INTEGRAL REPRESENTATION OF MEIJER TRANSFORMS

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1. Meijer (1941*a*, p. 727) introduced the integral equation

$$F(s) = \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2}, m} (st) f(t) dt, \tag{1.1}$$

and its inverse

$$f(t) = \lim_{\lambda\to\infty} \frac{\Gamma(1-k+m)}{2\pi i \Gamma(1+2m)} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st} (st)^{k-\frac{1}{2}} M_{k-\frac{1}{2}, m} (st) F(s)\, ds, \tag{1.2}$$

where $M_{k, m}(z)$ and $W_{k, m}(z)$ are the two Whittaker functions.

Jaiswal (1952, p. 385) denoted (1.1) symbolically as

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi(s),$$

where $\phi(s) \equiv sF(s)$.

For $k = \pm m$, (1) reduces to the known Laplace transform, due to the identity

$$e^{-\frac{1}{2}st} = (st)^{-m-\frac{1}{2}} W_{m+\frac{1}{2}, m}(st),$$

i.e.

$$\phi(s) = s \int_0^\infty e^{-st} f(t) dt,$$

which we will denote as

$$\phi(s) \doteqdot f(t).$$

The object of this paper is to establish some integral representations of the image of Meijer transforms with the help of known integral representations of $W_{k, m}(z)$ and $M_{k, m}(z)$.

We have, in general, imposed stringent conditions on the results given in this paper. These conditions may be relaxed by the help of analytic-continuation.

Further, in the case of functions ${}_pF_q \left[\begin{matrix}\alpha_1, \alpha_2 \ldots \alpha_p \\ \beta_1, \ldots \beta_q\end{matrix} ; -z\right]$, $p = q+1$, we have made a cut from o to $-\infty$ in the z-plane, so that the function may be extended to the cut z-plane from $|z| < |$.

2. *Theorem 1. If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi(s)$

and $$\psi(s, u) \doteqdot t^{-k} K_{2m}(2u\sqrt{t}) f(t),$$

*then**

$$\phi(s) = \frac{4}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-u^2/s} u^{-2k-1} \psi(s, u) du, \tag{2.1}$$

provided

(i) *$f(t)$ is continuous for $t > 0$,*

(ii) *$R(s) \geqslant s_0 > 0$, and $R(-k \pm m) > 0$,*

(iii) *$R(\mu - k + 1 \pm m) > 0$, where $f(t) = 0(t^\mu)$ for small t,*

(iv) $\int_T^\infty \left| t^{-k} e^{-st} K_{2m}^{(2u\sqrt{t})} f(t) \right| dt$ tends to zero as $T \to \infty$,

for $R(s) \geqslant s_0 > 0$ and $u > 0$, and

(5) *the integral* (2.1) *is absolutely convergent.*

Proof. We have

$$\phi(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2}, m}^{(st)} f(t) dt \tag{2.2}$$

Using the integral (Meijer, 1941*c*, p. 299)

$$W_{k+\frac{1}{2}, m}(st) = \frac{4(st)^{\frac{1}{2}} e^{-\frac{1}{2}st}}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-u^2} u^{-2k-1} K_{2m}(2u\sqrt{st}) du,$$

where $s \neq 0$ and $R(-k \pm m) > 0$,
we get

$$\phi(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} \left[\frac{4(st)^{\frac{1}{2}} e^{-\frac{1}{2}st}}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-u^2} u^{-2k-1} K_{2m}(2u\sqrt{st}) du \right] f(t) dt.$$

Replacing $u\sqrt{s}$ by u we get

$$\phi(s) = \frac{4s}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-st} (st)^{-k} \left[\int_0^\infty e^{-u^2/S} \frac{u^{-2k-1}}{s^{-k-\frac{1}{2}}} K_{2m}(2u\sqrt{t}) \frac{du}{\sqrt{s}} \right] f(t) dt$$

$$= \frac{4}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-u^2/s} u^{-2k-1} \left[s \int_0^\infty e^{-st} t^{-k} K_{2m}(2u\sqrt{t}) f(t) dt \right] du \tag{2.3}$$

$$= \frac{4}{\Gamma \times (-k \pm m)} \int_0^\infty e^{-u^2/s} u^{-2k-1} \psi(u, s) du,$$

provided the change of order of integration is justified.

Regarding the change of order of integration in (2.3), we note that the u-integral is absolutely convergent, if $R(-k \pm m) > 0$, $R(s) \geqslant s_0 > 0$, and the t-integral is absolutely convergent, if $R(\mu - k + 1 \pm m) > 0$, where $f(t) = 0\ (t^\mu)$ for small t, $\int_T^\infty \left| t^{-k} e^{-st} K_{2m}(2u\sqrt{t}) f(t) \right| dt$ tends to zero as $T \to \infty$, for $R(s) \geqslant s_0 > 0$,

* The symbol $\Gamma \times\ (\alpha \pm \beta)$ denotes $\Gamma(\alpha + \beta)\ \Gamma(\alpha - \beta)$.

and $f(t)$ is continuous for $t > 0$, and the repeated integral is absolutely convergent, if (2.1) is absolutely convergent.

Hence the change of order of integration is justified by de la Vallée Poussin's theorem (Bromwich, 1931, p. 504).

2.1. *Theorem 2.* *If* $t^n f(t) \frac{k+\frac{1}{2}}{m} \phi_{n, k+\frac{1}{2}, m}(s)$,

then

$$\phi_{0, k+\frac{1}{2}, m}(s) = 2s^{\frac{1}{2}-k} \int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u) \tanh^{\frac{1}{2}-k} u\, \phi_{\frac{1}{2}-k, m+\frac{1}{2}, m}(s \cosh^2 u)\, du, \tag{2.4}$$

provided

(i) $R(s) \geqslant s_0 > 0$, $R(k) < 1$,

(ii) $R(\mu - k + \frac{3}{2}) > 0$, *where* $f(t) = 0(t^\mu)$ *for small* t,

(iii) $f(t)$ *is continuous for* $t > 0$,

(iv) $\int_T^\infty \left| e^{-st} t^{\frac{1}{2}-k} f(t) \right| dt \to o$ as $T \to \infty$, *for* $R(s) \geqslant s_0 > 0$, *and*

(v) *the integral* (2.4) *is absolutely convergent.*

Proof. We have (Meijer, 1941*b*, p. 600)

$$W_{k+\frac{1}{2}, m}(st) = 2(st) \int_0^\infty e^{-\frac{1}{2}st \cosh 2u} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u) \sinh^{\frac{1}{2}-k} u \cosh^{3/2+k} u\, du,$$

where $s \neq 0$, $|\arg s| < \pi/2$ and $R(k) < 1$.

Using this integral in (2.2), we get

$$\phi_{0, k+\frac{1}{2}, m}(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} \left[2st \int_0^\infty e^{-\frac{1}{2}st \cosh 2u} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u) \times \sinh^{\frac{1}{2}-k} u \cosh^{3/2+k} u\, du \right] f(t)\, dt$$

$$= 2s^{\frac{1}{2}-k} \int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u) \tanh^{\frac{1}{2}-k} u \left[s \cosh^2 u \int_0^\infty e^{-st \cosh^2 u} t^{\frac{1}{2}-k} f(t)\, dt \right] du$$

$$= 2s^{\frac{1}{2}-k} \int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u) \tanh^{\frac{1}{2}-k} u\, \phi_{\frac{1}{2}-k, m+\frac{1}{2}, m}(s \cosh^2 u)\, du,$$

provided the change of order of integration is justified.

This can easily be shown if we follow the method of Theorem 1.

We may note here that we can find the behaviour of $\phi_{\frac{1}{2}-k, m+\frac{1}{2}, m}(s \cosh^2 u)$ for large values of u by the help of Watson's lemma.

