ON THE NEGATIVE ORDER SUMMABILITY OF A FOURIER SERIES AT A POINT

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1.1. Let $f(t)$ be integrable (L) over $(-\pi, \pi)$, and periodic with period 2π, and let its Fourier series be given by

$$\frac{1}{2}a_0+\Sigma(a_n \cos nt+b_n \sin nt) = \frac{1}{2}A_0+\sum_{1}^{\infty} A_n(t). \tag{1.1.1}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}.$$

We assume throughout that $t > 0$, and write

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du,\ \alpha > 0,$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\,\Phi_\alpha(t),\ \alpha \geqslant 0,$$

and

$$\Phi_\alpha(t) = \frac{d}{dt}\Phi_{\alpha+1}(t),\quad (-1 < \alpha < 0).$$

1.2. It is known that the summability (c, δ), $\delta \geqslant 0$, of the series (1.1.1) depends only on the properties of $f(t)$ near the point $t = x$. But when $\delta < 0$ this is no longer so (Kogbetliantz, 1919). A reason of this failure may be that the Lebesgue integrability of $f(t)$ over $(-\pi, \pi)$, by itself does not warrant anything beyond the asymptotic order estimate

$$A_n(t) = o\,(1),$$

as $n\to\infty$. Bosanquet and Offord (1936) have proved that this is the only reason for the failure of this property. They have shown, if

$$A_n(t) = o\left(n^{\delta}\right),\quad -1 < \delta < 0, \tag{1.2.1}$$

then the summability (c, δ) of the series (1.1.1) is a local property of the generating function.

As an extension of de la Valleé-Poussin's convergence criterion for Fourier series, Bosanquet proved the following theorem on Cesàro summability of positive order.

THEOREM A.* If $\phi_\alpha(t)$ $(\alpha\geqslant 1)$ is of bounded variation in an interval $(0, \eta)$, and $\phi_\alpha(t)\to 0$, as $t\to 0$, then the Fourier series of $f(t)$ is summable $(c, \alpha-1)$ at the point $t = x$.

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* Bosanquet (1934). See footnote on page 26.

The object of this paper is to extend Theorem A to the case in which $0 < \alpha < 1$, by imposing on the Fourier series a restriction of the type of (1.2.1).

2.1. We prove the following theorem:

THEOREM. *If* $\phi_\alpha(t)$ $(0 < \alpha < 1)$ *is of bounded variation in an interval* $(0, \eta)$, $\phi_\alpha(t) \to 0$, *as* $t \to 0$, *and*

$$A_n(x) = o\,(n^{\alpha-1}),$$

then the Fourier series of $f(t)$ *is summable* $(c, \alpha-1)$ *at the point* $t = x$.

2.2. We require the following Lemma for the proof of above theorem.

Lemma. Sufficient conditions for the Fourier series of $f(t)$ *to be summable* $(c, -\beta)$, $0 < \beta < 1$, *to sum s for* $t = x$ *are that*

$$A_n(x) = o\,(n^{-\beta}),$$

$$\phi_1(t) = o\,(1),$$

as $t \to 0$, *and*

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \left| n^{\beta} \int_{\frac{1}{n}}^{\delta} \phi(t) \frac{\sin(n\,;\,t)}{t^{1-\beta}}\, dt \right| = 0,$$

where

$$(n\,;\,t) = (n+\tfrac{1}{2}-\tfrac{1}{2}\beta)t+\tfrac{1}{2}\beta\pi.$$

This Lemma is proved by Bosanquet and Offord (1936).

2.3. Proof of the theorem: Since $\phi_\alpha(t) \to 0$ as $t \to 0$, therefore, $\phi_1(t) \to 0$ as $t \to 0$. Thus, after the Lemma, it is sufficient for the proof of the theorem to establish the following

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \left| n^{1-\alpha} \int_{\frac{1}{n}}^{\delta} \phi(t) t^{-\alpha} \sin(n\,;\,t)dt \right| = 0, \tag{2.3.1}$$

where

$$(n\,;\,t) = (n+\tfrac{1}{2}\alpha)t+\tfrac{1}{2}(1-\alpha)\pi.$$

Now

$$\begin{aligned} I &= \Gamma(1-\alpha) \int_{\frac{1}{n}}^{\delta} \phi(t) t^{-\alpha} \sin\,(n\,;\,t)dt \\ &= \int_{\frac{1}{n}}^{\delta} t^{-\alpha} \sin\,(n\,;\,t)dt \int_0^t (t-u)^{-\alpha}\, d\Phi_\alpha(u) \\ &= \int_0^{\frac{1}{n}} d\Phi_\alpha(u) \int_{\frac{1}{n}}^{\delta} (t-u)^{-\alpha}\, t^{-\alpha} \sin\,(n\,;\,t)dt \\ &\quad + \int_{\frac{1}{n}}^{\delta} d\Phi_\alpha(u) \int_u^{\delta} (t-u)^{-\alpha}\, t^{-\alpha} \sin\,(n\,;\,t)dt \\ &= \int_0^{\frac{1}{n}} J(n,\, u)d\Phi_\alpha(u) + \int_{\frac{1}{n}}^{\delta} K(n,\, u)d\Phi_\alpha(u) \\ &= I_1+I_2\,,\ \text{say}. \end{aligned} \tag{2.3.2}$$

The inversion of order of integration in the two parts of the repeated integral is justified by Fubini's theorem, since the resulting integrals are easily seen to be absolutely convergent.

