ON LOCALIZED AXISYMMETRIC TURBULENCE

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1. Introduction

In order of simplicity, the case of homogeneous and axisymmetric turbulence comes next to that of homogeneous and isotropic turbulence. Since turbulent fluctuations in the velocity components in a uniform stream are generally defined with reference to the local mean values, it is natural to start the investigation of turbulence in such a field with the idea of axisymmetry about the direction of mean flow. In such a case the mean value of any function of the fluctuating velocities and their derivatives should remain invariant for rotations about the preferential mean flow direction (say, λ) and for reflections in planes containing λ and perpendicular to λ. Here the direction $\vec{\lambda}$ is not localized so that all directions parallel to λ are equivalent and should have the same property. This type of homogeneous and axisymmetric turbulence has been studied by Batchelor (1946), and by Chandrasekhar (1950). Bass (1954) in his recent publication has considered sectionally homogeneous and cylindrical turbulence in stationary state. Apart from introducing the time factor in his correlations, this author has analysed axisymmetric turbulence with an added dependence of the velocity correlation values on a distance factor of the configuration, the distance being measured in the direction of the axis. The turbulence produced in a tunnel is expected to be really of this type. In this, if a configuration of the velocity vectors be translated in a direction perpendicular to $\vec{\lambda}$ (this preferential direction λ being obviously parallel to the axis of the tunnel), the correlation values of vectors are supposed to remain unchanged, whereas any translation parallel to the λ-direction would alter these correlations.

In the present paper we discuss an axisymmetric, stationary turbulent fluid field extending to infinity in all directions. The symmetry of the field of turbulence is supposed to be about a localized direction (say, λ), which is also the direction of mean flow. All velocity fluctuations are considered with reference to this mean flow. Unlike the case considered by Bass (1954), here this turbulence is assumed longitudinally homogeneous, and sectionally (i.e. transversely) non-homogeneous. The $\vec{\lambda}$-direction in our case is a localized axis which extends both ways to infinity and is fixed in space. A line source of turbulence in an infinite fluid at rest, or in uniform motion may be expected to produce a field of this type.

2. Mean Values of Velocity Products

In discussing axisymmetric, longitudinally homogeneous and stationary turbulence about a localized axis $\vec{\lambda}$ in an infinite turbulent fluid we choose any two points P and Q close to each other in the fluid field.

Let NX be the localized axis about which there is symmetry. This axis is the line source of turbulence. P, Q are two arbitrarily chosen points in the turbulent

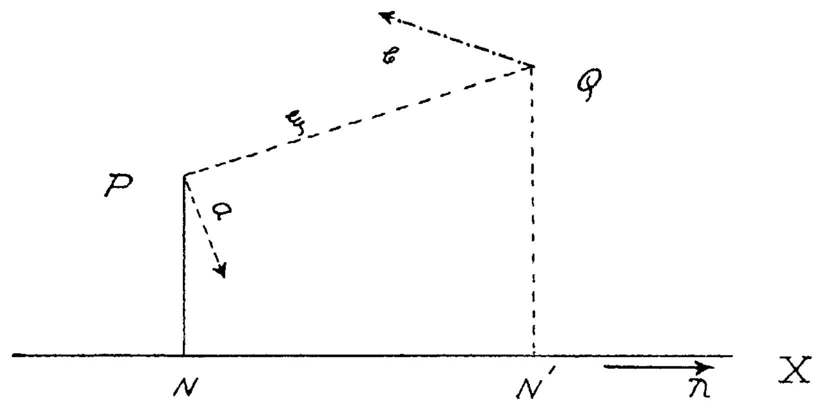

FIG. 1

fluid field, and PN the perpendicular from P on NX; NP is really the axial distance of P from the axis of turbulence. The join PQ is represented by the vector $\vec{\xi}$, and NP by $\vec{\mu}$, and the unit vector along NX will be called $\vec{\lambda}$. Let us now take $\vec{a}$ and $\vec{b}$ two unit vectors at P and Q respectively, and let u_a be the fluctuating component of velocity at P in the direction of $\vec{a}$ and u'_b the fluctuating component of velocity at Q in the direction of b. Next we write the axisymmetric form

$$\begin{aligned} F(a, b; \xi) &\equiv \overline{u_a u'_b} \\ &= \overline{u_i a_i u'_j b_j} \\ &= \overline{u_i u'_j} a_i b_j = F_{ij} a_i b_j \qquad \ldots \quad \ldots \quad (1) \end{aligned}$$

$F(a, b; \xi)$ in (1) has been expressed in a bilinear form of the vectors a, b, the coefficients F_{ij} being the axisymmetric tensor of the desired type, formed by the mean values of the products of velocity components u_i at P, and u'_j at Q.

A translation of the rigid frame of the three vectors a, b, and ξ, in the direction $\vec{\lambda}$, or rotation about it should not change F, but a translation in the direction of $\vec{\mu}$ certainly should. So in consideration of this type of symmetry the direction of $\vec{\lambda}$, and $|PN| = |\mu|$ ($= p$ say) are important. It will now be convenient, following the well-known procedure, to remove the restriction on the type of rotations and translation of the configuration formed with $\vec{\xi}, \vec{a}$ and $\vec{b}$, by attaching to it $\vec{PN}$, and the line NX in the direction λ, and allow the new configuration with $\vec{\xi}, \vec{a}, \vec{b}, \vec{\mu}$, $\vec{\lambda}$ all types of translation, and proper and improper rotations. Then the axisymmetric form $F(a, b; \xi)$ would be equivalent to isotropic form $F(a, b; \xi, \mu, \lambda)$. Following the method of Robertson (1940) we write down the isotropic form $F(a, b; \xi, \mu, \lambda)$ as follows:

$$\begin{aligned} F(a, b; \xi, \mu, \lambda) &= A(\xi.a)(\xi.b) + B(a.b) + C(\lambda.a)(\lambda.b) + D(\lambda.a)(\xi.b) \\ &+ E(\xi.a)(\lambda.b) + F(\lambda.a)(\mu.b) + G(\mu.a)(\lambda.b) \\ &+ H(\xi.a)(\mu.b) + I(\mu.a)(\xi.b) + J(\mu.a)(\mu.b) \\ &= F_{ij} a_i b_j \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2) \end{aligned}$$

where we have

$$F_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D\lambda_i\xi_j + E\xi_i\lambda_j + F\lambda_i\mu_j + G\mu_i\lambda_j + H\xi_i\mu_j + I\mu_i\xi_j + J\mu_i\mu_j \quad \ldots \quad (3)$$

$A, B, C, D, E, F, G, H, I, J$ being ten defining scalars. If $\vec{\psi}$ be the unit vector in the direction of the perpendicular NP (from λ-axis towards the first point P), the ten defining scalars are to be functions of

$$(\xi.\xi) \equiv r^2,\ (\xi.\lambda) \equiv s,\ (\xi.\psi) \equiv q, \text{ and } p\ (= y_\alpha \psi_\alpha)$$

y being the vector OP, where O is a fixed point on $\vec{\lambda}$.