If $t^{\frac{1}{2}-k} f(t)$ satisfies the conditions of Watson's lemma and if $f(t) = 0(t^\mu)$ for small t, then, by Watson's lemma

$$\phi_{\frac{1}{2}-k, m+\frac{1}{2}, m}(s \cosh^2 u) = 0\{(s \cosh^2 u)^{-\mu+k-\frac{1}{2}}\} \text{ for large } \cosh^2 u,$$

$$= 0\{e^{-2(\mu-k+\frac{1}{2})u}\}, \text{ for large } u.$$

Example. *If* $f(t) = J_v(at)$, *then* * (Jaiswal, 1952, p. 389)

$$\phi_{n,\,k+\frac{1}{2},\,m}(s) = \frac{a^{\nu}\,\Gamma\times(\nu-k+n+1\pm m)}{2^{\nu}s^{\nu+n}\,\Gamma(\nu+n-2k+1)\Gamma(\nu+1)}\times$$

$$\times\, 4F_3\left[\begin{matrix}\dfrac{\nu+n-k+1\pm m}{2},\dfrac{\nu+n-k+2\pm m}{2} \\ \nu+1,\dfrac{\nu-2k+1+n}{2},\dfrac{\nu+n-2k+2}{2}\end{matrix};-\frac{a^2}{s^2}\right]$$

$R(\nu-k+n+1\pm m) > 0$ and $R(s) > 0$.

Using the values of $\phi_{0,\,k+\frac{1}{2},\,m}(s)$ and $\phi_{\frac{1}{2}-k,\,m+\frac{1}{2},\,m}(s\cosh^2 u)$ in (2.4), we get

$$\int_0^{\infty} P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh 2u)\tanh^{\frac{1}{2}-k}u\,\operatorname{sech}^{2\nu-2k+1}u\; 2F_1\left[\begin{matrix}\dfrac{\frac{3}{2}-k+\nu}{2},\dfrac{\frac{5}{2}-k+\nu}{2} \\ \nu+1\end{matrix};-\frac{a^2}{s^2\cosh^4 u}\right]du$$

$$=\frac{\Gamma\times(\nu+1-k\pm m)}{2\Gamma(\nu-2k+1)\Gamma(\nu-k+\frac{3}{2})}\,4F_3\left[\begin{matrix}\dfrac{\nu-k+1\pm m}{2},\dfrac{\nu-k+2\pm m}{2} \\ \nu+1,\dfrac{\nu-2k+1}{2},\dfrac{\nu-2k+2}{2}\end{matrix};-a^2/s^2\right],$$

$R(\nu-k+1\pm m) > 0,\ R(\nu-k+\frac{3}{2}) > 0,\ R(k) < 1$ and $R(s) \geqslant s_0 > 0$.

2.2. *Theorem 3.* *If* $t^n f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi_{n,\,k+\frac{1}{2},\,m}(s)$, *then*

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = 2s^{2\lambda}\int_0^{\infty} P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,\times$$
$$\times\cosh^{2k+2\lambda-1}u\;\phi_{2\lambda,\,k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(s\cosh^2 u)\,du, \tag{2.5}$$

provided

(i) $R(\lambda) > 0,\ R(s) \geqslant s_0 > 0$,

(ii) $\displaystyle\int_T^{\infty}\left|t^{2\lambda}e^{-st}f(t)\right|dt \to 0$ as $T\to\infty$, *for* $R(s)\geqslant s_0 > 0$,

(iii) $R(\lambda+\mu-k+1\pm\overline{\tfrac{1}{2}-\lambda}) > 0$ *where* $f(t) = 0\,(t^{\mu})$ *for small* t,

(iv) $f(t)$ *is continuous for* $t > 0$, *and*

(v) *the integral* (2.5) *is absolutely convergent.*

Proof. Using the integral (Meijer, 1941*b*, p. 599)

$$W_{k+\frac{1}{2},\,m}(st) = 2(st)^{\lambda}\int_0^{\infty} e^{-\frac{1}{2}st\sinh^2 u}\,W_{k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(st\cosh^2 u)P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,du,$$

where $s\neq 0$, $|\arg s| < \pi/2$ and $R(\lambda) > 0$,

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* The symbol $pFq\left[\begin{matrix}\alpha_1\pm\beta_1,\alpha_2\pm\beta_2\cdots\cdots \\ \gamma_1\pm\delta_1,\gamma_2\pm\delta_2\cdots\cdots\end{matrix};Z\right]$ denotes

$$pFq\left[\begin{matrix}\alpha_1+\beta_1,\alpha_1-\beta_1,\alpha_2+\beta_2,\alpha_2-\beta_2,\cdots\cdots \\ \gamma_1+\delta_1,\gamma_1-\delta_1,\gamma_2+\delta_2,\gamma_2-\delta_2\cdots\cdots\end{matrix};Z\right].$$

in (2.2), we get

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}f(t)$$

$$\times\left[2(st)^{\lambda}\int_0^\infty e^{-\frac{1}{2}st\sinh^2 u}W_{k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(st\cosh^2 u)P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,du\right]dt$$

$$= 2s^{2\lambda}\int_0^\infty P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\cosh^{2k+2\lambda-1}u$$

$$\times\left[s\cosh^2 u\int_0^\infty e^{-\frac{1}{2}st\cosh^2 u}(st\cosh^2 u)^{-k-\lambda-\frac{1}{2}}W_{k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(st\cosh^2 u)t^{2\lambda}f(t)\,dt\right]du$$

$$= 2s^{2\lambda}\int_0^\infty P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\cosh^{2k+2\lambda-1}u\,\phi_{2\lambda,\,k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(s\cosh^2 u)\,du,$$

provided the change of order of integration is justified.

This can easily be shown, if we follow the method of Theorem 1.

Example. If $f(t) = e^{-at}$, then (Jaiswal, 1952, p. 387)

$$\phi_{n,\,k+\frac{1}{2},\,m}(s) = \frac{1}{s^n}\frac{\Gamma\times(n-k+1\pm m)}{\Gamma(n-2k+1)}\,{}_2F_1\left[\begin{matrix}n-k+1\pm m\\ n-2k+1\end{matrix};-\frac{a}{s}\right],$$

$R(n-k+1\pm m) > 0$ and $R(s) > 0$.

Substituting the values of $\phi_{0,\,k+\frac{1}{2},\,m}(s)$ and $\phi_{2\lambda,\,k+\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(s\cosh^2 u)$ in (2.5), we get

$$\int_0^\infty P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\cosh^{2k-2\lambda-1}u\,{}_2F_1\left[\begin{matrix}\lambda-k+1\pm\overline{\tfrac{1}{2}-\lambda}\\ 1-2k\end{matrix};-\frac{a}{s\cosh^2 u}\right]du$$

$$= \frac{1}{2}\frac{\Gamma\times(1-k\pm m)}{\Gamma\times(\lambda-k+1\pm\frac{1}{2}-\lambda)}\,{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};-\frac{a}{s}\right],$$

$R(1-k\pm m) > 0$, $R(\lambda-k+1\pm\overline{\tfrac{1}{2}-\lambda}) > 0$, $R(\lambda) > 0$, $R(s+a) > 0$ and $R(s) > 0$.

