ON THE ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES, ITS CONJUGATE SERIES AND THEIR DERIVED SERIES

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1.1. **Definitions.** Let Σa_n be a given infinite series, and let λ_n be a positive, monotonic increasing function of n, steadily tending to infinity with n. We write

$$A_\lambda(\omega) = A_\lambda^0(\omega) = \sum_{\lambda_n \leqslant \omega} a_n,$$

$$A_\lambda^r(\omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^r a_n.$$

The series Σa_n is said to be absolutely summable (R, λ, r), or summable $|R, \lambda, r|$, $r \geqslant 0$, if $A_\lambda^r(\omega)/\omega^r$ is of bounded variation in (A, ∞), where A is a finite positive number (Obrechkoff, 1928, 1929).

An equivalent definition is obtained, as follows, by a suitable definition of λ at non-integral points and a corresponding change of variable.

Let $\lambda = \lambda(\omega)$ be a continuous, differentiable and monotonic increasing function of ω in (K, ∞), where K is a positive constant and let $\lambda(\omega)$ tend to infinity with ω. We write

$$C_r(\omega) = \sum_{n \leqslant \omega} \{\lambda(\omega) - \lambda(n)\}^r a_n, \; r \geqslant 0.$$

Then Σa_n is said to be summable $|R, \lambda, r|$, $r \geqslant 0$, if the integral

$$\int_A^\infty |d[C_r(\omega)/\{\lambda(\omega)\}^r]|,$$

where A is a finite positive number, is convergent. Now, for $r > 0$, $m < \omega < m+1$,

$$\frac{d}{d\omega}[C_r(\omega)/\{\lambda(\omega)\}^r] = \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \sum_{n \leqslant \omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda(n) a_n.$$

Hence, summability $|R, \lambda, r|$, $r > 0$, is equivalent to the convergence of the integral

$$\int_A^\infty \left| \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \sum_{n \leqslant \omega} \{\lambda(\omega) - \lambda(n)\}^{r-1} \lambda(n) a_n \right| d\omega.$$

Evidently summability $|R, \lambda, 0|$ is equivalent to absolute convergence. For convenience we shall adopt throughout the present paper the alternative definition given above.

1.2. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we assume the Fourier series of $f(t)$ to be given by

$$\sum_{1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{1}^{\infty} A_n(t), \tag{1.21}$$

so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0. \tag{1.22}$$

Then the conjugate series of the Fourier series of $f(t)$ is given by

$$\sum_{1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} B_n(t). \tag{1.23}$$

Throughout we shall use the following notations

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\};$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\};$$

$$P(t) = \sum_{i=0}^{r-1}(\theta_i t^i/i\,!),$$

where the θ's are arbitrary;

$$g(t) = \tfrac{1}{2}[\{f(x+t)-P(t)\}+(-1)^r\{f(x-t)-P(-t)\}];$$

$$h(t) = \tfrac{1}{2}[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}];$$

$$\Phi_\sigma(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-u)^{\sigma-1}\phi(u)du,\ \sigma > 0;$$

$$\Phi_0(t) = \phi(t);$$

$$\phi_\sigma(t) = \Gamma(\sigma+1)t^{-\sigma}\Phi_\sigma(t),\ \sigma \geqslant 0;$$

$\Psi_\sigma(t)$, $\psi_\sigma(t)$, $G_\sigma(t)$, $g_\sigma(t)$, $H_\sigma(t)$, $h_\sigma(t)$ have similar meanings;

$$\gamma_{\alpha,r}(t) = g_{\alpha-r}(t)/t^r;\ \theta_{\alpha,r}(t) = h_{\alpha-r}(t)/t^r;$$

$$e(\omega) = \exp\{(\log\omega)^{1+1/\alpha}\};$$

$$E(\omega, t) = \sum_{n\leqslant\omega}\{e(\omega)-e(n)\}^{\alpha-1+\delta}e(n)\cos nt;\ E^{(r)}(\omega, t) = \frac{\partial^r}{\partial t^r}E(\omega, t);$$

$$\bar{E}(\omega, t) = \sum_{n\leqslant\omega}\{e(\omega)-e(n)\}^{\alpha-1+\delta}e(n)\sin nt;\ \bar{E}^{(r)}(\omega, t) = \frac{\partial^r}{\partial t^r}\bar{E}(\omega, t);$$

$$g(\omega, t) = \int_0^t \frac{u^\alpha}{\log(k/u)}E^{(\alpha)}(\omega, u)du;\ h(\omega, t) = \int_t^\pi \frac{u^\alpha}{\log(k/u)}E^{(\alpha)}(\omega, u)du;$$

$$(F(t))_r = \frac{\partial^r}{\partial t^r}F(t).$$

1.3. The author recently established (Pati, 1954) four theorems of a very general character concerning the absolute Riesz summability, of a rapidly increasing type, of the Fourier series, its conjugate series and their derived series. The object of the present paper is to improve upon these results by replacing the order of summability $\alpha+1$ by the more precise order $\alpha+\delta$, $0<\delta<1$. It will be seen that the special cases, for $\alpha=1$, of the theorems of this paper give sharper results than the theorems 4, 6, 7 and 8, respectively, of Mohanty (1951). Also, when ρ is an integer $\geqslant 1$, Lemma 11 of the present paper provides a deeper insight into the summability of the infinite series $\Sigma(-1)^n n^\rho$ than the result (Lemma 7 of (3)) obtained by Hyslop (1940) concerning its absolute Cesàro summability. Hyslop's result follows from our result by an application of a 'second theorem of consistency' due to Chandrasekharan (1942) (Lemma 12 of this paper). It may be mentioned that the particular case, for $\alpha=1$, of Theorem 1 of the present paper is the direct analogue for absolute summability of a theorem of Wang (1943) on the ordinary Riesz summability of Fourier series.

