PARTITIONS OF BIPARTITE NUMBERS WHEN THE SUMMANDS ARE UNEQUAL

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1. In a number of statistical-mechanical problems we have to deal with assemblies characterized by conservation of two or more parameters. A typical illustration is Fermi's (1951) discussion of the angular distribution of the pions produced in high energy nuclear collisions, where one takes into account the conservation of angular momentum in addition to the conservation of energy. Recently Zilsel (1953) has found that, in order to explain properties of liquid helium II, it is essential to consider, in addition to the conservation of total energy, the conservation of total momentum as well.

Thus, when there are two parameters, say E and P, which are conserved, and each particle of the assembly can occupy the levels (r, s) (r, s are non-negative integers), where the contribution of the level (r, s) to E being $r\epsilon_0$ and to P $s\eta_0$, then, we have to enumerate distinct number of ways $p(m, n)$ and also $q(m, n)$, in which an assembly of particles corresponding to given values of $E = m\epsilon_0$ and $P = n\eta_0$ can be realized. Auluck (1953) has considered the case $p(m, n)$ when the summands are r and s and are unrestricted. We, in this paper, consider the corresponding problem of finding the number of ways $q(m, n)$, in which the bipartite number (m, n) can be written as the sum of numbers (r, s) when the pairs of numbers r and s are not repeated. Further, we find asymptotic expressions for $q(m, n)$: (*a*) m is a fixed number, and (*b*) m and n are of the same order, that is, if there are positive numbers k_1 and k_2 such that $k_1 m < n < k_2 n$.

Case (*a*) is dealt in paragraph 2 and the case (*b*) in paragraph 4. In paragraph 3 asymptotic expansions of the generating function for $q(m, n)$ are dealt with, which are used in paragraph 4.

The partition function $q(m, n)$ is the number of summands, when summands are not repeated. As an illustration, the nine partitions of number (3, 2) are as follows :—

(3, 2) ; (3, 1) (0, 1) ; (3, 0) (0, 2) ; (2, 2) (1, 0) ; (2, 1) (1, 1) ;
(2, 1) (1, 0) (0, 1) ; (2, 0) (1, 2) ; (2, 0) (1, 1) (0, 1) ; (2, 0) (1, 0) (0, 2).

In the following table, the values of $q(m, n)$ for m, n up to 5 are given.

Table for $q(m, n)$

m/n	1	2	3	4	5
1	2	3	5	7	10
2	3	5	9	14	21
3	5	9	18	27	42
4	7	14	27	47	74
5	10	21	42	74	125

2. If one of the integers m is fixed, we can express $q(m, n)$ in terms of partition functions $q(r)$ of unipartite numbers.

The generating function of $q(m, n)$ is

$$Z(x, y) = \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} q(m, n)x^m y^n$$

$$= (1+x)(1+y)(1+x^2)(1+y^2) \dots$$

$$= \prod_{r=1}^{\infty} (1+x^r) \prod_{s=1}^{\infty} \prod_{t=0}^{\infty} (1+y^s x^t). \quad \dots \quad \dots \quad (1)$$

The product converges for $|x| < 1$, $|y| < 1$.
It is shown in Crystal's Algebra

$$(1+a)(1+ax)(1+ax^2)(1+ax^3) \dots$$

$$= 1+\frac{a}{1-x}+\frac{a^2x}{(1-x)(1-x^2)}+\frac{a^3x^3}{(1-x)(1-x^2)(1-x^3)}$$

$$+\frac{a^4x^6}{(1-x)(1-x^2)(1-x^3)(1-x^4)}+ \dots$$

$$+\frac{a^k x^{\frac{1}{2}k(k-)}}{(1-x)(1-x^2)(1-x^3) \dots (1-x^k)}+ \dots$$

Therefore, the generating function Z can be written in the form

$$Z(x, y) = \prod_{r=1}^{\infty} (1+x^r) \prod_{s=1}^{\infty} \left(\sum_{t=0} a_t y^{st} \right)$$

$$= \prod_{r=1}^{\infty} (1+x^r)(1+B_1y+B_2y^2+B_3y^3+ \dots), \quad \dots \quad \dots \quad (2)$$

where $a_0 = 1, \quad a_1 = \frac{1}{1-x}, \quad a_2 = \frac{x}{(1-x)(1-x^2)},$

$$a_3 = \frac{x^3}{(1-x)(1-x^2)(1-x^3)}, \quad a_4 = \frac{x^6}{(1-x)(1-x^2)(1-x^3)(1-x^4)},$$

$$\dots\dots\dots, \quad a_t = \frac{x^{\frac{1}{2}t(t-1)}}{(1-x)(1-x^2)(1-x^3) \dots (1-x^t)}, \dots$$

and $B_1 = a_1, \quad B_2 = (a_1+a_2), \quad B_3 = a_1+a_3+a_1^2,$

$$B_4 = a_1+a_2+a_1a_2+a_1^2+a_4, \quad B_5 = a_1+a_5+a_1a_3+2a_1a_2+2a_1^2,$$

$$\dots\dots\dots; \quad B_m = \sum_{\Sigma r\lambda_r = m} a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_3}. \quad (m = 0, 1, 2 \dots).$$

a_{λ_r} is the coefficient of $y^{s\lambda_r}$ from the series $\sum_{t=0}^{\infty} a_t y^{st}$.