We next show that, for $0 < u < \frac{1}{n}$,

(2.3.3) $$|J(n, u)| \leqslant An^{2\alpha-1};$$

and, for $$0 < u < \delta,$$

(2.3.4) $$|K(n, u)| \leqslant Au^{-\alpha} n^{\alpha-1};$$

where A denotes some number independent of δ, n and u.

Now, writing

$$\begin{aligned} J(n, u) &= \int_{\frac{1}{n}}^{\delta} (t-u)^{-\alpha} t^{-\alpha} \sin(n; t) dt \\ &= \int_{\frac{1}{n}}^{\frac{2}{n}} + \int_{\frac{2}{n}}^{\delta} \\ &= J_1 + J_2, \text{ say.} \end{aligned}$$

We have, if $0 < u < \frac{1}{n}$,

$$\begin{aligned} |J_1| &\leqslant An^{\alpha} \int_{\frac{1}{n}}^{\frac{2}{n}} (t-u)^{-\alpha} dt \\ &= O(n^{2\alpha-1}); \end{aligned}$$

and, by the second mean value theorem, $\frac{2}{n} < \xi < \xi' < \delta$,

$$\begin{aligned} |J_2| &\leqslant A\left(\frac{2}{n}\right)^{-\alpha} \left(\frac{1}{n}\right)^{-\alpha} \left| \int_{\xi}^{\xi'} \sin(n; t) dt \right| \\ &\leqslant A.n^{2\alpha-1}. \end{aligned}$$

This completes the proof of (2.3.3).

Again writing

$$\begin{aligned} K(n,u) &= \int_{u}^{\delta} (t-u)^{-\alpha} t^{-\alpha} \sin(n; t) dt \\ &= \int_{u}^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\delta} \\ &= K_1 + K_2, \text{ say.} \end{aligned}$$

We have, for $0 < u < \delta$,

$$\begin{aligned} |K_1| &\leqslant Au^{-\alpha} \int_{u}^{u+\frac{1}{n}} (t-u)^{-\alpha} dt \\ &= Au^{-\alpha} n^{\alpha-1}; \end{aligned}$$

and, by the second mean value theorem, $u+\frac{1}{n}<\xi<\delta$,

$$|K_2| \leqslant Au^{-\alpha} n^{\alpha} \left| \int_{u+\frac{1}{n}}^{\xi} \sin(n\,;\,t)dt \right|$$
$$\leqslant Au^{-\alpha} n^{\alpha-1}.$$

This completes the proof of (2.3.4).

Now, by integration by parts, we have

$$|I_1| = \left| \int_0^{\frac{1}{n}} J(n,u)d\Phi_\alpha(u) \right|$$
$$= \left| \Big[\Phi_\alpha(u)J(n,u) \Big]_0^{\frac{1}{n}} - \int_0^{\frac{1}{n}} \Phi_\alpha(u)\frac{d}{du}J(n,u)du \right|$$
$$\leqslant \frac{A}{\Gamma(\alpha+1)} \left| \phi_\alpha\left(\frac{1}{n}\right) \right| n^{\alpha-1} + \frac{1}{\Gamma(\alpha+1)} \left| \int_0^{\frac{1}{n}} \phi_\alpha(u)u^\alpha \frac{d}{du}J(n,u)du \right|.$$

Since $\phi_\alpha(u)$ is of bounded variation in an interval $(0,\eta)$ and $\phi_\alpha(u)\to 0$ as $u\to 0$, hence we may write

$$\phi_\alpha(u) = \psi_1(u)-\psi_2(u)$$

where $\psi_1(u)$ and $\psi_2(u)$ are positive increasing functions of u; each of these functions tends to the same limit as $u\to 0$; and we may, by subtracting a constant from each function, arrange that this limit shall be zero.

Now, we shall show that

(2.3.5) $$|M| \equiv \left| \int_0^{\frac{1}{n}} \psi_1(u)u^\alpha \frac{d}{du}J(n,u)du \right| \leqslant A\left|\psi_1\left(\frac{1}{n}\right)\right| n^{\alpha-1}.$$

By the second mean value theorem we have

$$|M| = \left| \psi_1\left(\frac{1}{n}\right) \int_\xi^{\frac{1}{n}} u^\alpha \frac{d}{du}J(n,u)du \right| \qquad \left(0<\xi<\frac{1}{n}\right)$$
$$= \left| \psi_1\left(\frac{1}{n}\right) \Big[J(n,u)u^\alpha \Big]_\xi^{\frac{1}{n}} - \alpha\,\psi_1\left(\frac{1}{n}\right) \int_\xi^{\frac{1}{n}} u^{\alpha-1} J(n,u)du \right|$$
$$\leqslant A\left|\psi_1\left(\frac{1}{n}\right)\right| \left\{ \left| \Big[n^{2\alpha-1}u^\alpha \Big]_\xi^{\frac{1}{n}} \right| + \left| n^{2\alpha-1} \int_\xi^{\frac{1}{n}} u^{\alpha-1}du \right| \right\}$$
$$\leqslant A\left|\psi_1\left(\frac{1}{n}\right)\right| n^{\alpha-1}.$$

This completes the proof of (2.3.5).