3. Relations among Scalars due to Divergence Equation

The velocity correlation tensor F_{ij} given in (3) is solenoidal with respect to both the indices i, and j, and by virtue of the equations of continuity at P and Q, we have the equations

$$\frac{\partial F_{ij}}{\partial x_i} = 0, \text{ and } \frac{\partial F_{ij}}{\partial x_j'} = 0 \quad \ldots \quad (4)$$

where x_i and x_j' are the co-ordinates of P and Q respectively. We consider F_{ij} to contain the x_i and x_i' co-ordinates as $x_i' - x_i = \xi_i$ and x_i; and this x_i to occur through the scalar $p\ (= x_i\psi_i)$. In the type of symmetry assumed by us F_{ij} is supposed to remain invariant with respect to translation along λ-axis (which can be equivalently said as z-axis of the cylindrical system of co-ordinates) and also with respect to rotations about this axis (which would mean independence with respect to the azimuthal θ-co-ordinate of the cylindrical system with $\vec{\lambda}$ as z-axis). The first equation of (4) can be written in a simple manner. If we regard for the moment the λ, ψ, and the direction perpendicular to them (x) as z, r, θ directions of cylindrical co-ordinates, the equation of continuity for P may be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial}{r\partial\theta}(u_\theta) + \frac{\partial}{\partial z}(u_z) = 0.$$

Multiplying the above equation by u_j' (velocity component at Q) we rewrite it as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r u_j') + \frac{\partial}{r\partial\theta}(u_\theta u_j') + \frac{\partial}{\partial z}(u_z u_j') = 0.$$

Now we put $u_r = \psi_i u_i$, $u_z = \lambda_i u_i$, $u_\theta = x_i u_i$ and ultimately form the mean throughout, noting that F_{ij} ($\equiv$ the double velocity product mean) is unaltered for translation along λ and for rotation about λ, we obtain

$$\frac{\psi_i}{p}\frac{\partial}{\partial p}(pF_{ij}) = 0, \text{ hence } \psi_i\frac{\partial}{\partial p}(pF_{ij}) = 0 \quad \ldots \quad (4.1)$$

The second equation of (4) we can now write as

$$\frac{\partial F_{ij}}{\partial \xi_j} = 0 \cdot \quad \ldots \quad (4.2)$$

Substituting for F_{ij} from (3) in (4.1) we get

$$\psi_i\frac{\partial}{\partial p}\left[p\{A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D\lambda_i\xi_j + E\xi_i\lambda_j + F\lambda_i\mu_j + G\mu_i\lambda_j + H\xi_i\mu_j + I\mu_i\xi_j + J\mu_i\mu_j\}\right] = 0$$

or

$$\xi_j\left[q\frac{\partial}{\partial p}(Ap)+p\frac{\partial}{\partial p}(Ip)+Ip\right]+\lambda_j\left[q\frac{\partial}{\partial p}(Ep)+p\frac{\partial}{\partial p}(Gp)+Gp\right]+\psi_j\left[\frac{\partial}{\partial p}(Bp)+\right.$$
$$\left.+qp\frac{\partial}{\partial p}(Hp)+qHp+p^2\frac{\partial}{\partial p}(Jp)+2Jp^2\right]=0$$

since ξ_j, λ_j, and ψ_j are independent, we must have

$$\left.\begin{aligned}&\frac{\partial}{\partial p}[qpA+p^2I]=0\\&\frac{\partial}{\partial p}[qpE+p^2G]=0\\&\frac{\partial}{\partial p}[Bp+qp^2H+p^3J]=0\end{aligned}\right\}\quad \ldots\quad\ldots\quad\ldots\quad (5.1)$$

Similarly substituting for F_{ij} from (3) in (4.2), we have

$$4A\xi_i+3D\lambda_i+E\lambda_i+H\mu_i+3I\mu_i$$
$$+\left\{\frac{\xi_j}{r}\frac{\partial}{\partial r}+\lambda_j\frac{\partial}{\partial s}+\psi_j\frac{\partial}{\partial q}\right\}[A\xi_i\xi_j+B\delta_{ij}+C\lambda_i\lambda_j+D\lambda_i\xi_j+E\xi_i\lambda_j+F\lambda_i\mu_j+G\mu_i\lambda_j+$$
$$+H\xi_i\mu_j+I\mu_i\xi_j+J\mu_i\mu_j]=0$$

or

$$4A\xi_i+3D\lambda_i+E\lambda_i+H\mu_i+3I\mu_i+\xi_i r\frac{\partial A}{\partial r}+\xi_i s\frac{\partial A}{\partial s}+\xi_i q\frac{\partial A}{\partial q}+\frac{\xi_i}{r}\frac{\partial B}{\partial r}+\lambda_i\frac{\partial B}{\partial s}+\frac{\mu_i}{p}\frac{\partial B}{\partial q}+$$
$$+\lambda_i\cdot\frac{s}{r}\cdot\frac{\partial C}{\partial r}+\lambda_i\frac{\partial C}{\partial s}+\lambda_i r\frac{\partial D}{\partial r}+\lambda_i s\frac{\partial D}{\partial s}+\lambda_i q\frac{\partial D}{\partial q}+\xi_i\cdot\frac{s}{r}\frac{\partial E}{\partial r}+\xi_i\frac{\partial E}{\partial s}+\lambda_i\frac{pq}{r}\frac{\partial F}{\partial r}+$$
$$+\lambda_i p\frac{\partial F}{\partial q}+\mu_i\frac{s}{r}\frac{\partial G}{\partial r}+\mu_i\frac{\partial G}{\partial s}+\xi_i\frac{pq}{r}\frac{\partial H}{\partial r}+\xi_i p\frac{\partial H}{\partial q}+\mu_i r\frac{\partial I}{\partial r}+\mu_i s\frac{\partial I}{\partial s}+\mu_i q\frac{\partial I}{\partial q}+$$
$$+\frac{pq}{r}\mu_i\frac{\partial J}{\partial r}+p\mu_i\frac{\partial J}{\partial q}=0.$$