2.3. *Theorem 4. If* $t^n f(t)\xrightarrow[m]{k+\frac{1}{2}}\phi_{n,\,k+\frac{1}{2},\,m}(s)$, *then*

(2.6) $$\phi_{0,\,k+\frac{1}{2},\,m}(s) = (2s)^{2\lambda}\int_0^\infty P^{1-2\lambda}_{2m-\frac{1}{2}}(\cosh u)\sinh^{2\lambda}u\cosh^{2k+2\lambda-3/2}u\,\times$$

$$\times\phi_{2\lambda,\,k+\lambda+\frac{1}{2},\,\pm\frac{1}{4}}(s\cosh^2 u)du,$$

provided

(i) $R(\lambda) > 0$ *and* $R(s) \geqslant s_0 > 0$,

(ii) $f(t)$ *is continuous for* $t > 0$,

(iii) $R(\mu+\lambda-k+\frac{3}{4}) > 0$, *where* $f(t) = 0\,(t^{\mu})$ *for small* t,

(iv) $\int_T^\infty \left| e^{-st} t^{2\lambda} f(t) \right| dt \to 0$ *as* $T \to \infty$ *for* $R(s) \geqslant s_0 > 0$, *and*

(v) *the integral* (2.6) *is absolutely convergent.*

Proof. We have (Meijer, 1941*b*, p. 600)

$$W_{k+\frac{1}{2}, m}(st) = 2^{\lambda-k-\frac{1}{4}} (st)^{\lambda+\frac{1}{4}} \int_0^\infty e^{-\frac{1}{2}st \sinh^2 u} D_{2k+2\lambda+\frac{1}{2}} (\sqrt{2st \cosh^2 u}) \times$$
$$\times P^{1-2\lambda}_{2m-\frac{1}{2}} (\cosh u) \sinh^{2\lambda} u \, du,$$

$s \neq 0$, $|\arg s| < \pi/2$ and $R(\lambda) > 0$.
Using this in (2.2), we get

$$\phi_{0, k+\frac{1}{2}, m}(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} \left[2^{\lambda-k-\frac{1}{4}} (st)^{\lambda+\frac{1}{4}} \int_0^\infty e^{-\frac{1}{2}st \sinh^2 u} \times \right.$$
$$\left. \times D_{2k+2\lambda+\frac{1}{2}}(\sqrt{2st \cosh^2 u}) \, P^{1-2\lambda}_{2m-\frac{1}{2}} (\cosh u) \sinh^{2\lambda} u du \right] f(t) dt,$$
$$= 2^{\lambda-k-\frac{1}{4}} s^{2\lambda} \int_0^\infty P^{1-2\lambda}_{2m-\frac{1}{2}} (\cosh u) \sinh^{2\lambda} u \cosh^{2k+2\lambda+\frac{1}{2}} u \left[s \int_0^\infty e^{-\frac{1}{2}st \cosh^2 u} \times \right.$$
$$\left. \times (st \cosh^2 u)^{-k-\lambda-\frac{1}{4}} D_{2k+2\lambda+\frac{1}{2}} \{(2ts \cosh^2 u)^{\frac{1}{2}}\} t^{2\lambda} f(t) dt \right] du$$
$$= (2s)^{2\lambda} \int_0^\infty P^{1-2\lambda}_{2m-\frac{1}{2}} (\cosh u) \sinh^{2\lambda} u \cosh^{2k+2\lambda-3/2} u \, \phi_{2\lambda, k+\lambda+\frac{1}{2}, \pm\frac{1}{4}} (s \cosh^2 u) du,$$

provided the change of order of integration is justified.
This can be shown if we follow the method of Theorem 1.

Example. We have (Jaiswal, 1952, p. 132)

$$t^n H_\nu(2t) \xrightarrow[m]{k+\frac{1}{2}} \frac{\Gamma_\chi(\nu+n-k+2\pm m)}{s^{\nu+n+1} \Gamma(\frac{3}{2}) \Gamma(\nu+\frac{3}{2}) \Gamma(\nu+n-2k+2)} \times$$
$$\times {}_5F_4 \left[\begin{matrix} \frac{1}{2}(\nu+n-k+2\pm m), \frac{1}{2}(\nu+n-k+3\pm m), 1 \\ \frac{3}{2}, \frac{1}{2}(\nu+n-2k+2), \frac{1}{2}(\nu+n-2k+3), \nu+\frac{3}{2} \end{matrix} ; -\frac{4}{s^2} \right],$$

$R(\nu+n-k+2\pm m) > 0$ and $R(s) > 0$.
Substituting the values of $\phi_{0, k+\frac{1}{2}, m}(s)$ and $\phi_{2\lambda, k+\lambda+\frac{1}{2}, \frac{1}{4}}(s \cosh^2 u)$ in the integral (2.6), we get

$$\int_0^\infty P^{1-2\lambda}_{2m-\frac{1}{2}} (\cosh u) \sinh^{2\lambda} u \cosh^{2k-2\lambda-2\nu-7/2} u \times$$
$$\times {}_5F_4 \left[\begin{matrix} \frac{1}{2}(\nu+\lambda-k+2\pm\frac{1}{4}), \frac{1}{2}(\nu+\lambda-k+3\pm\frac{1}{4}), 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{1}{2}(\nu-2k+2), \frac{1}{2}(\nu-2k+3) \end{matrix} ; -\frac{4}{s^2 \cosh^4 u} \right] du$$
$$= \frac{\Gamma_\chi(\nu-k+2\pm m)}{\Gamma_\chi(\nu+\lambda-k+2\pm\frac{1}{4})} {}_5F_4 \left[\begin{matrix} \frac{\nu-k+2\pm m}{2}, \frac{\nu-k+3\pm m}{2}, 1 \\ \frac{3}{2}, \frac{1}{2}(2\nu+3), \frac{1}{2}(\nu-2k+2), \frac{1}{2}(\nu-2k+3) \end{matrix} ; -\frac{4}{s^2} \right]$$

$R(\lambda) > 0$, $R(\nu-k+2\pm m) > 0$, $R(\lambda-k+\nu+7/4) > 0$ and $R(s) > 0$.

2.4. *Theorem 5. If* $t^n f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi_{n,\,k+\frac{1}{2},\,m}(s)$, *then*

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = 2\sqrt{\frac{s}{\pi}}\int_0^\infty \cosh 2mu \cosh^{2k-1}u\, \phi_{\frac{1}{2},\,k+3/2,\,\pm\frac{1}{2}}(s\cosh^2 u)\, du, \tag{2.7}$$

provided

(i) $R(s) \geqslant s_0 > 0$,

(ii) $R(\mu-k+1) > 0$, *where* $f(t) = 0\,(t^\mu)$ *for small* t,

(iii) $f(t)$ *is continuous for* $t > 0$,

(iv) $\int_T^\infty |e^{-st}t^{\frac{1}{2}}f(t)|\,dt \to 0$ *as* $T \to \infty$ *for* $R(s) \geqslant s_0 > 0$, *and*

(v) *the integral* (2.7) *is absolutely convergent.*

Proof. We have (Meijer, 1941*b*, p. 599)

$$W_{k+\frac{1}{2},\,m}(st) = \frac{2^{\frac{1}{2}-k}(st)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\int_0^\infty e^{-\frac{1}{2}st\sinh^2 u} D_{2k+1}(\sqrt{2st\cosh^2 u})\cosh 2mu\, du,$$

$s \neq 0$, $|\arg s| < \pi/2$.
Using this in (2.2), we get

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\left[\frac{2^{\frac{1}{2}-k}(st)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\int_0^\infty e^{-\frac{1}{2}st\sinh^2 u}\times\right.$$
$$\left.\times D_{2k+1}(\sqrt{2st\cosh^2 u})\cosh 2mu\, du\right] f(t)\, dt$$

$$= 2\sqrt{\frac{s}{\pi}}\int_0^\infty \cosh 2mu\cosh^{2k-1}u\left[2^{-k-\frac{1}{2}}s\cosh^2 u\int_0^\infty e^{-\frac{1}{2}st\cosh^2 u}\times\right.$$
$$\left.\times(st\cosh^2 u)^{-k-\frac{1}{2}}D_{2k+1}(\sqrt{2st\cosh^2 u})t^{\frac{1}{2}}f(t)\, dt\right] du$$

$$= 2\sqrt{\frac{s}{\pi}}\int_0^\infty \cosh 2mu\cosh^{2k-1}u\, \phi_{\frac{1}{2},\,k+\frac{3}{2},\,\pm\frac{1}{2}}(s\cosh^2 u)\, du,$$

provided the change of order of integration is justified.
This can be easily shown if we follow the method of Theorem 1.

