\,,$$

$$(h_{11}+h_{33})_{\,,\,24}-h_{41\,,\,21}-2\left(\frac{m}{r}\right)_{,\,12}\phi^2-4\left(\frac{m}{r}\right)_{,\,2}\phi\phi_{\,,\,1}=0\,,$$

$$(h_{33\,,\,11}+h_{33\,,\,22}-h_{33\,,\,44})+(h_{11}+h_{22}-h_{44})_{\,,\,33}-4\left(\frac{m}{r}\right)_{,33}\phi^2=0\,,$$

$$(h_{11}+h_{22})_{\,,\,34}-h_{41\,,\,31}-2\left(\frac{m}{r}\right)_{,13}\phi^2-4\left(\frac{m}{r}\right)_{,\,3}\phi\phi_{\,,\,1}=0\,,$$

$$\tfrac{1}{2}(h_{44\,,\,11}+h_{44\,,\,22}+h_{44\,,\,33})-h_{41\,,\,41}+\tfrac{1}{2}(h_{11}+h_{22}+h_{33})_{\,,\,44}$$

$$+4\left(\frac{m}{r}\right)_{,\,1}\phi\phi_{\,,\,1}=0.$$

It is now possible to obtain from (6.6)

$$h_{11}=4m\{\phi^2\log(x+r)\}_{\,,\,1}\,,$$

$$h_{44}=-4m\{\phi^2\log(x+r)\}_{,\,4}\,, \qquad (6.7)$$

$$h_{14}=-2m\{\phi^2\log(x+r)\}_{,\,1}+2m\{\phi^2\log(x+r)\}_{,\,4}\,,$$

$$h_{22}=h_{33}=0.$$

By virtue of the boundary condition that when $\phi=0$, the field reduces to the field of a point mass in classical relativity we see that the complementary functions which otherwise would have been attached to the h's in (6.7) are zero. Substituting for α, γ, etc., in (6.5) we get

$$h_{12}=2m\phi^2\{\log(x+r)\}_{,\,2}=-h_{24}, \qquad (6.8)$$

$$h_{13}=2m\phi^2\{\log(x+r)\}_{,\,3}=-h_{34},$$

$$h_{23}=0.$$

Thus we have

$$
\begin{aligned}
& h_{11} = 4m \{\phi^2 \log (x+r)\}_{,1}, \quad h_{12}=2m\phi^2 \{\log (x+r)\}_{,2}, \\
& h_{13} = 2m\phi^2 \{\log (x+r)\}_{,3}, \\
& h_{14} = -2m \{\phi^2 \log (x+r)\}_{,1}+2m \{\phi^2 \log (x+r)\}_{,4}, \\
& h_{22} = 0, \qquad h_{23} = 0, \\
& h_{24} = -2m\phi^2 \{\log (x+r)\}_{,2}, \quad h_{33} = 0, \\
& h_{34} = -2m\phi^2 \{\log (x+r)\}_{,3}, \quad h_{44} = -4m \{\phi^2 \log (x+r)\}_{,4}.
\end{aligned} \qquad (6.9)
$$

It is obvious from (6.9) that there exist relations among h's:

$$
\begin{aligned}
& h_{14} = -\tfrac{1}{2}(h_{11}+h_{44}), \\
& h_{12} = -h_{24}, \qquad h_{13} = -h_{34}, \\
& h_{22} = h_{23} = h_{33} = 0.
\end{aligned} \qquad (6.10)
$$

The solution obtained in (6.9) happens to be a particular solution of the gravitational equations. Before some meaning is given to the new singularities in (6.9) it has to be seen whether the singularities have arisen out of a singularity in a transformation. By means of an infinitesimal transformation of co-ordinates

$$x^\alpha = x'^\alpha + \underset{(3)}{T^\alpha} \qquad (6.11)$$

where $\underset{(3)}{T^\alpha}$ ($\alpha = 1, 2, 3, 4$) are four functions of the third order it is possible to transform away these singularities, provided we take

$$\underset{(3)}{T^1} = \underset{(3)}{T^4} = 2\phi^2 m \log (x+r), \quad \underset{(3)}{T^2} = \underset{(3)}{T^3} = 0. \qquad (6.12)$$

We get the unexpected result that the solution (6.9) is a transform of Schwarzschild's external solution through the singular transformation (6.12). It may be mentioned that the expressions (4.13) for the charge and current density remain unaffected up to the fourth order by the transformation, since its departure from the identity transformation is of the third order of smallness.

The two noteworthy features of the investigation are (1) the automatic cancellation of the explicit electromagnetic terms from (6.1), and (2) a solution of the reduced equations (6.1) with electromagnetic effects which, however, is a transform of Schwarzschild's solution through a singular transformation. Thus the electromagnetic effects can be traced to the singularity in the transformation and are therefore not real.

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Abstract

A particular solution of the gravitational field equations in Einstein's generalized theory of gravitation has been obtained. The field set up is such that it reduces to the field of a point mass in classical relativity in the absence of the electromagnetic field, and also up to the first order of approximations it reduces to the field of uniform monochromatic radiation ($E_y = H_z = -A \cos 2\pi (x-t)/\lambda$, $E_x = E_z = H_x = H_y = 0$) in the absence of the point mass. The charge

density and the first component of the current vector density vanish all along the x-axis while the remaining components of the current vector density vanish on the positive side of this axis and become infinite on its negative side. The electric and magnetic intensities are affected by the presence of the gravitational field of the point mass. The electromagnetic field possesses the properties of a plane wave. There are four partial differential equations of the second order (4.19) for the four electromagnetic vector potentials subject to the condition of continuity (4.7). When the values obtained for the electromagnetic field quantities are substituted in the ten gravitational equations (6.1) they are reduced to those of general relativity for a weak field. The solution obtained from (6.1) contains a line of singularities. What is rather curious is that this solution is a transform of the Schwarzschild solution through a singular transformation.

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