We arrange the expression on the left as follows:

$$\xi_i\left[4A+r\frac{\partial A}{\partial r}+s\frac{\partial A}{\partial s}+q\frac{\partial A}{\partial q}+\frac{1}{r}\frac{\partial B}{\partial r}+\frac{s}{r}\frac{\partial E}{\partial r}+\frac{\partial E}{\partial s}+\frac{qp}{r}\frac{\partial H}{\partial r}+p\frac{\partial H}{\partial q}\right]$$
$$+\lambda_i\left[\frac{\partial B}{\partial s}+\frac{s}{r}\frac{\partial C}{\partial r}+\frac{\partial C}{\partial s}+3D+r\frac{\partial D}{\partial r}+s\frac{\partial D}{\partial s}+q\frac{\partial D}{\partial q}+E+\frac{qp}{r}\frac{\partial F}{\partial r}+p\frac{\partial F}{\partial q}\right]$$
$$+\psi_i\left[\frac{1}{p}\frac{\partial B}{\partial q}+\frac{s}{r}\frac{\partial G}{\partial r}+\frac{\partial G}{\partial s}+H+3I+r\frac{\partial I}{\partial r}+s\frac{\partial I}{\partial s}+q\frac{\partial I}{\partial q}+\frac{qp}{r}\frac{\partial J}{\partial r}+p\frac{\partial J}{\partial q}\right]=0.$$

Since ξ_i, λ_i, and μ_i are linearly independent, we get the equations

$$\left.\begin{aligned}&4A+r\frac{\partial A}{\partial r}+s\frac{\partial A}{\partial s}+q\frac{\partial A}{\partial q}+\frac{1}{r}\frac{\partial B}{\partial r}+\frac{s}{r}\frac{\partial E}{\partial r}+\frac{\partial E}{\partial s}+\frac{qp}{r}\frac{\partial H}{\partial r}+p\frac{\partial H}{\partial q}=0\\&\frac{\partial B}{\partial s}+\frac{s}{r}\frac{\partial C}{\partial r}+\frac{\partial C}{\partial s}+3D+r\frac{\partial D}{\partial r}+s\frac{\partial D}{\partial s}+q\frac{\partial D}{\partial q}+E+\frac{qp}{r}\frac{\partial F}{\partial r}+p\frac{\partial F}{\partial q}=0\\&\frac{1}{p}\frac{\partial B}{\partial q}+\frac{s}{r}\frac{\partial G}{\partial r}+\frac{\partial G}{\partial s}+H+3I+r\frac{\partial I}{\partial r}+s\frac{\partial I}{\partial s}+q\frac{\partial I}{\partial q}+\frac{qp}{r}\frac{\partial J}{\partial r}+p\frac{\partial J}{\partial q}=0\end{aligned}\right\}\quad\ldots\quad\ldots\quad (5.2)$$

4. Limited Expansions of the Defining Scalars

Let us next assume expansions of the defining scalars A, B, C, D, E, F, G, H, I, J in power series of s, q, r^2 up to such an order that when these expressions are substituted in (3) we shall have only terms up to the second order in r, s, or q. If we propose to keep terms only up to the second order in the distance of separation of the points P and Q, in F_{ij}, we may write the expansions of the defining scalars as follows:

$$\left.\begin{aligned}
A &= A_0\\
B &= B_0+B_1s+B_2q+B_{11}r^2+B_{22}s^2+B_{33}q^2+2B_{23}sq\\
C &= C_0+C_1s+C_2q+C_{11}r^2+C_{22}s^2+C_{33}q^2+2C_{23}sq\\
D &= D_0+D_1s+D_2q\\
E &= E_0+E_1s+E_2q\\
F &= F_0+F_1s+F_2q+F_{11}r^2+F_{22}s^2+F_{33}q^2+2F_{23}sq\\
G &= G_0+G_1s+G_2q+G_{11}r^2+G_{22}s^2+G_{33}q^2+2G_{23}sq\\
H &= H_0+H_1s+H_2q\\
I &= I_0+I_1s+I_2q\\
J &= J_0+J_1s+J_2q+J_{11}r^2+J_{22}s^2+J_{33}q^2+2J_{23}sq
\end{aligned}\right\} \quad \ldots \quad (6)$$

where the *48 coefficients* A_0, B_0, B_1, ... are functions of p alone. Thus the ten defining scalars are made to depend (for approximation up to second order) on 48 functions of p. On substitution of the expansions (6) in (5.1) and (5.2) we get the following two sets of conditions:

(1)

$$\begin{aligned}
&p^2I_0, p^2I_1, pA_0+p^2I_2;\\
&p^2G_0, p^2G_1, pE_0+p^2G_2, p^2G_{11}, p^2G_{22}, p^2G_{33}+pE_2,\ 2p^2G_{23}+pE_1;\\
&pB_0+p^3J_0, pB_1+p^3J_1, pB_2+p^2H_0+p^3J_2, pB_{11}+p^3J_{11}, pB_{22}+p^3J_{22},\\
&\qquad pB_{33}+p^2H_2+p^3J_{33}, 2pB_{23}+p^2H_1+2p^3J_{23}, \quad \ldots \quad (7.1)
\end{aligned}$$

are constants independent of p, and

(2)

$$\left.\begin{aligned}
&4A_0+2B_{11}+E_1+pH_2 = 0\\
&B_1+C_1+3D_0+E_0+pF_2 = 0\\
&2B_{22}+2C_{11}+2C_{22}+4D_1+E_1+2pF_{23} = 0\\
&2B_{23}+2C_{23}+4D_2+E_2+2pF_{11}+2pF_{33} = 0\\
&\frac{1}{p}B_2+G_1+H_0+3I_0+pJ_2 = 0\\
&\frac{2}{p}B_{33}+2G_{23}+H_2+4I_2+2pJ_{11}+2pJ_{33} = 0\\
&\frac{2}{p}B_{23}+2G_{11}+2G_{22}+H_1+4I_1+2pJ_{23} = 0
\end{aligned}\right\} \quad \ldots \quad (7.2)$$

The incompressibility conditions derived in the last section are equivalent to condition (7.1) and (7.2).