Example. Let $f(t) = J_\mu(2at)\,J_\nu(2at)$.
We know (Jain, 1955, p. 132)

$$\phi_{n,\,k+\frac{1}{2},\,m}(s) = \frac{\Gamma_{\mathrm{X}}(n+\mu+\nu-k+1\pm m)a^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+n-2k+1)s^{n+\mu+\nu}}\times$$

$$\times{}_6F_5\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2),\ \frac{1}{2}(n+\mu+\nu-k+1\pm m),\ \frac{1}{2}(n+\mu+\nu-k+2\pm m)\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{1}{2}(\mu+\nu+n-2k+1),\ \frac{1}{2}(\mu+\nu+n-2k+2)\end{matrix};\frac{-16a^2}{s^2}\right]$$

$R(n+\mu+\nu-k+1\pm m) > 0$ and $R(s) > 0$.

Substituting the values of $\phi_{0,\,k+\frac{1}{2},\,m}(s)$ and $\phi_{\frac{1}{2},\,k+3/2,\,+\frac{1}{4}}(s\cosh^2 u)$ in (2.7) we get

$$\int_0^\infty \cosh 2mu \cosh^{2(k-\mu-\nu-1)} u \times$$

$$\times {}_6F_5\left[\begin{matrix} \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2),\ \frac{1}{2}(\mu+\nu-k+\frac{5}{4}\pm\frac{1}{4}),\ \frac{\mu+\nu-k+\frac{9}{4}\pm\frac{1}{4}}{2} \\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{\mu+\nu-2k+1}{2},\ \frac{\mu+\nu-2k+2}{2};\ \frac{-16a^2}{s^2\cosh^4 u}\end{matrix}\right] du$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma\mathrm{x}(\mu+\nu-k+1\pm m)}{\Gamma\mathrm{x}(\mu+\nu-k+\frac{5}{4}\pm\frac{1}{4})} \times$$

$$\times {}_6F_5\left[\begin{matrix} \frac{\mu+\nu+1}{2},\ \frac{\mu+\nu+2}{2},\ \frac{\mu+\nu-k+1\pm m}{2},\ \frac{\mu+\nu-k+2\pm m}{2} \\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{\mu+\nu-2k+1}{2},\ \frac{\mu+\nu-2k+2}{2}\end{matrix};\ -\frac{16a^2}{s^2}\right],$$

$R(\mu+\nu-k+1) > 0$, $R(\mu+\nu-k+1\pm m) > 0$ and $R(s) > 0$.

2.5. *Theorem 6. If* $t^r f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi_{r,\,k+\frac{1}{2},\,m}(s)$, *then*

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = \frac{2^{2k+2}\,\Gamma(-2k)}{\Gamma\mathrm{x}(-k\pm m)}\int_0^\infty \cosh 2mu \cosh^{2(k-1)} u\ \psi(s,u)\, du. \tag{2.8}$$

where

$$\psi(s,u) = s\cosh^2 u\int_0^\infty \exp(-\tfrac{1}{2}st\cosh^2 u + st\sinh^2 u)(2st\cosh^2 u)^{-k}\times$$

$$\times D_{2k}(\sqrt{2st\cosh^2 u})\, f(t)\, dt$$

$$= \sum_{r=0}^{\infty}\frac{s^r\sinh^{2r} u}{r!}\,\phi_{r,\,k+\frac{1}{2},\,\pm\frac{1}{4}}(s\cosh^2 u),$$

provided

(i) $R(-k\pm m) > 0$ *and* $R(s) \geqslant s_0 > 0$,

(ii) $R(\mu-k+1) > 0$, *where* $f(t) = 0\,(t^\mu)$ *for small* t,

(iii) $e^{-R(s)t}|f(t)| \to 0$ *as* $t\to\infty$ *for* $R(s) \geqslant s_0 > 0$,

(iv) $f(t)$ *is continuous for* $t > 0$, *and*

(v) *the integral* (2.8) *is absolutely and uniformly convergent.*

Proof. We have (Meijer, 1941*b*, p. 601)

$$W_{k+\frac{1}{2},\,m}(st) = \frac{2^{k+2}(st)^{\frac{1}{2}}\,\Gamma(-2k)}{\Gamma\mathrm{x}(-k\pm m)}\int_0^\infty e^{\frac{1}{2}st\sinh^2 u}\, D_{2k}(\sqrt{2st\cosh^2 u})\cosh 2mu\, du$$

$s\neq 0$, $|\arg s| > 3\pi/2$ and $R(-k\pm m) > 0$.

Using this integral in (2.2), we get

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = s\int_0^\infty e^{-\frac{1}{2}(st)}(st)^{-k-\frac{1}{2}}\left[\frac{2^{k+2}(st)^{\frac{1}{2}}\,\Gamma(-2k)}{\Gamma(-k\pm m)}\int_0^\infty e^{\frac{1}{2}st\sinh^2 u}\times\right.$$

$$\left.\times D_{2k}(\sqrt{2st\cosh^2 u})\cosh 2mu\,du\right]f(t)\,dt$$

$$=\frac{2^{2(k+1)}\,\Gamma(-2k)}{\Gamma x(-k\pm m)}\int_0^\infty \cosh 2mu\cosh^{2k}u\left[s\int_0^\infty e^{-\frac{1}{2}st\cosh^2 u}\times\right.$$

$$\left.\times(2st\cosh^2 u)^{-k}D_{2k}(\sqrt{2st\cosh^2 u})\,e^{st\sinh^2 u}f(t)\,dt\right]du$$

$$=\frac{2^{2(k+1)}\,\Gamma(-2k)}{\Gamma x(-k\pm m)}\int_0^\infty \cosh 2mu\cosh^{2(k-1)}u\,\psi(s,u)\,du,$$

provided the change of order of integration is justified.

Now expanding $e^{st\sinh^2 u}$ and changing the order of integration and summation, we get the required result.

The change of order of integration and summation easily follows if we note the following:

(i) the series $e^{st\sinh^2 u} = \sum_{r=0}^{\infty}\frac{(st\sinh^2 u)^r}{r!}$ is uniformly convergent in any fixed interval $o < t \leqslant a$, a being arbitrary,

(ii) $f(t)$ is continuous for all values of $t > 0$,

(iii) the integral $\int_0^\infty e^{-\frac{1}{2}R(s)\cosh^2 u}\left|t^{r-k}f(t)D_{2k}(\sqrt{2st\cosh^2 u})\right|dt$ is uniformly convergent if $R(\mu-k+1) > 0$ and $e^{-R(s)t}|f(t)|\to 0$ as $t\to\infty$ for $R(s) \geqslant s_0 > 0$, and

(iv) the integral $\int_0^\infty \exp\{-\frac{1}{2}st(1-\sinh^2 u)\}\,|D_{2k}(\sqrt{2st\cosh^2 u})t^{-k}f(t)|\,dt$ is uniformly convergent under the above conditions.