2.1. We establish the following theorems.

Theorem 1. *If α is an integer $\geqslant 1$, and $\phi_\alpha(t)\log(k/t)$ is of bounded variation in $(0,\pi)$, then the Fourier series of $f(t)$, at $t=x$, is summable $|R, e(\omega), \alpha+\delta|$ for every $\delta>0$.*

Theorem 2. *If α is an integer $\geqslant 1$, and if (i) $\phi_\alpha(t)\log(k/t)$ is of bounded variation in $(0,\pi)$ and (ii) $|\psi_\alpha(t)|/t$ is integrable (L) over $(0,\pi)$, then the conjugate series of the Fourier series of $f(t)$, at $t=x$, is summable $|R, e(\omega), \alpha+\delta|$ for every $\delta>0$.*

Theorem 3. *If r is an integer $\geqslant 1$, and $\gamma_{\alpha,r}(t)\log(k/t)$ is of bounded variation in $(0,\pi)$, then the rth derived series of the Fourier series of $f(t)$, at $t=x$, is summable $|R, e(\omega), \alpha+\delta|$ for every integral $\alpha\geqslant r$, and $\delta>0$.*

Theorem 4. *If r is an integer $\geqslant 1$, and if (i) $\theta_{\alpha,r}(t)\log(k/t)$ is of bounded variation in $(0,\pi)$ and (ii) $|\theta_{\alpha,r}(t)|/t$ is integrable (L) over $(0,\pi)$, then the rth derived series of the conjugate series of the Fourier series of $f(t)$, at $t=x$, is summable $|R, e(\omega), \alpha+\delta|$ for every integral $\alpha\geqslant r$ and $\delta>0$.*

In the enunciations of these theorems it is sufficient to take $k>\pi$. Since it is immaterial what particular value k has, for the sake of convenience we assume $k>e^{\alpha+2}\pi$ in the proofs of Theorems 1 and 3.

2.2. We shall require the following lemmas for the proof of our theorems.

Lemma 1. (Obrechkoff, 1928, 1929). *If a series Σa_n is summable $|R, \lambda, r|$, $r\geqslant 0$, then it is summable $|R, \lambda, r'|$ for every $r'>r$.*

Lemma 2.* *The Fourier series of the special functions*

$$\left(\log\left|\frac{k}{t}\right|\right)^{-1}, \left(\log\left|\frac{k}{t}\right|\right)^{-2}, \ldots, \left(\log\left|\frac{k}{t}\right|\right)^{-(\alpha+1)},$$

are all absolutely convergent at $t=0$.

Lemma 3.† *Let $C_n^{(k)}$, $S_n^{(k)}$ and $\bar{S}_n^{(k)}$ denote the nth Cesàro-sums of order k $(k\geqslant 0)$ corresponding to the series*

$$\sum_1^\infty(-1)^n n^\rho, \sum_1^\infty(\cos nt)_\rho \text{ and } \sum_1^\infty(\sin nt)_\rho$$

* Pati (1954), Lemma 2
† Pati (1954), Lemma 3

respectively. Then

(i) $S_n^{(k)} = O\left(n^{\rho+k+1}\right),\ 0 < t \leqslant 1/n;$

(ii) $S_n^{(k)} = O\left(n^{\rho} t^{-(k+1)}\right) + O\left(n^{k-1} t^{-(\rho+1)}\right),\ 1/n < t \leqslant \pi;$

(iii) $\bar{S}_n^{(k)} = O\left(n^{\rho+k+1}\right),\ 0 < t \leqslant 1/n;$

(iv) $\bar{S}_n^{(k)} = O\left(n^{\rho} t^{-(k+1)}\right) + O\left(n^{k} t^{-(\rho+1)}\right),\ 1/n < t \leqslant \pi;$

(v) *when* ρ *is an even integer* $\geqslant 2$,

$$C_n^{(k)} = O\left(n^{\max(\rho, k-1)}\right).$$

Lemma 4. (Hardy and Riesz, 1915). *Let*

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \leqslant x} a_n,$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n \leqslant x} (x-\lambda_n)^r a_n,\ r > 0.$$

Then, if k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!}\left(\frac{d}{dx}\right)^k A_\lambda^k(x).$$

Lemma 5. *The nth derivative of* $\{F(x)\}^m$ *is a sum of constant multiples of terms of the form*

$$\{F(x)\}^{m-r}\{F^{(1)}(x)\}^{\alpha_1}\{F^{(2)}(x)\}^{\alpha_2}\ldots\{F^{(n)}(x)\}^{\alpha_n},$$

where $r \leqslant n$ *and the* α*'s are positive integers or zeros such that*

$$\sum_{\nu=1}^{n} \alpha_\nu = r; \quad \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

This is a particular case of a result, due to Faa di Bruno (Vallée Poussin, 1923), on the successive derivative of a function of a function.