Hence, we have

(2.3.6) $$\overline{\lim_{n\to\infty}}\, n^{1-\alpha}|I_1| < A \overline{\lim_{n\to\infty}} \left\{ \left|\phi_\alpha\left(\frac{1}{n}\right)\right| + \left|\psi_1\left(\frac{1}{n}\right)\right| + \left|\psi_2\left(\frac{1}{n}\right)\right| \right\} = 0.$$

Again, by integration by parts, we have

$$|I_2| = \left|\int_{\frac{1}{n}}^{\delta} K(n, u)d\Phi_\alpha(u)\right|$$

$$= \left| \Big[\Phi_\alpha(u)K(n, u)\Big]_{\frac{1}{n}}^{\delta} - \int_{\frac{1}{n}}^{\delta} \Phi_\alpha(u) \frac{d}{du} K(n, u)du \right|$$

$$\leqslant An^{\alpha-1}\left\{ |\phi_\alpha(\delta)| + \left|\phi_\alpha\left(\frac{1}{n}\right)\right| \right\}$$

$$+A\left|\int_{\frac{1}{n}}^{\delta} \phi_\alpha(u)u^\alpha \frac{d}{du} K(n, u)du\right|.$$

If now we prove that

(2.3.7) $$|P| \equiv \left|\int_{\frac{1}{n}}^{\delta} \psi_1(u)u^\alpha \frac{d}{du} K(n, u)du\right| \leqslant A|\psi_1(\delta)|n^{\alpha-1},$$

then we shall have

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}}\, n^{1-\alpha}|I_2|$$

$$\leqslant A \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \left\{ |\phi_\alpha(\delta)| + \left|\phi_\alpha\left(\frac{1}{n}\right)\right| + |\psi_1(\delta)| + |\psi_2(\delta)| \right\}$$

$$= 0.$$

Thus (2.3.1) follows from (2.3.2), (2.3.6), (2.3.7) and the Lemma.

We proceed now to establish (2.3.7). We have, by the second mean value theorem,

$$|P| = \left|\psi_1(\delta)\int_{\xi}^{\delta} u^\alpha \frac{d}{du} K(n, u)du\right| \qquad \left(\frac{1}{n} < \xi < \delta\right)$$

$$= \left|\psi_1(\delta)\Big[u^\alpha K(n, u)\Big]_{\xi}^{\delta} - \alpha\,\psi_1(\delta)\int_{\xi}^{\delta} u^{\alpha-1}K(n, u)du\right|$$

(2.3.8) $$\leqslant An^{\alpha-1}|\psi_1(\delta)| + A|\psi_1(\delta)|\left|\int_{\xi}^{\delta} u^{\alpha-1}K(n, u)du\right|.$$

Now

$$Q = \int_{\xi}^{\delta} u^{\alpha-1} K(n, u) du$$

$$= \int_{\xi}^{\delta} u^{\alpha-1} du \int_{u}^{\delta} (t-u)^{-\alpha} t^{-\alpha} \sin(n; t) dt$$

$$= \int_{\xi}^{\delta} t^{-\alpha} \sin(n; t) dt \int_{\xi}^{t} u^{\alpha-1} (t-u)^{-\alpha} du$$

$$= \int_{\xi}^{\delta} t^{-\alpha} \sin(n; t) dt \int_{\frac{\xi}{t}}^{1} V^{\alpha-1} (1-V)^{-\alpha} dV$$

$$= \left(\int_{\frac{\xi}{\delta}}^{1} V^{\alpha-1} (1-V)^{-\alpha} dV \right) \left(\int_{\eta}^{\delta} t^{-\alpha} \sin(n; t) dt \right)$$

$$\left(0 < \frac{1}{n} < \xi < \eta < \delta \right).$$

Again, by the second mean value theorem, we have

$$\left| \int_{\eta}^{\delta} t^{-\alpha} \sin(n; t) dt \right| \leqslant A \eta^{-\alpha} n^{-1} \leqslant A n^{\alpha-1}.$$

Hence, we have

(2.3.9) $$|Q| \leqslant A n^{\alpha-1}.$$

Thus, finally, from (2.3.8) and (2.3.9), we have

$$|P| \leqslant A |\psi_1(\delta)| n^{\alpha-1}$$

as stated in (2.3.7).

This completes the proof of the theorem.

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