5. Symmetry Conditions of the Bilinear Form F

Apart from the incompressibility conditions (7.1) and (7.2) we shall derive further relations between the coefficients A_0, B_0, B_1, etc., from the symmetry conditions of the bilinear form $F(a, b\,;\,\xi)$. Keeping the frame (λ, NP, a, ξ, b) fixed we can rename the vectors a, b, and ξ, renaming a as b, and b as a, ξ being renamed with reference to its relations to a and b.

The form $F(a, b\,;\,\xi)$ should thereby remain unaltered. This alteration is equivalent to calling Q the first point and P the second, so that ξ is changed to $-\xi$, the symbol a is replaced by b, and b by a. We represent the new scalars by A', B', etc., and have

$$\begin{aligned} &A(\xi.a)(\xi.b)+B(a.b)+C(\lambda.a)(\lambda.b)+D(\lambda.a)(\xi.b)+E(\xi.a)(\lambda.b)\\ &\quad+F(\lambda.a)(\mu.b)+G(\mu.a)(\lambda.b)+H(\xi.a)(\mu.b)+I(\mu.a)(\xi.b)+J(\mu.a)(\mu.b)\\ &=A'(-\xi.b)(-\xi.a)+B'(b.a)+C'(\lambda.b)(\lambda.a)+D'(\lambda.b)(-\xi.a)+E'(-\xi.b)(\lambda.a)\\ &\quad+F'(\lambda.b)(\mu.a)+G'(\mu.b)(\lambda.a)+H'(-\xi.b)(\mu.a)+I(\mu.b)(-\xi.a)\\ &\quad+J(\mu.b)(\mu.a) \qquad \ldots \quad (8)\end{aligned}$$

Here we consider the symmetry condition on the basis of the rigid frame constituted by $\vec{\lambda}$, $\vec{\mu}$, $\vec{a}$, $\vec{b}$, and $\vec{\xi}$. The vector $\vec{\mu}$ as chosen here is not symmetrically placed with respect to the points P and Q. One may, however, take the vector perpendicular to $\vec{\lambda}$ from the middle point of PQ as the vector $\vec{\mu}$, which would now be symmetrically placed with respect to P and Q. This would not make any difference in our final analysis and the ultimate results represented by equations (12*a*–*f*) can be reached in this manner also.

Now equating out the coefficients of $a_i b_j$ from both sides of (8) we get

$$\begin{aligned} &A\xi_i\xi_j+B\delta_{ij}+C\lambda_i\lambda_j+D\lambda_i\xi_j+E\xi_i\lambda_j+F\lambda_i\mu_j+G\mu_i\lambda_j+H\xi_i\mu_j+I\mu_i\xi_j+J\mu_i\mu_j\\ &=A'\xi_i\xi_j+B'\delta_{ij}+C'\lambda_i\lambda_j-D'\xi_i\lambda_j-E'\lambda_i\xi_j+F'\mu_i\lambda_j+G'\lambda_i\mu_j-H'\mu_i\xi_j-I'\xi_i\mu_j+J'\mu_i\mu_j \qquad \ldots \quad (9)\end{aligned}$$

Now comparing both sides we write down the symmetry conditions as

$$\begin{aligned} &A=A',\ B=B',\ C=C',\ D=-E',\ E=-D',\\ &F=G',\ G=F',\ H=-I',\ I=-H',\ J=J' \qquad \ldots \quad (10)\end{aligned}$$

The unprimed A, B, C, etc., are functions of r^2, s, q, and p, whereas the primed A, B, C, etc., are respectively the same functions of $r^2, -s$, q' and p', when $q'[=(-\xi\cdot\psi')]$ $p'[=(y'.\psi')]$ by definition; $\vec{\psi'}$ is the unit vector in the direction reverse to the perpendicular drawn from Q on λ-axis, and y'_α $(=y_\alpha+\xi_\alpha)$ represent the co-ordinates of Q. From the geometry of the configuration, it can be readily deduced that

$$\left.\begin{aligned} p' &\simeq p+q+\frac{1}{2p}(r^2-s^2-q^2)\\ \text{and}\qquad q' &\simeq -q-\frac{1}{p}(r^2-s^2-q^2)\end{aligned}\right\} \qquad \ldots \quad (11)$$

correct up to the second order in r, s, and q (s and q both involve r).

From the first one of the sets of relations in (10) we get no information about A_0

(the only term up to the second order in r, in the limited expansion of A). The second one of the set we write as

$$B(r^2, s, q\,;p) = B(r^2, -s, q'\,;p')$$

$$= B\left(r^2, -s, -q-\frac{1}{p}(r^2-s^2-q^2);\, p+q+\frac{1}{2p}(r^2-s^2-q^2)\right)$$

replacing q' and p' by the approximate expressions given in (11). Introducing the expansions (6) and using Taylor's expansion up to the required order we derive on equating out coefficients of s, q, r^2, etc.

$$B_1 = 0,\ B_2 = \frac{1}{2}\frac{\partial B_0}{\partial p} \quad \ldots \quad (12a)$$

Similarly from the other equations of (10) we obtain in the same manner the following relations

$$C_1 = 0,\ C_2 = \frac{1}{2}\frac{\partial C_0}{\partial p} \quad \ldots \quad (12b)$$

$$D_0 = -E_0,\ D_1 = E_1,\ D_2 = E_2 - \frac{\partial E_0}{\partial p} \quad \ldots \quad (12c)$$

$$F_0 = G_0,\ F_1 = -G_1,\ F_2 = -G_2 + \frac{\partial G_0}{\partial p},\ F_{11} = G_{11} + \frac{1}{2p}\frac{\partial G_0}{\partial p} - \frac{1}{p}G_2$$

$$F_{22} = G_{22} - \frac{1}{2p}\frac{\partial G_0}{\partial p} + \frac{1}{p}G_2,\ F_{33} = G_{33} - \frac{1}{2p}\frac{\partial G_0}{\partial p} + \frac{1}{2}\frac{\partial^2 G_0}{\partial p^2} + \frac{1}{p}G_2 - \frac{\partial G_2}{\partial p},$$

$$F_{23} = G_{23} - \frac{1}{2}\frac{\partial G_1}{\partial p} \quad \ldots \quad (12d)$$

$$H_0 = -I_0,\ H_1 = I_1,\ H_2 = I_2 - \frac{\partial I_0}{\partial p} \quad \ldots \quad (12e)$$

$$J_1 = 0,\ J_2 = \frac{1}{2}\frac{\partial J_0}{\partial p} \quad \ldots \quad (12f)$$