Lemma 6. *If* ρ *is an even integer such that* $2 \leqslant \rho \leqslant \alpha - 1$, *then*

$$\int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega\, e^{\alpha+\delta}(\omega)} \left| E^{(\rho)}(\omega, \pi) \right| d\omega < \infty.$$

Proof.

We have

$$E^{(\rho)}(\omega, \pi) = (-1)^{\frac{\rho}{2}} \sum_{n \leqslant \omega} \{e(\omega) - e(n)\}^{\alpha+\delta-1} e(n)(-1)^n n^\rho.$$

Evidently it suffices to consider the convergence of the integrals

$$\int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}} e(\omega)}{\omega e^{\alpha+\delta}(\omega)} \left| \sum\nolimits_1 \right| d\omega, \quad \int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| \sum\nolimits_2 \right| d\omega,$$

where

$$\sum\nolimits_1 = \sum_{n \leqslant \omega} \{e(\omega)-e(n)\}^{\alpha+\delta-1}(-1)^n n^\rho$$

and

$$\sum\nolimits_2 = \sum_{n \leqslant \omega} \{e(\omega)-e(n)\}^{\alpha+\delta}(-1)^n n^\rho.$$

We adopt the notations

$$C(x) = \sum_{n \leqslant x} (-1)^n n^\rho;$$

$$C^k(x) = \sum_{n \leqslant x} (x-n)^k(-1)^n n^\rho \ (k = 1, 2, \ldots).$$

We then have *

$$(2.21) \qquad \begin{cases} C^{\alpha-1}(x) = O(x^{\alpha-1}); \\ C^{\alpha}(x) = O(x^{\alpha-1}). \end{cases}$$

Now

$$\sum\nolimits_1 = -\int_1^\omega C(x)\frac{d}{dx}\{e(\omega)-e(x)\}^{\alpha+\delta-1}dx$$

$$= -\left(\int_1^e + \int_e^\omega\right) C(x)\frac{d}{dx}\{e(\omega)-e(x)\}^{\alpha+\delta-1}dx$$

$$= -(I_1+I_2), \text{ say}.$$

We easily have

$$e(\omega)I_1 = O\{e^{\alpha+\delta-1}(\omega)\}.$$

Next

$$I_2 = \int_e^\omega C(x)\frac{d}{dx}\{e(\omega)-e(x)\}^{\alpha+\delta-1}dx$$

$$= \frac{1}{(\alpha-1)!}\int_e^\omega \left(\frac{d}{dx}\right)^{\alpha-1} C^{\alpha-1}(x)\frac{d}{dx}\{e(\omega)-e(x)\}^{\alpha+\delta-1}dx.$$

Integrating by parts $(\alpha-1)$ times, we have

$$I_2 = O\{e^{\alpha+\delta-2}(\omega)\}$$

$$+O\left(\left|\int_e^\omega C^{\alpha-1}(x)\left(\frac{d}{dx}\right)^{\alpha}\{e(\omega)-e(x)\}^{\alpha+\delta-1}dx\right|\right).$$

* Cf. Pati (1954), (2.24)

Now, by Lemma 5, the last integral is a sum of constant multiples of integrals of the type

$$\mathscr{I}=\int_{\epsilon}^{\omega} C^{\alpha-1}(x)\{e(\omega)-e(x)\}^{\alpha+\delta-1-r}\{e^{(1)}(x)\}^{\beta_1}\cdots\{e^{(\alpha)}(x)\}^{\beta_\alpha}dx,$$

where

(2·22) $$\beta_1+\beta_2+\cdots+\beta_\alpha=r\leqslant\alpha;\ \beta_1+2\beta_2+\cdots+\alpha\beta_\alpha=\alpha.$$

Case (i): $r<\alpha$. In this case, since both r and α are positive integers, $r\leqslant\alpha-1$, so that

$$\mathscr{I}=\int_{\epsilon}^{\omega} C^{\alpha-1}(x)\{e(\omega)-e(x)\}^{\alpha-1-r+\delta}\{e^{(1)}(x)\}^{\beta_1}\cdots\{e^{(\alpha)}(x)\}^{\beta_\alpha}dx$$

$$=O\left[\{e(\omega)\}^{\alpha-1-r+\delta}\frac{\{(\log\omega)^{\frac{1}{\alpha}}\}^{\beta_1+\cdots+\alpha\beta_\alpha}}{\omega^{\beta_1+\cdots+\alpha\beta_\alpha}}\{e(\omega)\}^{\beta_1+\cdots+\beta_\alpha}\right.$$

$$\left.\times\max_{\epsilon\leqslant\xi\leqslant\xi'\leqslant\omega}\left|\int_{\xi}^{\xi'}C^{\alpha-1}(x)dx\right|\right]$$

$$=O\left\{e^{\alpha+\delta-1}(\omega)\frac{\log\omega}{\omega^{\alpha}}\omega^{\alpha-1}\right\}$$

$$=O\left\{\frac{\log\omega}{\omega}e^{\alpha+\delta-1}(\omega)\right\}.$$

Hence

$$e(\omega)I_2=O\left\{\frac{\log\omega}{\omega}e^{\alpha+\delta}(\omega)\right\}.$$