2.6. *Theorem 7. If* $t^r f(t)\xrightarrow[m]{k+\frac{1}{2}}\phi_{r,\,k+\frac{1}{2},\,m}(s)$, *then*

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = \frac{2\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}{\Gamma x(-k\pm m)}\int_0^\infty P_{m-\frac{1}{2}}^{1-2\lambda}(\cosh 2u)\sinh^{2\lambda}u\times \tag{2.9}$$

$$\times\cosh^{2k-2\lambda-1}u\,G(s,u)\,du,$$

where

$$G(s,u) = s\cosh^2 u\int_0^\infty \exp\{-\tfrac{1}{2}st(\cosh^2 u-2\sinh^2 u)\}(st\cosh^2 u)^{\lambda-k-\frac{1}{2}}\times$$

$$\times W_{k-\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(st\cosh^2 u)f(t)\,dt$$

$$=\sum_{r=0}^{\infty}\frac{s^r\sinh^{2r}u}{r!}\,\phi_{r,\,k-\lambda+\frac{1}{2},\,\frac{1}{2}-\lambda}(s\cosh^2 u),$$

provided

(i) $R(s) \geqslant s_0 > 0,\ R(-k\pm m) > 0,\ R(\lambda) > 0,$

(ii) $R(\mu-k+\lambda+1\pm\overline{\tfrac{1}{2}-\lambda}) > 0,$ *where* $f(t) = 0(t^{\mu}),$

(iii) $f(t)$ *is continuous for* $t > 0,$

(iv) $e^{-R(s)t}\,|f(t)| \to 0$ *as* $t\to\infty$ *for* $R(s) \geqslant s_0 > 0,$ *and*

(v) *the integral* (2.9) *is absolutely and uniformly convergent.*

Proof. We have (Meijer, 1941*b*, p. 601)

$$W_{k+\frac{1}{2},m}(st) = \frac{2(st)^{\lambda}\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}{\Gamma\mathrm{x}(-k\pm m)}\int_0^{\infty} e^{\frac{1}{2}st\sinh^2 u}\, W_{k-\lambda+\frac{1}{2},\frac{1}{2}-\lambda}(st\cosh^2 u)\times$$
$$\times P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,du$$

$s\neq 0$, $|\arg s| < 3\pi/2$, $R(-k\pm m) > 0$ and $R(\lambda) > 0$.
Using this in (2.2), we get

$$\phi_{0,k+\frac{1}{2},m}(s) = s\int_0^{\infty} e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\left[\frac{2(st)^{\lambda}\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}{\Gamma\mathrm{x}(-k\pm m)}\int_0^{\infty} e^{\frac{1}{2}st\sinh^2 u}\times\right.$$
$$\left.\times W_{k-\lambda+\frac{1}{2},\frac{1}{2}-\lambda}(st\cosh^2 u)\,P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,du\right] f(t)\,dt$$
$$= \frac{2\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}{\Gamma\mathrm{x}(-k\pm m)}\int_0^{\infty} P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\cosh^{2k-2\lambda-1}u\,\sinh^{2\lambda}u\times$$
$$\times\left[s\cosh^2 u\int_0^{\infty} e^{-\frac{1}{2}st\cosh^2 u}(st\cosh^2 u)^{-k+\lambda-\frac{1}{2}}\,W_{k-\lambda+\frac{1}{2},\frac{1}{2}-\lambda}(st\cosh^2 u)\times\right.$$
$$\left.\times e^{st\sinh^2 u} f(t)\,dt\right] du$$
$$= \frac{2\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}{\Gamma\mathrm{x}(-k\pm m)}\int_0^{\infty} P^{1-2\lambda}_{m-\frac{1}{2}}(\cosh 2u)\sinh^{2\lambda}u\,\cosh^{2k-2\lambda-1}u\,G(s,u)\,du.$$

Now, expanding $e^{st\sinh^2 u}$ and changing the order of integration and summation the result follows, provided the change of order of integration and the change of order of integration and summation is justified.

If we follow the methods of Theorems 1 and 6, these changes can easily be justified.

Example. If $f(t) = t^n$, then (Jaiswal, 1952, p. 387)

$$\phi_{r,k+\frac{1}{2},m}(s) = \frac{\Gamma\mathrm{x}(n+r-k+1\pm m)}{\Gamma(n+r-2k+1)s^{n+r}};$$

and

$$G(s,u) = s\cosh^2 u\int_0^{\infty} e^{-\frac{1}{2}st\cosh^2 u}(st\cosh^2 u)^{\lambda-k-\frac{1}{2}}\,W_{k-\lambda+\frac{1}{2},\frac{1}{2}-\lambda}(st\cosh^2 u)\times$$
$$\times e^{st\sinh^2 u}\,t^n\,dt,\ R(n-k+\lambda+1\pm\overline{\tfrac{1}{2}-\lambda}) > 0 \text{ and } R(s) > 0.$$

Substituting the values of $\phi_{0,\, k+\frac{1}{2},\, m}(s)$ in (2.9) we get

$$\int_0^\infty P_{m-\frac{1}{2}}^{1-2\lambda}(\cosh 2u)\sinh^{2\lambda}u \cosh^{2(k-\lambda-\frac{1}{2})}u\, G(s,u)\,du$$
$$= \frac{\Gamma\mathrm{x}(n-k+1\pm m)\Gamma\mathrm{x}(-k\pm m)s^{-n}}{2\Gamma(n-2k+1)\Gamma(\frac{1}{2}-k)\Gamma(2\lambda-k-\frac{1}{2})}$$

$$R(n-k+\lambda+1\pm\overline{\tfrac{1}{2}-\lambda})>0,\quad R(-k\pm m)>0,\quad R(\lambda)>0 \text{ and } R(s)>0.$$

2.7. *Theorem 8. If* $t^r f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi_{r,\, k+\frac{1}{2},\, m}(s)$, *then*

$$\phi_{0,\, k+\frac{1}{2},\, m}(s) = \frac{\Gamma\mathrm{x}(-k+\lambda\pm\mu)}{\Gamma(2\lambda)\Gamma\mathrm{x}(-k\pm m)}\int_1^\infty {}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right]\times$$
$$\times(u-1)^{2\lambda-1}u^{\mu+k-\lambda-1}G_1(s,u)\,du, \tag{2.10}$$

where

$$G_1(s,u) = \sum_{r=0}^{\infty}\frac{s^r(u-1)^r}{r!}\phi_{r,\, k-\lambda+\frac{1}{2},\, \mu}(su),$$

provided

(i) $R(-k\pm m)>0$, $R(\lambda)>0$ *and* $R(s)\geqslant s_0>0$,

(ii) $R(\mu_1-k+\lambda+1\pm\mu)>0$ *where* $f(t)=0(t^{\mu_1})$ *for small t*,

(iii) $e^{-R(s)t}|f(t)|\to 0$ *as* $t\to\infty$ *for* $R(s)\geqslant s_0>0$,

(iv) $f(t)$ *is continuous for* $t>0$, *and*

(v) *the integral* (2.10) *is absolutely and uniformly convergent.*

Proof. We have (Meijer, 1941*b*, p. 601)

$$W_{k+\frac{1}{2},\, m}(st) = \frac{(st)^\lambda e^{-\frac{1}{2}st}\Gamma\mathrm{x}(-k+\lambda\pm\mu)}{\Gamma\mathrm{x}(-k\pm m)\Gamma(2\lambda)}\int_1^\infty e^{\frac{1}{2}stu}W_{k-\lambda+\frac{1}{2},\, \mu}(stu)\times$$
$$\times{}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right](u-1)^{2\lambda-1}u^{\mu-\frac{1}{2}}\,du,$$

$s\neq 0$, $|\arg s|<3\pi/2$, $R(-k\pm m)>0$ and $R(\lambda)>0$.
Using this in (2.2), we get

$$\phi_{0,\, k+\frac{1}{2},\, m}(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\left\{\frac{(st)^\lambda e^{-\frac{1}{2}st}\Gamma\mathrm{x}(-k+\lambda\pm\mu)}{\Gamma\mathrm{x}(-k\pm m)\Gamma(2\lambda)}\int_1^\infty e^{\frac{1}{2}stu}W_{k-\lambda+\frac{1}{2},\, \mu}(stu)\times\right.$$
$$\left.\times{}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right](u-1)^{2\lambda-1}u^{\mu-\frac{1}{2}}\,du\right\}f(t)\,dt$$
$$= \frac{\Gamma\mathrm{x}(-k+\lambda\pm\mu)}{\Gamma\mathrm{x}(-k\pm m)\Gamma(2\lambda)}\int_1^\infty {}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right](u-1)^{2\lambda-1}u^{\mu+k-\lambda-1}\times$$
$$\times\left[su\int_0^\infty e^{-\frac{1}{2}stu}(stu)^{-k+\lambda-\frac{1}{2}}W_{k-\lambda+\frac{1}{2},\, \mu}(stu)e^{-(1-u)st}f(t)\,dt\right]du.$$

Expanding $e^{(u-1)st}$ and changing the order of integration and summation, we get

$$\phi_{0,\,k+\frac{1}{2},\,m}(s) = \frac{\Gamma\chi(-k+\lambda\pm\mu)}{\Gamma(2\lambda)\Gamma\chi(-k\pm m)} \int_1^\infty {}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right] \times$$

$$\times (u-1)^{2\lambda-1} u^{\mu+k-\lambda-1} G_1(s,u)\,du,$$

provided the change of order of integration and change of order of integration and summation is justified. This can easily be done if we follow the method of Theorems 1 and 6 respectively.