6. Reduction of the 48 Coefficients to Thirteen Absolute Constants and Nine Functions of p

In (7.1) we have, in general, to introduce seventeen integration constants, but from some of the relations (7.2), three constants, namely those introduced for p^2G_1, $pB_{33}+p^2H_2+p^3J_{33}$, $2pB_{23}+p^2H_1+2p^3J_{23}$ can be written in terms of some of the other remaining constants. Moreover, introducing relations of (12a) and (12f) we get one of the relations of (7.1) as $pB_1+p^3J_1 = 0$. As a consequence of (7.1), (7.2) and (12a–f) we can express the scalar functions of p in the expansion (6) in terms of only 13 integration constants, $k_1, k_2, k_3, \ldots, k_{13}$, and nine scalar coefficients, namely A_0, B_{22}, B_{33}, B_{23}, C_0, C_{11}, C_{22}, C_{33}, C_{23} (which are functions of p). One may note the following relations:

$$B_0 = (k_1+k_{11}) + \frac{3}{2}\frac{k_{10}}{p};\quad B_1 = 0;\quad B_2 = \frac{1}{2}\frac{\partial B_0}{\partial p};$$

$$B_{11} = \frac{1}{4}(B_{22}+C_{11}+C_{22}) - \frac{3}{2}A_0 - \frac{k_1}{4p^2} - \frac{k_8}{2p} + \frac{k_9}{8p} + \frac{k_{11}}{4p^2};$$

$$C_1 = 0\,;\ C_2 = \frac{1}{2}\frac{\partial C_0}{\partial p}\,;$$

$$E_0 = -D_0 = -\frac{2k_4}{p^2} - \frac{k_5}{p}\,;\quad E_1 = D_1 = -\frac{1}{2}(B_{22}+C_{11}+C_{22}) - \frac{1}{2p^2}(3k_1+k_{11}) - \frac{k_9}{4p}\,;$$

$$E_2 = -\frac{2}{3}\left[B_{23}+C_{23}+\frac{k_4}{p^3}+\frac{2k_5}{p^2}+\frac{k_6}{p}+\frac{k_8}{p}\right];\quad D_2 = E_2 - \frac{\partial E_0}{\partial p}\,;$$

$$F_0 = G_0 = \frac{k_4}{p^2}\,;\quad F_1 = -G_1 = -\frac{3k_1}{p^2} - \frac{k_{11}}{p^2}\,;\quad F_2 = -\frac{1}{p^2}(k_5 - pE_0) - \frac{2k_4}{p^3}\,;$$

$$F_{11} = \frac{k_6}{p^2} - \frac{2k_5}{p^3} - \frac{3k_4}{p^4}\,;\quad F_{22} = \frac{k_7}{p^2}+\frac{2k_5}{p^3}+\frac{3k_4}{p^4}\,;\quad F_{33} = \frac{1}{p^2}(k_8 - pE_2)+\frac{6k_5}{p^3}+\frac{12k_4}{p^4}\,;$$

$$F_{23} = \frac{1}{2p^2}(k_9 - pE_1) + \frac{1}{p^3}(3k_1+k_{11})\,;$$

$$G_2 = \frac{2k_5}{p^2}+\frac{2k_4}{p^3}\,;\quad G_{11} = \frac{k_6}{p^2}\,;\quad G_{22} = \frac{k_7}{p^2}\,;\quad G_{33} = \frac{1}{p^2}(k_8 - pE_2)\,;$$

$$G_{23} = \frac{1}{2p^2}(k_9 - pE_1)\,;$$

$$H_0 = -I_0 = -\frac{k_1}{p^2}\,;\quad H_1 = I_1 = \frac{k_2}{p^2}\,;\quad H_2 = \frac{1}{p^2}(k_3 - pA_0) + \frac{2k_1}{p^3}\,;$$

$$I_2 = \frac{1}{p^2}(k_3 - pA_0)\,;$$

$$J_0 = \frac{1}{p^3}(k_{10} - pB_0)\,;\quad J_1 = 0\,;\quad J_2 = \frac{1}{2}\frac{\partial J_0}{\partial p}\,;\quad J_{11} = \frac{1}{p^3}(k_{12} - pB_{11})\,;$$

$$J_{22} = \frac{1}{p^3}(k_{13} - pB_{22})\,;\quad J_{33} = \frac{1}{p^3}\left[-\frac{2k_1}{p} - 3k_3 - \frac{k_9}{2} - k_{12} + pA_0 - pB_{33}\right];$$

$$J_{23} = -\frac{k_6+k_7+pB_{23}}{p^3} - \frac{5}{2}\frac{k_2}{p^3}. \qquad (13)$$

From the above analysis we can infer as follows:

(*a*) The pairs of coefficients (D, E), (F, G) and (H, I) are not in general identical. Hence the velocity *correlation tensor* F_{ij} as given in (3) *is not symmetrical in the two indices i and j even up to the second order of smallness in the distance r*. It was found by Bass (1954) that for turbulence with a preferential direction the tensor F_{ij} (in his paper this is denoted by $R_{\alpha\beta}$) with transverse homogeneity is symmetrical in the indices up to the second order in r. The lack of symmetry in the present case is due to the vector λ being localized and the consequent dependence of the tensor F_{ij} on $|\mu| = p$.