Case (ii): $r=\alpha$. In this case, by a subtraction, we get from the relations (2.22)

$$\beta_2+2\beta_3+\ldots+(\alpha-1)\beta_\alpha=0,$$

which implies that

$$\beta_2=\beta_3=\ldots=\beta_\alpha=0;\ \beta_1=\alpha.$$

Thus

$$\mathscr{I}=\int_{\epsilon}^{\omega} C^{\alpha-1}(x)\{e(\omega)-e(x)\}^{\delta-1}\{e^{(1)}(x)\}^{\alpha}dx$$

$$=\left(\int_{\epsilon}^{\omega_1}+\int_{\omega_1}^{\omega}\right)C^{\alpha-1}(x)\{e(\omega)-e(x)\}^{\delta-1}\{e^{(1)}(x)\}^{\alpha}dx,$$

where $$(\log\omega_1)^{1+\frac{1}{\alpha}}=(\log\omega)^{1+\frac{1}{\alpha}}-1,$$

$$=\mathscr{I}_1+\mathscr{I}_2,\text{ say.}$$

Now

$$\mathscr{I}_1 = O\left[\{e(\omega)-e(\omega_1)\}^{\delta-1}\{e^{(1)}(\omega)\}^{\alpha}\max\left|\int_{\xi}^{\xi'} C^{\alpha-1}(x)dx\right|\right] \quad (\xi' \leqslant \omega_1)$$

$$= O\left\{\frac{\log\omega}{\omega^{\alpha}}\ e^{\alpha+\delta-1}(\omega)\omega^{\alpha-1}\right\}$$

$$= O\left\{\frac{\log\omega}{\omega}\ e^{\alpha+\delta-1}(\omega)\right\}.$$

Also, using the transformation

$$u = (\log\omega)^{1+\frac{1}{\alpha}} - (\log x)^{1+\frac{1}{\alpha}} = \omega' - (\log x)^{1+\frac{1}{\alpha}},$$

$$\mathscr{I}_2 = K\int_0^1 \{e(\omega)-e(\omega)e^{-u}\}^{\delta-1}e^{\alpha}(\omega)e^{-\alpha u}\frac{(\omega'-u)^{\alpha(1-1/\alpha)/(\alpha+1)}}{e^{(\alpha-1)(\omega'-u)^{\alpha/(\alpha+1)}}}C^{\alpha-1}\left(e^{(\omega'-u)^{\alpha/(\alpha+1)}}\right)du$$

$$= K\left(\int_0^{(\log\omega)^{1/\alpha}/\omega} + \int_{(\log\omega)^{1/\alpha}/\omega}^1\right)$$

$$= K(\mathscr{I}_{2,1}+\mathscr{I}_{2,2}), \text{ say.}$$

Now

$$\mathscr{I}_{2,1} = O\left[e^{\alpha}(\omega)(\log\omega)^{1-1/\alpha}e^{\delta-1}(\omega)\int_0^{(\log\omega)^{1/\alpha}/\omega} u^{\delta-1}du\right] \qquad \text{(by (2.21))}$$

$$= O\left[e^{\alpha+\delta-1}(\omega)(\log\omega)^{1-1/\alpha}\left\{\frac{(\log\omega)^{1/\alpha}}{\omega}\right\}^{\delta}\right]$$

$$= O\left\{e^{\alpha+\delta-1}(\omega)(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}\Big/\omega^{\delta}\right\}.$$

Also

$$\mathscr{I}_{2,2} = O\left[e^{\alpha+\delta-1}(\omega)\left|\int_{(\log\omega)^{1/\alpha}/\omega}^1 (1-e^{-u})^{\delta-1}\frac{e^{-\alpha u}(\omega'-u)^{\alpha/(\alpha+1)}}{e^{\alpha(\omega'-u)^{\alpha/(\alpha+1)}}}C^{\alpha-1}\left(e^{(\omega'-u)^{\alpha/(\alpha+1)}}\right)\times d\left(e^{(\omega'-u)^{\alpha/(\alpha+1)}}\right)\right|\right]$$

$$= O\left[e^{\alpha+\delta-1}(\omega)\left\{\frac{(\log\omega)^{1/\alpha}}{\omega}\right\}^{\delta-1}\frac{\log\omega}{\omega^{\alpha}}\max\left|\int_{\xi}^{\xi'}C^{\alpha-1}(x)dx\right|\right]$$

$$(\omega_1 \leqslant \xi \leqslant \xi' \leqslant x_1,\ (\log x_1)^{1+1/\alpha} = \omega' - (\log\omega)^{1/\alpha}/\omega)$$

$$= O\left\{e^{\alpha+\delta-1}(\omega)(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}\Big/\omega^{\delta}\right\}.$$

Hence, finally,

$$e(\omega)I_2 = O\left\{\frac{\log\omega}{\omega}\,e^{\alpha+\delta}(\omega)\right\} + O\left\{\frac{(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^{\delta}}\,e^{\alpha+\delta}(\omega)\right\}.$$

Thus

$$\int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} e(\omega) \left|\sum\nolimits_1\right| d\omega < \infty .$$

Similarly

$$\int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left|\sum\nolimits_2\right| d\omega < \infty .$$

This completes the proof of Lemma 6.

Lemma 7. *If ρ is zero or a positive integer, then*

$$\sum_{n\leqslant\omega} \{e(\omega)-e(n)\}^{\alpha+\delta-1} e(n)n^\rho = O\left\{\frac{\omega^{\rho+1}}{(\log\omega)^{1/\alpha}} e^{\alpha+\delta}(\omega)\right\}.$$

Proof.