Example. If $f(t) = t^l$, then

$$\phi_{r,\,k+\frac{1}{2},\,m}(s) = \frac{\Gamma\chi(r+l-k+1\pm m)}{\Gamma(r+l-2k+1)s^{r+l}},\ R(l-k+1\pm m) > 0 \text{ and } R(s) > 0.$$

Substituting the value of $\phi_{r,\,k-\lambda+\frac{1}{2},\,\mu}(s)$ in the expression for $G_1(s,u)$, we get

$$G_1(s,u) = \frac{\Gamma\chi(l-k+\lambda+1\pm\mu)}{\Gamma(l+2\lambda-2k+1)s^l u^l}\,{}_2F_1\left[\begin{matrix}l-k+\lambda+1\pm\mu\\ l+2\lambda-2k+1\end{matrix};\frac{u-1}{u}\right],$$

$$R(l-k+\lambda+1\pm\mu) > 0,\ R(s) > 0 \text{ and } u > 1.$$

(2.10) then gives

$$\int_1^\infty {}_2F_1\left[\begin{matrix}\lambda+\mu\pm m\\ 2\lambda\end{matrix};1-u\right] {}_2F_1\left[\begin{matrix}l-k+\lambda+1\pm\mu\\ l+2\lambda-2k+1\end{matrix};\frac{u-1}{u}\right] \times$$

$$\times (u-1)^{2\lambda-1} u^{\mu-l+k-\lambda-1}\,du$$

$$= \frac{\Gamma\chi(l-k+1\pm m)\Gamma(-k\pm m)\Gamma(l+2\lambda-2k+1)\Gamma(2\lambda)}{\Gamma\chi(l-k+\lambda+1\pm\mu)\Gamma(l-2k+1)\Gamma\chi(-k+\lambda\pm\mu)},\ R(-k\pm m) > 0,$$

$$R(\lambda) > 0,\ R(l-k+\lambda+1\pm\mu) > 0,\ R(l-k+1\pm m) > 0 \text{ and } R(s) > 0.$$

2.8. *Theorem 9. If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi(s)$, *then*

$$(2.11)\quad \phi(s) = \frac{\pi^{3/2} s^{-k/2}}{\Gamma\chi(-k\pm m)\sin 2m\pi} \int_0^\infty e^{-\frac{1}{2}\frac{u}{s}} u^{-\frac{k}{2}-1} W_{\frac{1}{2}-\frac{1}{2}k,\,\frac{1}{2}+\frac{1}{2}k}(u/s)\,\chi(u,s)\,du,$$

where

$$\chi(u,s) \doteqdot \left\{I^2_{-m}(\sqrt{ut}) - I^2_m(\sqrt{ut})\right\} t^{-k} f(t),$$

provided

(i) $R(-k\pm m) > 0,\ R(1\pm m) > 0,\ R(s) \geqslant s_0 > 0,$

(ii) $R(\mu-k+1\pm m) > 0$ *where* $f(t) = 0(t^\mu)$ *for small* t,

(iii) $\int_T^\infty |e^{-st} t^{-k} f(t)|\,dt \to 0$ *as* $T\to\infty$ *for* $R(s) \geqslant s_0 > 0$,

(iv) $f(t)$ *is continuous for* $t > 0$, *and*

(v) *the integral* (2.11) *converges absolutely.*

Proof. We have (Meijer, 1941*d*, p. 448)

$$W_{k+\frac{1}{2},m}(st) = \frac{2\pi^{3/2}(st)^{\frac{1}{2}}e^{-\frac{1}{2}st}}{\Gamma\chi(-k\pm m)\sin 2m\pi}\int_0^\infty e^{-\frac{1}{2}u^2}u^{-k-1}$$

$$W_{\frac{1}{2}-\frac{1}{2}k,\frac{1}{2}+\frac{1}{2}k}(u^2)\left\{I^2_{-m}(u\sqrt{st})-I^2_m(u\sqrt{st})\right\}du,$$

where $s \neq 0$, $R(1\pm m) > 0$ and $R(-k\pm m) > 0$.
Replacing u^2s by u and substituting in (2.2), we get

$$\phi(s) = s\int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\left[\frac{\pi^{3/2}e^{-\frac{1}{2}st}(st)^{\frac{1}{2}}s^{k/2}}{\Gamma\chi(-k\pm m)\sin 2m\pi}\int_0^\infty e^{-\frac{1}{2}\frac{u}{s}}u^{-\frac{k}{2}-1}\times\right.$$

$$\left.\times W_{\frac{1}{2}-\frac{k}{2},\frac{1}{2}+\frac{k}{2}}(u|s)\left\{I^2_{-m}(\sqrt{ut})-I^2_m(\sqrt{ut})\right\}du\right]f(t)\,dt$$

$$= \frac{\pi^{3/2}s^{-\frac{k}{2}}}{\Gamma\chi(-k\pm m)\sin 2m\pi}\int_0^\infty e^{-\frac{1}{2}\frac{u}{s}}u^{-\frac{k}{2}-1}W_{\frac{1}{2}-\frac{1}{2}k,\frac{1}{2}+\frac{1}{2}k}(u|s)\,du\times$$

$$\times\left[s\int_0^\infty e^{-st}\left\{I^2_{-m}(\sqrt{ut})-I^2_m(\sqrt{ut})\right\}t^{-k}f(t)\,dt\right]$$

$$= \frac{\pi^{3/2}s^{-\frac{k}{2}}}{\Gamma\chi(-k\pm m)\sin 2m\pi}\int_0^\infty e^{-\frac{1}{2}\frac{u}{s}}u^{-\frac{k}{2}-1}W_{\frac{1}{2}-\frac{1}{2}k,\frac{1}{2}+\frac{1}{2}k}(u/s)\,\chi(u,s)\,du,$$

provided the change of order of integration is justified.
This can be shown if we follow the method of Theorem 1.