(*b*) From (3), restricting ourselves to the limiting case when $r \to 0$, we get the turbulent energy per unit of volume associated with the longitudinal direction $(\frac{1}{2}\overline{u_1^2})$ given by ϵ_{11} where

$$2\epsilon_{11} = [F_{11}]_{r=0} = B_0 + C_0 = (k_1+k_{11}) + \tfrac{3}{2}\frac{k_{10}}{p} + C_0, \qquad (14.1)$$

where 1-direction defines the λ-direction, and $[B]_{r=0} = B_0$, $[C]_{r=0} = C_0$ are the same as in (6). Similarly energy associated with the direction of $\vec{\mu}$ is given by ϵ_{22} where

$$2\epsilon_{22} = [F_{22}]_{r=0} = B_0+p^2J_0 = \frac{k_{10}}{p}, \quad \ldots \quad (14.2)$$

and energy associated with the azimuthal direction given by ϵ_{33} where

$$2\epsilon_{33} = [F_{33}]_{r=0} = B_0 = (k_1+k_{11})+\frac{3}{2}\frac{k_{10}}{p}. \quad \ldots \quad (14.3)$$

Hence twice the turbulent energy per unit volume is

$$3B_0+C_0+p^2J_0 = 2(k_1+k_{11})+4\frac{k_{10}}{p}+C_0,$$

which in general is a function of p only. One may expect that the turbulent energy at large distance from the axis should vanish. Further as none of the above three energy parts $[F_{11}]_{r=0}$, $[F_{22}]_{r=0}$, $[F_{33}]_{r=0}$ can be negative, all of three parts should vanish as $p \to \infty$. This requires that $k_1+k_{11} = 0$ and the expansion of C_0 in powers of p should not contain any constant term. If these conditions are not satisfied the sideway extension of the medium to infinity is not permissible, and a limitation of the liquid by a cylindrical boundary would become necessary.

7. Iso-mean Surfaces and Curves

We shall now study the distribution of the double velocity product mean values in the fluid field induced by a turbulence characterized by (3), with the restrictions on the defining scalars worked out before. With this view we shall find out the nature of the iso-mean surfaces for the longitudinal, perpendicular and azimuthal velocity products and study their plane sections. From the physical idea that the two point velocity correlation values should sharply tend to zero value as the distance of separation of the two points increases it is clear that the same should be true for the mean velocity products and so the iso-mean surfaces and their section curves should be closed. The longitudinal, perpendicular and azimuthal velocity product means can be calculated from (3) as follows:

(*a*) Mean of the product of longitudinal velocity components at P and Q

$$\omega_{\parallel} = \overline{\lambda_i\lambda_j\, u_i u_j'} = As^2+B+C+(D+E)s \quad \ldots \quad (15a)$$

(*b*) Mean of the product of transverse velocity components (in $\vec{\psi}$ direction) at P and Q

$$\omega_{\perp} = \overline{\psi_i\psi_j\, u_i u_j'} = Aq^2+B+pq(H+I)+p^2J \quad \ldots \quad (15b)$$

(*c*) Mean of the product of azimuthal velocity components at P and Q

$$\omega_{az} = \overline{\chi_i\chi_j\, u_i u_j'} = B \quad \ldots \quad (15c)$$

The directions (λ, ψ, χ) we shall henceforth call directions (1, 2, 3) and call X_1, X_2, X_3 as the components of $\vec{\xi}$ in 1, 2, 3 directions respectively, regarding momentarily P as the origin of co-ordinates. We substitute from (6) for the defining scalars in (15) and obtain

$$\omega_{\parallel} = m_0+m_1X_2+m_2X_1^2+m_3X_2^2+m_4X_3^2+m_5X_1X_2,$$

where

$m_0 = B_0+C_0$, $m_1 = B_2+C_2$, $m_2 = B_{11}+C_{11}+A_0+B_{22}+C_{22}+2E_1$, $m_3 = B_{11}+C_{11}+B_{33}+C_{33}$, $m_4 = B_{11}+C_{11}$, and $m_5 = 2B_{23}+2C_{23}+D_2+E_2$, ($m_i$'s being in general functions of p), ... (15a′)

$$\omega_\perp = q_0+q_1X_2+q_2X_1^2+q_3X_2^2+q_4X_3^2+q_5X_1X_2,$$

where

$q_0 = B_0+p^2J_0$, $q_1 = B_2+p^2J_2$, $q_2 = B_{11}+p^2J_{11}+B_{22}+p^2J_{22}$, $q_3 = A_0+B_{11}+p^2J_{11}+B_{33}+p^2J_{33}+pH_2+pI_2$, $q_4 = B_{11}+p^2J_{11}$, $q_5 = 2B_{23}+2p^2J_{23}+2pI_1$, ($q_i$'s being in general functions of p), ... (15b′)

and

$$\omega_{az} = n_0+n_1X_2+n_2X_1^2+n_3X_2^2+n_4X_3^2+n_5X_1X_2,$$

where

$$n_0 = B_0,\ n_1 = B_2,\ n_2 = B_{11}+B_{22},\ n_3 = B_{11}+B_{33},\ n_4 = B_{11},$$

and $n_5 = 2B_{23}$, (n_i's being in general functions of p). .. .. (15c′)

In (15a′–c′) the terms containing zero coefficients, such as $B_1+C_1+D_0+E_0$, $B_1+p^2J_1$, B_1 (whose zero values have been demonstrated) have been omitted. We have taken the origin of X_1, X_2, X_3 at P which now is kept fixed, while Q is supposed to wander in space. Then all the coefficients on the right of (15a′), (15b′), (15c′) are constants. If we write $\omega_\parallel$ = constant, $\omega_\perp$ = constant, ω_{az} = constant in (15a′), (15b′), (15c′) respectively, we get the equations of the surfaces on which Q must lie in order that the corresponding velocity product mean may have the chosen constant value.

Special cases :

(a) Let us take the surface obtained by putting $\omega_\parallel$ = constant in (15a′) ; it is a conicoid. We now consider some sections of this surface, which are iso-mean curves for the longitudinal velocity product. When $X_1 = 0$, we have the iso-mean curves lying on the section through P perpendicular to λ (on ψX-plane)

$$m_1X_2+m_3X_2^2+m_4X_3^2 = \omega_\parallel - m_0 = \text{constant} \quad .. \quad .. \quad (16.1)$$

where m_0, m_1, m_3, m_4 depend only on the value of the perpendicular p drawn from P or λ. For this to be an ellipse, as one should expect, m_3 and m_4 should be of the same sign ; the axes of the ellipse are along X_2 and X_3 directions but the centre is not the origin P but it is a point on X_2-axis. As the mean product values should weaken away from the λ-axis one may expect m_1 to be of the sign opposite to that of m_3, m_4.