For $m \leqslant \omega < m+1$, $\alpha \geqslant 2$,

$$\sum_{n\leqslant\omega} \{e(\omega)-e(n)\}^{\alpha+\delta-1} e(n)n^\rho = \sum_1^m \{e(\omega)-e(n)\}^{\alpha+\delta-1} e(n)n^\rho$$

$$= O\left[e^\delta(\omega)\sum_1^m \{e(\omega)-e(n)\}^{\alpha+\delta-1} e(n)n^\rho\right]$$

$$= O\left\{\frac{\omega^{\rho+1}}{(\log\omega)^{1/\alpha}} e^{\alpha+\delta}(\omega)\right\}$$

$$= O\left\{\omega^{\rho+1}e^{\alpha+\delta}(\omega)/(\log\omega)^{1/\alpha}\right\},$$

by Lemma 7 of (7). If $\alpha = 1$, $\sum_1^m \{e(\omega)-e(n)\}^{\alpha+\delta-1} e(n)n^\rho$

$$= \sum_1^m \{e(\omega)-e(n)\}^{\delta} e(n)n^\rho = O\{\omega^{\rho+1}e^{1+\delta}(\omega)/\log\omega\}.$$

Lemma 8. *If α is a positive integer $\geqslant 1$, then*

$$E^{(\alpha-1)}(\omega, t) = O\left\{\frac{\log\omega}{\omega} e^{\alpha+\delta}(\omega)t^{-(\alpha+1)}\right\} + O\left\{\frac{(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^\delta} e^{\alpha+\delta}(\omega)t^{-(\alpha+\delta)}\right\}.$$

Proof. The proof proceeds along the same lines as that of Lemma 6. In the analysis $C(x)$ and $C^{\alpha-1}(x)$ will be replaced by $S(x)$ and $S^{\alpha-1}(x)$, where

$$S(x) = \sum_{n\leqslant x} (\cos nt)_{\alpha-1},$$

$$S^{\alpha-1}(x) = \sum_{n\leqslant x} (x-n)^{\alpha-1}(\cos nt)_{\alpha-1},$$

and

$$S^{\alpha}(x) = \sum_{n\leqslant x} (x-n)^{\alpha}(\cos nt)_{\alpha-1}.$$

We use the estimates:

$$\begin{cases} S^{\alpha-1}(x) = O(x^{\alpha-1}t^{-\alpha}); \\ S^{\alpha}(x) = O(x^{\alpha-1}t^{-(\alpha+1)}). \end{cases}$$

The integral corresponding to $\mathscr{I}_2$ will be broken up into integrals over $(0, K(\log \omega)^{1/\alpha}/\omega t)$, $(K(\log \omega)^{1/\alpha}/\omega t, 1)$, where K is a suitable constant in view of the inequality $\omega > \tau = \frac{k}{t}\left(\log \frac{k}{t}\right)^{1/\alpha}$, and the treatment of these will be similar to that of $\mathscr{I}_{2,1}$ and $\mathscr{I}_{2,2}$ respectively.

Lemma 9. *If ρ is zero or a positive integer $\leqslant \alpha-1$, then*

$$\bar{E}^{(\rho)}(\omega,t) = O\left\{\frac{\log \omega}{\omega} e^{\alpha+\delta}(\omega)t^{-(\rho+2)}\right\} + O\left\{\frac{(\log \omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^{\delta}} e^{\alpha+\delta}(\omega)t^{-(\rho+1+\delta)}\right\}.$$

The proof of this lemma is parallel to that of Lemma 9 of (7). Application of the same technique as that employed in the proofs of Lemmas 6 and 8 of this paper will yield the result.

Lemma 10. *If ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha-1$, then*

$$\int_1^{\infty} \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \,|\bar{E}^{(\rho)}(\omega, \pi)|\, d\omega < \infty.$$

The result follows immediately on application of Lemma 9.

Lemma 11. *If ρ is an integer such that $1 \leqslant \rho \leqslant \alpha-1$, then the infinite series $\Sigma(-1)^n n^{\rho}$ is summable $|R, e(\omega), \alpha+\delta|$ for every $\delta > 0$.*

This is essentially a combination of the results of Lemmas 6 and 10.

Lemma 12. (Chandrasekharan, 1942). *If the series Σa_n is summable, $|R, \lambda, r|$, $r \geqslant 0$, and μ is a logarithmico-exponential function of λ such that $\mu = O(\lambda^{\Delta})$, where Δ is a constant, then the series Σa_n is summable $|R, \mu, r|$.*

Lemma 13. (Mohanty, 1951). *The necessary and sufficient conditions that* (i) *$F(t) \log (k/t)$ be of bounded variation in $(0, \eta)$ and* (ii) *$|F(t)|/t$ be integrable (L) in $(0, \eta)$, η being positive, are that $\int_0^{\eta} \log (k/t)\,|dF(t)| < \infty$ and $F(+0) = 0$.*

Lemma 14. (Mohanty, 1951). *If $F(+0) = 0$ and $\int_0^{\pi} \log (k/t)\,|dF(t)| < \infty$, then the series Σv_n, where*

$$v_n = \int_0^{\pi} F(t)\sin nt\, dt = -F(\pi)\frac{\cos n\pi}{n} + \int_0^{\pi} \frac{\cos nt}{n}\, dF(t),$$

is summable $|R$, $\exp(\omega^{\delta})$, $1|$, where $0 < \delta < 1$.