2.9. *Theorem 10. If* $f(t)\xrightarrow[m]{k+\frac{1}{2}}\phi(s)$, *and*

$$\chi_1(u,s) = \int_0^\infty K_m(st\cosh u)t^{1-k}f(t)\,dt, \text{ then}$$

$$\phi(s) = \sqrt{\frac{2}{\pi}}s^{2-k}\int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh u)\sinh^{\frac{1}{2}-k}u\cosh^{1+k}u\,\chi_1(u,s)\,du, \tag{2.12}$$

provided

(i) $R(k) < 1$ *and* $R(s) \geqslant s_0 > 0$,

(ii) $R(\mu-k+2\pm m) > 0$, *where* $f(t) = 0(t^\mu)$ *for small* t,

(iii) $\int_T^\infty |e^{-st}t^{\frac{1}{2}-k}f(t)|\,dt \to 0$ *as* $T\to\infty$ *for* $R(s)\geqslant s_0 > 0$,

(iv) $f(t)$ *is continuous for* $t > 0$, *and*

(v) *the integral* (2.12) *is absolutely convergent.*

Proof. We have (Meijer, 1941*b*, p. 603)

$$W_{k+\frac{1}{2},m}(st) = \frac{2^{\frac{1}{2}}(st)^{3/2}e^{\frac{1}{2}st}}{\pi^{\frac{1}{2}}}\int_0^\infty K_m(st\cosh u)P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh u)\sinh^{\frac{1}{2}-k}u\cosh^{1+k}u\,du$$

$$s \neq 0, |\arg s| < \pi/2 \text{ and } R(k) < 1.$$

Using this in (2.2), we get

$$\phi(s) = s\int_0^\infty (st)^{1-k}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty K_m(st\cosh u)P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh u)\sinh^{\frac{1}{2}-k}u\cosh^{1+k}u\,du\right]f(t)\,dt$$

$$= \sqrt{\frac{2}{\pi}}s^{2-k}\int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh u)\sinh^{\frac{1}{2}-k}u\cosh^{1+k}u\left[\int_0^\infty K_m(st\cosh u)t^{1-k}f(t)\,dt\right]du$$

$$= \sqrt{\frac{2}{\pi}}s^{2-k}\int_0^\infty P^{k+\frac{1}{2}}_{m-\frac{1}{2}}(\cosh u)\sinh^{\frac{1}{2}-k}u\cosh^{1+k}u\,\chi_1(u,s)\,du,$$

provided the change of order of integration is justified. This can be justified if we follow the method of Theorem 1.

3. *Theorem 11. If* $f(t)\xrightarrow[m]{k+\frac{1}{2}}\phi(s)$, *and*

$$h(u) \doteqdot s^{k+m}\phi(s), \text{ then}$$

(3.1)
$$f(t) = \frac{t^{2k-1}}{\Gamma(m+k)}\int_0^t (1-u|t)^{m+k-1}u^{m-k}h(u)\,du,$$

provided

(i) $R\left(\frac{1}{2}+m\pm\overline{k-\frac{1}{2}}\right) > 0$, $R(s) \geqslant s_0 > 0$,

(ii) *$h(u)$ is regular for $0 < u < t$, and*

(iii) *the integral (3.1) is absolutely convergent.*

Proof. We have (Watson, 1952, p. 352)

$$M_{k,m}(z) = \frac{\Gamma(1+2m)z^{m+\frac{1}{2}}2^{-2m}}{\Gamma\text{x}(\frac{1}{2}+m\pm k)}\int_{-1}^1 (1+u)^{-\frac{1}{2}+m-k}(1-u)^{-\frac{1}{2}+m+k}e^{\frac{1}{2}zu}\,du,$$

Using this in (1.2), we get

$$f(t) = \lim_{\lambda\to\infty}\frac{\Gamma(1-k\pm m)}{\Gamma(1+2m)2\pi i}\int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st}(st)^{k-\frac{1}{2}}\frac{\phi(s)}{s}\left[\frac{\Gamma(1+2m)2^{-2m}(st)^{m+\frac{1}{2}}}{\Gamma(m+k)\Gamma(1+m-k)}\times\right.$$

$$\left.\times\int_{-1}^1 (1+u)^{m-k}(1-u)^{m+k-1}e^{\frac{1}{2}stu}\,du\right]ds.$$

Replacing $t(1+u)$ by $2u$, we get

$$f(t) = \lim_{\lambda\to\infty} \frac{2^{-m+k}t^{-1}}{\Gamma(m+k)2\pi i}\int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st}(st)^{k+m}\frac{\phi(s)}{s}\left[\int_0^t \left(\frac{2u}{t}\right)^{m-k}\left(1-\frac{u}{t}\right)^{m+k-1}\times e^{\frac{1}{2}st\left(\frac{2u}{t}-1\right)}du\right]ds$$

$$= \frac{t^{2k-1}}{\Gamma(m+k)}\int_0^t \left(1-\frac{u}{t}\right)^{m+k-1} u^{m-k}\left[\frac{1}{2\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} e^{su}s^{k+m}\frac{\phi(s)}{s}\,ds\right]du$$

$$= \frac{t^{2k-1}}{\Gamma(m+k)}\int_0^t (1-u|t)^{m+k-1}u^{m-k}h(u)\,du$$

where $h(u) \doteqdot s^{k+m}\phi(s)$,
provided the change of order of integration is justified.

The change of order of integration can easily be justified, if $h(u)$ is regular for $0 < u < t$, $R(s) \geqslant s_0 > 0$, and

$$\int_0^t \left|\left(1-\frac{u}{t}\right)^{m+k-1} u^{m-k}h(u)\right| du \text{ is convergent.}$$

Example. We have (Jain, 1955, p. 122),

$$\frac{t^{\nu-\mu}\Gamma(\nu-\mu-2k+1)}{\Gamma\mathrm{x}(\nu-\mu-k+1\pm m)}\,{}_2F_2\left[\begin{matrix}\nu,\,\nu-\mu-2k+1\\ \nu-\mu-k+1\pm m\end{matrix};\,-at\right]\xrightarrow[m]{k+\frac{1}{2}}\frac{s^{\mu}}{(s+a)^{\nu}},$$

also we have (Humbert, 1941, p. 56)

$$e^{-u}L_{m+k-1}(u) \doteqdot \frac{s^{m+k}}{(1+s)^{m+k}}.$$

Taking $\mu = 0$, $a = 1$ and $\nu = k+m$ and substituting in (3.1), we get

$$\int_0^t e^{-u}u^{m-k}\left(1-\frac{u}{t}\right)^{m+k-1} L_{m+k-1}(u)\,du = \frac{t^{m-k+1}\Gamma(m+k)\Gamma(m-k+1)}{\Gamma(2m+1)}\times {}_2F_2\left[\begin{matrix}m+k,\,m-k+1\\ 1,\,2m+1\end{matrix};\,-t\right]$$

$(m+k)$ is a positive integer, $m-k+1 > 0$ and $m > -\frac{1}{2}$.

3.1. *Theorem 12. If* $f(t)\xrightarrow[m]{k+\frac{1}{2}}\phi(s)$, *and*

$$h(u,t) \doteqdot J_{2m}(2\sqrt{us})\,s^k\phi(s), \text{ then}$$

$$f(t) = \frac{\Gamma(1-k+m)}{\Gamma(m+k)}\int_0^\infty e^{-u/t}u^{k-1}h(u,t)\,du \tag{3.2}$$

provided

(i) $R(k+m) > 0$, $R(s) \geqslant s_0 > 0$,

(ii) $h(u, t)$ *is regular for* $u > 0$ *and* $t > 0$, *and*

(iii) *the integral* (3.2) *is absolutely convergent.*

Proof. We have (Meijer, 1941*c*, p. 301)

$$M_{k-\frac{1}{2}, m}(st) = \frac{2\Gamma(1+2m)(st)^{\frac{1}{2}} e^{\frac{1}{2}st}}{\Gamma(k+m)} \int_0^\infty u^{2k-1} e^{-u^2} J_{2m}(2u\sqrt{st})\, du,$$

$$R(k+m) > 0 \text{ and } s \neq 0.$$

Replacing u^2t by u and substituting in (1.2), we get

$$f(t) = \lim_{\lambda \to \infty} \frac{\Gamma(1-k+m)}{\Gamma(1+2m)2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st}(st)^{k-\frac{1}{2}} \left[\frac{e^{\frac{1}{2}st} 2(st)^{\frac{1}{2}} \Gamma(1+2m)}{\Gamma(m+k)} \times \right.$$

$$\left. \times \int_0^\infty e^{-u/t} \left(\frac{u}{t}\right)^{k-\frac{1}{2}} J_{2m}(2\sqrt{us}) \frac{du}{2\sqrt{tu}} \right] \frac{\phi(s)}{s}\, ds,$$

$$= \frac{\Gamma(1-k+m)}{\Gamma(m+k)} \int_0^\infty e^{-u/t} u^{k-1} \left[\lim_{\lambda \to \infty} \frac{1}{2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{st} s^k J_{2m}(2\sqrt{us}) \frac{\phi(s)}{s}\, ds \right] du,$$

$$= \frac{\Gamma(1-k+m)}{\Gamma(m+k)} \int_0^\infty e^{-u/t} u^{k-1} h(u, t)\, du,$$

provided the change of order of integration is permissible.