When $X_2 = 0$ in (15a′), we have iso-mean curves on the plane through P perpendicular to p,

$$m_2X_1^2+m_4X_3^2 = \omega_\parallel - m_0 = \text{constant} \quad .. \quad .. \quad .. \quad (16.2)$$

where m_2, m_4 are functions of p. This, as is to be expected from symmetry, is an ellipse with P as centre and X_1, X_3 as directions of its axes. From considerations stated above one should expect m_2 and m_4 to be of the same sign.

Thirdly, when $X_3 = 0$, we get the iso-mean curves on the ($\psi\lambda$)-plane.

$$m_1X_2+m_2X_1^2+m_3X_2^2+m_5X_1X_2 = \omega_\parallel - m_0 = \text{constant} \quad .. \quad (16.3)$$

where m_5 is also a function of p. For this to be an ellipse, further relation to be satisfied is $4m_2m_3 > m_5^2$. We note that the axes of the ellipse are inclined to the X_1, X_2-axes. Two directions in the azimuthal plane making equal angles with p on two sides are not really equivalent.

(*b*) Let us take the surface $\omega_\perp =$ constant given by (15*b*′); it is a conicoid. As usual we consider different sections of the iso-mean surface for the transverse velocity components as follows:

When $X_1 = 0$, i.e. $\vec{\xi}$ is on (X_2, X_3)-plane, the iso-mean curves are of the nature

$$q_1X_2+q_3X_2^2+q_4X_3^2 = \omega_\perp - q_0 = \text{constant} \quad \ldots \quad \ldots \quad \ldots \quad (17.1)$$

where q_i's are functions of p (17.1), to be an ellipse, q_3, q_4 must be of the same sign; further q_1 should be of a sign opposite to that of q_3, q_4 as the turbulence should weaken away from the λ-axis. The curves are of the type (16.1). When $X_2 = 0$, the iso-mean curves are homothetic to

$$q_2X_1^2+q_4X_3^2 = \omega_\perp - q_0 = \text{constant} \quad (17.2)$$

For (17.2) to be an ellipse, q_2, q_4 should necessarily be of the same sign. These ellipses are of the type (16.2).

Again when $X_3 = 0$, the iso-mean curves are of the type

$$q_1X_2+q_2X_1^2+q_3X_2^2+q_5X_1X_2 = \omega_\perp - q_0 = \text{constant} \quad \ldots \quad \ldots \quad (17.3)$$

and for this to be an ellipse we should have $4q_2q_3 > q_5^2$ in addition to the restrictions on the sign of q_2, q_3. The curves are like those given by (16.3).

(*c*) Next we take the iso-mean surfaces for the azimuthal velocity components both at the reference point P and at the wandering point by putting $\omega_{az} =$ constant in (15*c*′). We consider different plane sections of such surfaces.

When $X_1 = 0$, the iso-mean surfaces are ellipses given by

$$n_1X_2+n_3X_2^2+n_4X_3^2 = \omega_{az} - n_0 = \text{constant} \quad \ldots \quad \ldots \quad (18.1)$$

where n_3, n_4 should have the same sign and n_1 should have the sign opposite to that of n_3, n_4 as usual. The curves are similar to those of (16.1), (17.1).

When $X_2 = 0$, the iso-mean curves are ellipses

$$n_2X_1^2+n_4X_3^2 = \omega_{az} - n_0 = \text{constant} \quad \ldots \quad \ldots \quad \ldots \quad (18.2)$$

where n_2, n_4 should have the same sign. The ellipses (18.2) are similarly stated as those of (16.2), (17.2). Further when $X_3 = 0$ the iso-mean curves for azimuthal velocity components at the reference point and at the wandering point we have

$$n_1X_2+n_2X_1^2+n_3X_2^2+n_5X_1X_2 = \omega_{az} - n_0 = \text{constant} \quad \ldots \quad (18.3)$$

For this to be an ellipse, further relation to be satisfied is $2n_2n_3 > n_5^2$ and we expect this to be satisfied. The curves (18.3) are similarly situated to those given by (16.3), (17.3).

8. Suggestion for Numerical Evaluation of some Unknown Coefficients from Experimentally Traceable Curves

If it be possible to trace the iso-mean curves from experiments in such a turbulence field (as envisaged by us) with the help of anemometer, this would open a way for calculation of many of the coefficients and constants which have remained undetermined up to now.

For instance, let us consider the iso-mean curves of the type (16.1)

$$\omega_{\parallel}-m_0 = m_1X_2+m_3X_2^2+m_4X_3^2$$

of which the method of plotting is as follows:

Suppose one of the anemometers is placed properly directed at the point P, while the second wire of the anemometer directed in the same way be made to move about in the (X_2, X_3)-plane through P in such a manner that the mean longitudinal velocity product value recorded $\omega_{\parallel}$ remains constant and the corresponding positions of the second anemometer are noted. In this way suppose the iso-mean curve corresponding to that specific constant value of $\omega_{\parallel}$ is traced on the (X_2, X_3)-plane through P. Keeping the anemometer at P fixed and placed in the same way, we can by a similar procedure draw other iso-mean curves in (X_2, X_3)-plane corresponding to other constant values of $\omega_{\parallel}$.

From such curves it is possible to obtain by least square method, the best numerical values for m_0, m_1, m_3, m_4 corresponding to P as reference point whose distance from the λ-axis is $\bar{p}$ (say). Similarly the iso-mean curves (with respect to the same reference point P) on the planes $X_2 = 0$, $X_3 = 0$ given by equations

$$\omega_{\parallel}-m_0 = m_2X_1^2+m_4X_3^2$$

and

$$\omega_{\parallel}-m_0 = m_1X_2+m_2X_1^2+m_3X_2^2+m_5X_1X_2$$

can be traced for different constant values for $\omega_{\parallel}$. The same method thus offers a means for finding the values of the m_i coefficients with respect to P, involved in the above equations.