3.1. Proof of Theorem 1.

Since

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \phi(t)\cos nt\, dt,$$

we have only to establish that

$$\int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| \int_0^\pi \phi(t) E(\omega, t) dt \right| d\omega < \infty.$$

Integrating by parts α times, we have

$$\int_0^\pi \phi(t) E(\omega, t) dt = \left[\sum_1^\alpha (-1)^{\rho-1} \Phi_\rho(t) E^{(\rho-1)}(\omega, t) \right]_0^\pi + (-1)^\alpha \int_0^\pi \Phi_\alpha(t) E^{(\alpha)}(\omega, t) dt.$$

Again

$$\int_0^\pi \Phi_\alpha(t) E^{(\alpha)}(\omega, t) dt = \frac{1}{\Gamma(\alpha+1)} \left[\phi_\alpha(t) \log (k/t) g(\omega, t) \right]_0^\pi$$

$$- \frac{1}{\Gamma(\alpha+1)} \int_0^\pi d\{\phi_\alpha(t) \log (k/t)\} g(\omega, t).$$

Hence, finally,

$$\int_0^\pi \phi(t) E(\omega, t) dt = \left[\sum_1^\alpha (-1)^{\rho-1} \Phi_\rho(t) E^{(\rho-1)}(\omega, t) \right]_0^\pi + \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \phi_\alpha(\pi) \log (k/\pi) g(\omega, \pi)$$

$$+ \frac{(-1)^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\pi d\{\phi(t) \log (k/t)\} g(\omega, t).$$

We observe that $\Phi_1(\pi) = 0$ and $E^{(r)}(\omega, \pi) = 0$, whenever r is odd. Hence it suffices to prove only the following.

$$\text{(I)} \quad \int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| E^{(\rho)}(\omega, \pi) \right| d\omega < \infty,$$

where ρ is an even integer such that $2 \leqslant \rho \leqslant \alpha - 1$;

$$\text{(II)} \quad \int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| g(\omega, \pi) \right| d\omega < \infty;$$

and

$$\text{(III)} \quad \int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| g(\omega, t) \right| d\omega = O(1) \quad \text{for } 0 < t < \pi.$$

The result (I) has been established in Lemma 6.

Again, since $g(\omega, t) = g(\omega, \pi) - h(\omega, t)$,

$$\int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| g(\omega, t) \right| d\omega$$

$$< \int_1^\tau \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| g(\omega, t) \right| d\omega + \int_1^\infty \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} \left| g(\omega, \pi) \right| d\omega$$

$$+\int_{\tau}^{\infty} \frac{(\log \omega)^{\frac{1}{\alpha}}}{\omega e^{\alpha+\delta}(\omega)} |h(\omega, t)| \, d\omega.$$

$$\left(\tau = \frac{k}{t}\left(\log \frac{k}{t}\right)^{\frac{1}{\alpha}}\right)$$

Hence, Theorem 1 will be established if only

$$(3.11) \qquad I_1 = \int_1^{\infty} \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |g(\omega, \pi)| \, d\omega < \infty;$$

$$(3.12) \qquad I_2 = \int_1^{\tau} \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |g(\omega, t)| \, d\omega = O(1) \text{ for } 0 < t < \pi;$$

$$(3.13) \qquad I_3 = \int_{\tau}^{\infty} \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |h(\omega, t)| \, d\omega = O(1) \text{ for } 0 < t < \pi.$$

Proof of (3.11).

Since

$$|g(\omega, \pi)| = \begin{cases} \left| \sum_{n \leqslant \omega} \{e(\omega) - e(n)\}^{\alpha-1+\delta} e(n) n^{\alpha} \int_0^{\pi} \frac{u^{\alpha}}{\log (k/u)} \cos nu \, du \right| (\alpha \text{ even}), \\ \left| \sum_{n \leqslant \omega} \{e(\omega) - e(n)\}^{\alpha-1+\delta} e(n) n^{\alpha} \int_0^{\pi} \frac{u^{\alpha}}{\log (k/u)} \sin nu \, du \right| (\alpha \text{ odd}), \end{cases}$$

proving the convergence of I_1 is the same thing as proving the summability $|R, e(\omega), \alpha+\delta|$, $\delta > 0$, of $\Sigma n^{\alpha}\lambda_n$, where λ_n is the Fourier cosine constant of the even function $u^{\alpha}/\log |k/u|$, defined by periodicity outside $(-\pi, \pi)$, or of the series $\Sigma n^{\alpha}\mu_n$, where μ_n is the Fourier sine constant of the odd function $u^{\alpha}/\log |k/u|$, defined by periodicity outside $(-\pi, \pi)$, according as α is even or odd. The proof now follows the same lines as the proof of (3.11) in (7), with application of Lemma 11 of the present paper instead of Lemma 11 of (7).