This can be easily shown if we follow the method of Theorem 11.

3.2. *Theorem 13. If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi(s)$, *and*

$$h_1(u, t) \doteqdot H_{2m-\frac{1}{2}}(2\sqrt{us})\, s^{k-\frac{1}{4}} \phi(s), \text{ then}$$

$$(3.3)\; f(t) = \frac{\Gamma(1-k+m)}{\Gamma(k+m)} t^{\frac{1}{4}} \int_0^\infty e^{-\frac{u}{2t}} \left(\frac{u}{t}\right)^{-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{4}} W_{\left\{\substack{\frac{1}{2}k+\frac{1}{2}m+\frac{1}{4} \\ \frac{1}{2}k+\frac{1}{2}m-3/4}\right.}(u\,|\,t) u^{k-\frac{3}{4}} h_1(u, t)\, du,$$

provided

(i) $R(k+m) > 0$, $R(s) \geqslant s_0 > 0$,

(ii) $h_1(u, t)$ *is regular for* $u > 0$ *and* $t > 0$, *and*

(iii) *the integral* (3.3) *is absolutely convergent.*

Proof. We have (Meijer, 1941*c*, p. 307)

$$M_{k-\frac{1}{2}, m}(st) = \frac{2(st)^{\frac{1}{4}} e^{\frac{1}{2}st} \Gamma(1+2m)}{\Gamma(k+m)} \int_0^\infty e^{-\frac{1}{2}u^2} u^{k-m-1} W_{\left\{\substack{\frac{1}{2}k+\frac{1}{2}m+\frac{1}{4} \\ \frac{1}{2}k+\frac{1}{2}m-\frac{1}{2}-\frac{1}{4}}\right.}(u^2) H_{2m-\frac{1}{2}}(2u\sqrt{st})\, du,$$

$s \neq 0$ and $R(k+m) > 0$.

Replacing u^2t by u and substituting in (1.2), we get

$$f(t) = \lim_{\lambda\to\infty} \frac{\Gamma(1-k+m)}{\Gamma(1+2m)2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st}(st)^{k-\frac{1}{2}} \left[\frac{2(st)^{\frac{1}{2}} e^{\frac{1}{2}st}\Gamma(1+2m)}{\Gamma(k+m)} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \left(\frac{u}{t}\right)^{\frac{1}{2}(k-m-1)} \times \right.$$

$$\left. \times W_{\left\{\substack{\frac{1}{2}k+\frac{1}{2}m+\frac{1}{4} \\ \frac{1}{2}k+\frac{1}{2}m-3/4}\right.} (u/t) H_{2m-\frac{1}{2}}(2\sqrt{us}) \frac{du}{2\sqrt{tu}}\right] \frac{\phi(s)}{s} ds$$

$$= \frac{\Gamma(1-k+m)t^{-\frac{1}{2}}}{\Gamma(k+m)} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \left(\frac{u}{t}\right)^{-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{4}} W_{\left\{\substack{\frac{1}{2}k+\frac{1}{2}m+\frac{1}{4} \\ \frac{1}{2}k+\frac{1}{2}m-3/4}\right.} (u/t)\, u^{k-\frac{3}{4}} \times$$

$$\times \left[\lim_{\lambda\to\infty} \frac{1}{2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{st} s^{k-\frac{1}{4}} \frac{\phi(s)}{s} H_{2m-\frac{1}{2}}(2\sqrt{us})\, ds\right] du$$

$$= \frac{\Gamma(1-k+m)t^{\frac{1}{2}}}{\Gamma(k+m)} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \left(\frac{u}{t}\right)^{-\frac{1}{2}k-\frac{1}{2}m-\frac{1}{4}} W_{\left\{\substack{\frac{1}{2}k+\frac{1}{2}m+\frac{1}{4} \\ \frac{1}{2}k+\frac{1}{2}m-3/4}\right.} (u/t)\, u^{k-3/4} h_1(u, t)\, du,$$

provided the change of order of integration is justified.

This can be shown if we follow the method of Theorem 11.

3.3. *Theorem 14. If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \phi(s)$, *and*

$$h_2(u, t) \doteqdot s^{\frac{1}{2}} J_{2k-1}(\sqrt{us}) J_{2m}(\sqrt{us})\phi(s), \textit{ then}$$

$$f(t) = \frac{\pi^{\frac{1}{2}}\Gamma(1-k+m)}{\Gamma(m+k)} t^{-(2k-\frac{1}{2})} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \left(\frac{u}{t}\right)^{-k} W_{k,m}\left(\frac{u}{t}\right) u^{2k-1} h_2(u, t)\, du \tag{3.4}$$

provided

(i) $R(k) > 0$, $R(k+m) > 0$, $R(s) \geqslant s_0 > 0$,

(ii) $h_2(u, t)$ *is regular for* $t > 0$, $u > 0$, *and*

(iii) *the integral* (3.4) *is absolutely convergent.*

Proof. We have (Meijer, 1941*d*, p. 444)

$$M_{k-\frac{1}{2},m}(st) = \frac{2\sqrt{\pi}(st)^{1-k} e^{\frac{1}{2}st}\Gamma(1+2m)}{\Gamma(k+m)} \int_0^\infty u^{2k-1} e^{-\frac{1}{2}u^2} \times$$

$$\times W_{k,m}(u^2) J_{2k-1}(u\sqrt{st}) J_{2m}(u\sqrt{st})\, du,$$

$z \neq 0$, $R(k) > 0$ and $R(k+m) > 0$.

Replacing u^2t by u and substituting in (1.2), we get

$$f(t) = \lim_{\lambda\to\infty} \frac{\Gamma(1-k+m)}{\Gamma(1+2m)2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st}(st)^{k-1/2} \frac{\phi(s)}{s} \left[\frac{2\sqrt{\pi}(st)^{1-k} e^{\frac{1}{2}st}\Gamma(1+2m)}{\Gamma(k+m)} \times \right.$$

$$\left. \times \int_0^\infty \left(\frac{u}{t}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}\frac{u}{t}} W_{k,m}(u/t) J_{2k-1}(\sqrt{us}) J_{2m}(\sqrt{us}) \frac{du}{2\sqrt{tu}}\right] ds$$

$$= \frac{\Gamma(1-k+m)\pi^{\frac{1}{2}}}{\Gamma(k+m)} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \frac{u^{k-1}}{t^{k-\frac{1}{2}}} W_{k,m}\left(\frac{u}{t}\right) \left[\lim_{\lambda\to\infty} \frac{1}{2\pi i} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{st} s^{\frac{1}{2}} \times \right.$$

$$\left. \times J_{2k-1}(\sqrt{us})\, J_{2m}(\sqrt{us})\, \frac{\phi(s)}{s}\, ds \right] du$$

$$= \frac{\pi^{\frac{1}{2}}\Gamma(1-k+m)t^{-(2k-\frac{1}{2})}}{\Gamma(m+k)} \int_0^\infty e^{-\frac{1}{2}\frac{u}{t}} \left(\frac{u}{t}\right)^{-k} W_{k,m}\left(\frac{u}{t}\right) u^{2k-1} h_2(u,t)\, du,$$

provided the change of order of integration is justified.

This can be easily shown if we follow the method of Theorem 11.

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