Next by properly changing the direction of the wire of the anemometer both at the reference point P and at the wandering point Q, we can trace the transverse and azimuthal iso-mean curves experimentally, following the same procedure, and consequently determine q_0, q_1, q_2, q_3, q_4, q_5 from (17.1–17.3) numerically for the reference point P, as also the n_i's for the same reference point from (18.1–18.3).

The m_i's, q_i's and n_i's being numerically known for the same reference point P for which $p = \bar{p}$, the values of the following combinations of coefficients can be numerically evaluated as follows:

$$(B_2+C_2)_{p=\bar{p}} = (m_1)_{p=\bar{p}} = a_1 \text{ (say)}$$

$$(B_{11}+C_{11}+A_0+B_{22}+C_{22}+2E_1)_{p=} = (m_2)_{p=\bar{p}} = b_1 \text{ (say)}$$

$$(B_{11}+C_{11}+B_{33}+C_{33})_{p=\bar{p}} = (m_3)_{p=\bar{p}} = c_1 \text{ (say)}$$

$$(B_{11}+C_{11})_{p=\bar{p}} = (m_4)_{p=\bar{p}} = d_1 \text{ (say)}$$

$$(2B_{23}+2C_{23}+D_2+E_2)_{p=\bar{p}} = (m_5)_{p=\bar{p}} = e_1 \text{ (say)}$$

$$(B_2+p^2J_2)_{p=\bar{p}} = (q_1)_{p=\bar{p}} = a_2 \text{ (say)}$$

$$(B_{11}+p^2J_{11}+B_{22}+p^2J_{22})_{p=\bar{p}} = (q_2)_{p=\bar{p}} = b_2 \text{ (say)}$$

$$(A_0+B_{11}+p^2J_{11}+B_{33}+p^2J_{33}+pH_2+pI_2)_{p=\bar{p}} = (q_3)_{p=\bar{p}} = c_2 \text{ (say)}$$

$$(B_{11}+p^2J_{11})_{p=\bar{p}} = (q_4)_{p=\bar{p}} = d_2 \text{ (say)}$$

$$(2B_{23}+2p^2J_{23}+2pI_1)_{p=\bar{p}} = (q_5)_{p=\bar{p}} = e_2 \text{ (say)}$$

$$(B_2)_{p=\bar{p}} = (n_1)_{p=\bar{p}} = a_3 \text{ (say)}$$

$$(B_{11}+B_{22})_{p=\bar{p}} = (n_2)_{p=\bar{p}} = b_3 \text{ (say)}$$

$$(B_{11}+B_{33})_{p=\bar{p}} = (n_3)_{p=\bar{p}} = c_3 \text{ (say)}$$

$$(B_{11})_{p=\bar{p}} = (n_4)_{p=\bar{p}} = d_3 \text{ (say)}$$

and

$$(2B_{23})_{p=\bar{p}} = (n_5)_{p=\bar{p}} = e_3 \text{ (say)}$$

From the above equations the numerical values of B_2, C_2, J_2, B_{33}, C_{33}, B_{23}, B_{22}, J_{22}, J_{11}, B_{11}, C_{11}, $2p^2J_{23}+2pI_1$, $A_0+C_{22}+2E_1$, $2C_{23}+D_2+E_2$, $A_0+p^2J_{33}+pH_2+pI_2$ can be calculated for $p=\bar{p}$. Further m_0, q_0, n_0 being known we know the three energy parts ϵ_{11}, ϵ_{22}, and ϵ_{33} at P. These determine the values of B_0+C_0, $B_0+p^2J_0$, B_0 at P, and hence the numerical values of B_0, C_0, J_0 are separately known.

Next repeating the experiments and calculations as described above for different reference points (i.e. for different values of $p\neq o$) we can know B_2, C_2, . . ., etc., for different values of p. So from a fairly large number of data obtained in this manner, we can derive the functional relations for B_0, C_0, J_0, B_2, C_2, . . ., etc., in terms of the variable p, by fitting a curve of the form $l_0+\frac{l_1}{p}+\frac{l_2}{p^2}+\ldots$ for each one of B_0, C_0, J_0, . . ., etc., by the method of least squares. We can now compare these experimentally determined expansions of the above quantities with their theoretically obtained expansions in negative powers of p as given in (13) and find out the *eight* constants k_1, k_2, k_3, k_9, k_{10}, k_{11}, k_{12}, k_{13} in terms of the above calculated quantities and of the yet unknown *five* constants k_4, k_5, k_6, k_7, k_8. We note further that the nine unknown coefficients A_0, B_{22}, B_{33}, B_{23}, C_0, C_{11}, C_{22}, C_{33}, C_{23} which are functions of p are also known completely from those experimentally obtainable values. In this manner by a combination of theoretical formulae with observational results the problem of localized axisymmetric turbulence in stationary state is determined except for five integration (absolute) constants.

Further the results

$$B_2=\frac{1}{2}\frac{\partial B_0}{\partial p},\quad C_2=\frac{1}{2}\frac{\partial C_0}{\partial p},\quad \text{and}\quad J_2=\frac{1}{2}\frac{\partial J_0}{\partial p}$$

may be used as check conditions for the experimentally determined functions B_2, B_0; C_2, C_0; J_2 and J_0. The experimentally evaluated functions B_0, C_0 in terms of p will help us to conclude whether the sidewise extension of turbulence to infinity as envisaged in the above analysis is a permissible hypothesis. It is so if only the expansions for B_0 and C_0 in negative powers of p do not contain any non-zero constant terms.

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Abstract

In this paper a stationary field of turbulence with a localized axis of symmetry, and infinite in all dimensions has been analysed. In this field the second order means of the products of velocities at two points may be assumed to remain unchanged for translation parallel to the axis but these means will vary for translation perpendicular to the axis. It is shown that the tensor F_{ij} describing the invariance of such mean values is not symmetrical in the indices i and j even for approximations up to the second order in the distance r between the points. There exists a possibility for vanishing of the turbulence energy at large distances in directions perpendicular to the axis.

The expansion of the tensor F_{ij} up to the second order in r can be written in terms of thirteen absolute constants, and nine other scalar coefficients which depend on the distance of one of the points from the localized fixed axis. Some tentative suggestions have been made for experimental determination of seventeen of these twenty-two quantities which leave undetermined only five of the constants. Three check equations have been provided in these calculations.

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