Proof of (3.12).

$$g(\omega, t) = \int_0^t \frac{u^{\alpha}}{\log (k/u)} E^{(\alpha)}(\omega, u) du$$

$$= \frac{t^{\alpha}}{\log (k/t)} \int_{\eta}^{t} \frac{\partial}{\partial u} E^{(\alpha-1)}(\omega, u) du, \; 0 < \eta < t,$$

$$= O\left[\frac{t^{\alpha}}{\log (k/t)} \sum_{n \leqslant \omega} \{e(\omega) - e(n)\}^{\alpha+\delta-1} e(n) n^{\alpha-1}\right]$$

$$= O\left\{\frac{t^{\alpha}}{\log (k/t)} \omega^{\alpha} e^{\alpha+\delta}(\omega)/(\log \omega)^{1/\alpha}\right\},$$

by Lemma 7. Therefore

$$I_2 = O\left\{\int_1^{\tau} \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \frac{t^{\alpha}}{\log(k/t)} \frac{\omega^{\alpha}}{(\log\omega)^{1/\alpha}} e^{\alpha+\delta}(\omega) d\omega\right\}$$

$$= O\left\{\frac{t^{\alpha}}{\log(k/t)} \int_1^{\tau} \omega^{\alpha-1} d\omega\right\}$$

$$= O(1) \qquad \text{for } 0 < t < \pi.$$

Proof of (3.13).

$$h(\omega, t) = O\left[\,|E^{(\alpha-1)}(\omega, \pi)|\,\right] + O\left[\frac{t^{\alpha}}{\log(k/t)} |E^{(\alpha-1)}(\omega, t)|\right]$$

$$+ O\left[\int_t^{\pi} \frac{u^{\alpha-1}}{\log(k/u)} |E^{(\alpha-1)}(\omega, u)| du\right]$$

$$= O\left\{|E^{(\alpha-1)}(\omega, \pi)|\right\} + O\left\{\frac{1}{t\log(k/t)} \frac{\log\omega}{\omega} e^{\alpha+\delta}(\omega)\right\}$$

$$+ O\left\{\frac{(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^{\delta}} e^{\alpha+\delta}(\omega) \frac{1}{t^{\delta}\log(k/t)}\right\}$$

$$+ O\left\{\frac{\log\omega}{\omega} e^{\alpha+\delta}(\omega) \int_t^{\pi} \frac{du}{u^2\log(k/u)}\right\}$$

$$+ O\left\{\frac{(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^{\delta}} e^{\alpha+\delta}(\omega) \int_t^{\pi} \frac{du}{u^{1+\delta}\log(k/u)}\right\}, \text{ by Lemma 8,}$$

$$= O\{|E^{(\alpha-1)}(\omega, \pi)|\} + O\left\{\frac{1}{t\log(k/t)} \frac{\log\omega}{\omega} e^{\alpha+\delta}(\omega)\right\}$$

$$+ O\left\{\frac{1}{t^{\delta}\log(k/t)} \frac{(\log\omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^{\delta}} e^{\alpha+\delta}(\omega)\right\}.$$

Now, observing that $E^{(\alpha-1)}(\omega, \pi) = 0$, when α is even, and applying Lemmas 6 and 8, when α is odd, we obtain

$$I_3 = O(1) + O\left\{\frac{1}{t\log(k/t)} \int_{\tau}^{\infty} \frac{(\log\omega)^{1+1/\alpha}}{\omega^2} d\omega\right\}$$

$$+ O\left\{\frac{1}{t^{\delta}\log(k/t)} \int_{\tau}^{\infty} \frac{(\log\omega)^{1+\delta/\alpha}}{\omega^{1+\delta}} d\omega\right\}$$

$$= O(1) \text{ for } 0 < t < \pi.$$

This completes the proof of Theorem 1.

3.2. **Proof of Theorem 2.** In view of Lemma 13, Theorem 2 can be put in the following equivalent form.

Theorem 2a. *If α is an integer $\geqslant 1$, and if* (i) $\int_0^\pi \log (k/t)\,|d\psi_\alpha(t)| < \infty$ *and* (ii) $\psi_\alpha(+0) = 0$, *then the conjugate series of the Fourier series of $f(t)$, at $t = x$, is summable $|R, e(\omega), \alpha+\delta|$ for every $\delta > 0$.*

We proceed to prove Theorem 2*a*. Since

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,dt,$$

we have only to show that, under the hypotheses of the theorem, the integral

$$\int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)}\left|\int_0^\pi \psi(t)\bar{E}(\omega, t)dt\right| d\omega$$

is convergent. Integrating

$$\int_0^\pi \psi(t)\bar{E}(\omega, t)dt$$

by parts α times, as in the proof of Theorem 2*a* of (7), we find that it will suffice for our purpose to prove the following.

$$\int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)}\,|\bar{E}^{(\rho)}(\omega, \pi)|\,d\omega < \infty, \tag{3.21}$$

where ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha-1$;

$$J = \int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)}\left|\int_0^\pi d\psi_\alpha(t)\,\Lambda(t)\right| d\omega < \infty, \tag{3.22}$$

where

$$\Lambda(t) = t^\alpha\bar{E}^{(\alpha-1)}(\omega, t) - \alpha t^{\alpha-1}\bar{E}^{(\alpha-2)}(\omega, t) + \dots$$
$$+ (-1)^{\alpha-1}\alpha(\alpha-1)\dots 2\,t\,\bar{E}(\omega, t);$$

$$\int_1^\infty \frac{(\log\omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)}\left|\sum_{n\leqslant\omega}\{e(\omega)-e(n)\}^{\alpha-1+\delta}e(n)\right.$$
$$\left.\times\left\{-\psi_\alpha(\pi)\frac{\cos n\pi}{n} + \int_0^\pi \frac{\cos nt}{n}\,d\psi_\alpha(t)\right\}\right| d\omega < \infty. \tag{3.23}$$

Proof of (3.21). The result has already been established in Lemma 10.

Proof of (3.22).

$$J \leqslant \left(\int_1^\tau + \int_\tau^\infty\right) \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left\{\int_0^\pi |d\psi_\alpha(t)|\,|\Lambda(t)|\right\} d\omega \quad \left(\tau = (k/t)\{\log (k/t)\}^{1/\alpha}\right)$$

$$= J_1 + J_2, \text{ say.}$$

The proof of the convergence of J_1 follows exactly the same lines as that of its counterpart in (7), and requires the application of Lemma 7 of the present paper.

To prove the convergence of J_2 we observe that

$$\Lambda(t) = O\left\{\frac{\log \omega}{\omega} e^{\alpha+\delta}(\omega) t^{-1}\right\}$$
$$+ O\left\{\frac{(\log \omega)^{1+\frac{\delta}{\alpha}-\frac{1}{\alpha}}}{\omega^\delta} e^{\alpha+\delta}(\omega) t^{-\delta}\right\}.$$

Hence

$$J_2 = O\left\{\int_0^\pi |d\psi_\alpha(t)|\, t^{-1} \int_\tau^\infty \frac{(\log \omega)^{1+1/\alpha}}{\omega^2} d\omega\right\}$$
$$+ O\left\{\int_0^\pi |d\psi_\alpha(t)|\, t^{-\delta} \int_\tau^\infty \frac{(\log \omega)^{1+1/\alpha}}{\omega^{1+\delta}} d\omega\right\}$$
$$= O\left\{\int_0^\pi \log (k/t)\, |d\psi_\alpha(t)|\, \{t \log (k/t)\}^{-1} \int_\tau^\infty \frac{(\log \omega)^{1+1/\alpha}}{\omega^2} d\omega\right\}$$
$$+ O\left\{\int_0^\pi \log (k/t)\, |d\psi_\alpha(t)|\, \{t^\delta \log (k/t)\}^{-1} \int_\tau^\infty \frac{(\log \omega)^{1+\delta/\alpha}}{\omega^{1+\delta}} d\omega\right\}$$
$$= O(1).$$

Proof of (3.23).

This proceeds on exactly the same lines as the proof of (3.23) in (7), and requires the use of Lemmas 14, 12 and 1 of the present paper.

This completes the proof of Theorem 2.

3.3. **Proof of Theorem 3.** Let r be even. As in the proof of Theorem 3 of (7), we need only prove the following.

(3.31)
$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left|\int_0^\pi \tfrac{1}{2}\{P(t)+P(-t)\}\, E^{(r)}(\omega, t)\,dt\right| d\omega < \infty$$

and

(3.32)
$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left|\int_0^\pi g(t) E^{(r)}(\omega, t)\,dt\right| d\omega < \infty.$$

Proving (3.31) is the same thing as proving the summability $|R, e(\omega), \alpha+\delta|$, $\delta > 0$, of $\Sigma n^r p_n$, where p_n is the Fourier cosine constant of the even function $\frac{1}{2}\{P(t)+P(-t)\}$. This can be easily proved by making use of Lemma 11.

To prove (3.32), following the same line of argument as in the proof of (3.32) in (7), we need only show that

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |g(\omega, \pi)| d\omega < \infty, \tag{3.33}$$

$$\int_1^\tau \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |g(\omega, t)| d\omega = O(1) \text{ for } 0 < t < \pi, \tag{3.34}$$

and

$$\int_\tau^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |h(\omega, t)| d\omega = O(1) \text{ for } 0 < t < \pi. \tag{3.35}$$

All these have been proved in § 3.1. The case in which r is odd can be treated similarly. This completes the proof of Theorem 3.

3.4. **Proof of Theorem 4.** Let r be even. As in the proof of Theorem 4 of (7), we need only prove the following.

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left| \int_0^\pi \tfrac{1}{2}\{P(t)+P(-t)\}\bar{E}^{(r)}(\omega, t) dt \right| d\omega < \infty; \tag{3.41}$$

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left| \int_0^\pi h(t)\bar{E}^{(r)}(\omega, t) dt \right| d\omega < \infty. \tag{3.42}$$

Proving (3.41) is the same thing as proving the summability $|R, e(\omega), \alpha+\delta|$, $\delta > 0$, of $\Sigma n^r q_n$, where q_n is the Fourier sine constant of the odd function $\frac{1}{2}\{P(t)-P(-t)\}$. This can be easily proved like (3.31).

To prove (3.42), following the same line of argument as in the proof of (3.42) in (7), we need only show that

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} |\bar{E}^{(\rho)}(\omega, \pi)| d\omega < \infty, \tag{3.43}$$

where ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha - 1$;

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left| \int_0^\pi d\theta_{\alpha, r}(t) \Lambda(t) \right| d\omega < \infty, \tag{3.44}$$

and

$$\int_1^\infty \frac{(\log \omega)^{1/\alpha}}{\omega e^{\alpha+\delta}(\omega)} \left| \sum_{n \leqslant \omega} \{e(\omega)-e(n)\}^{\alpha-1+\delta} e(n) \times \left\{ -\theta_{\alpha, r}(\pi) \frac{\cos n\pi}{n} + \int_0^\pi \frac{\cos nt}{n} d\theta_{\alpha, r}(t) \right\} \right| d\omega < \infty. \tag{3.45}$$

In the arguments used in the proof of Theorem 2 we have only to replace $\psi_\alpha(t)$ by $\theta_{\alpha,r}(t)$ to establish these results. The case in which r is odd can be treated similarly. This completes the proof of Theorem 4.

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