Now, there are $p(m)$ terms in B_m, and the only term in B_m which has the maximum number of m factors is

$$\frac{x^{\frac{1}{2}m(m-1)}}{(1-x)(1-x^2) \dots (1-x^m)},$$

and therefore for $|x| < 1$ and $m \geqslant 3$

$$\left| B_m - \frac{x^{\frac{1}{2}m(m-1)}}{(1-x)(1-x^2) \dots (1-x^m)} \right| < \frac{p(m)}{(1-|x|)^{m-1}}$$

It follows that for $x = e^{-\lambda}$, $\lambda = \sigma + it$,

$$B_m = \frac{1}{m!\,\lambda^m}\{1+(o|\lambda|)\}, \quad \ldots \quad (3)$$

where $\lambda \to 0$ in the stolz angle $|t| \leqslant \sigma$ $(o < \Delta < \infty)$, provided m is a fixed positive integer. In this case

$$B_m \prod_{r=1}^{\infty}(1+x^r) = B_m \frac{\prod_{r=1}^{\infty}(1-x^r)^{-1}}{\prod_{r=1}^{\infty}(1-x^{2r})^{-1}} \sim \frac{\exp(\pi^2/12\lambda)}{m!\,\lambda^m\sqrt{2}} \quad \ldots \quad (4)$$

since $q(m, n)$ is the coefficient of x^n in the expansion of $B_m \prod_{r=1}^{\infty}(1+x^r)$, Ingham's (1941) Tauberian Theorem (4) can be applied, if we can prove that $q(m, n)$ is an increasing function of n for a fixed m. Now from (2) we have identically

$$q(m, n) = \sum_{r=0}^{n} C_{m,r}\, q(n-r) \quad \ldots \quad (5)$$

where the coefficients $C_{m,r}$ are positive functions of m and r only. It follows that $q(m, n)$ for a fixed m, is an increasing function of n. Thus we obtain finally for $n \to \infty$ and a fixed m

$$q(m, n) \sim \left(\frac{\sqrt{12n}}{\pi}\right)^m \frac{1}{4\,.\,3^{\frac{1}{4}}\,.\,n^{\frac{3}{4}}}\,.\,\exp\left[\pi\left(\frac{n}{3}\right)^{\frac{1}{2}}\right]. \quad \ldots \quad (6)$$

3. We now obtain an asymptotic expansion for the generating function $Z(x, y)$. Substituting $x = e^{-\lambda}$, $y = e^{-\mu}$ in $z(x, y)$, we obtain for $R(\lambda) > 0$, $R(\mu) > 0$,

$$\log z = \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \log(1+e^{-\lambda r-\mu s})$$

$$= \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} (-1)^{t-1} \frac{e^{-t(\lambda r+\mu s)}}{t}$$

$$= \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t}\,.\,e^{-(r\lambda+s\mu)t} - \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t}\,.\,e^{-(r\lambda+s\mu)2t}$$

Applying the result (15) of paragraph (3) from Auluck's (1953) paper on Partitions of Bipartite Numbers

$$\log Z = \frac{3}{4}\frac{\zeta(3)}{\lambda\mu} + \frac{\zeta(2)}{4}\left(\frac{1}{\lambda}+\frac{1}{\mu}\right) + \frac{1}{12}\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right)\log 2$$

$$-\frac{3}{4}\log 2 + \frac{1}{48}(\lambda+\mu) + o\left(\frac{\lambda^3}{\mu}, \frac{\mu^3}{\lambda}, \lambda^2, \mu^2, \lambda\mu\right). \quad \ldots \quad (7)$$

In particular, when $\lambda = \mu$, we have

$$\frac{3}{4}\frac{\zeta(3)}{\lambda^2} + \frac{1}{2}\frac{\zeta(2)}{\lambda} - \frac{7}{12}\log 2 + \frac{\lambda}{24} + o(\lambda^2). \quad \ldots \quad (8)$$

4. The partition function $q(m, n)$ is given by the integral

$$q(m, n) = \frac{1}{(2\pi i)^2} \int \int \frac{z(x, y)}{x^{m+1}\, y^{n+1}}\, dx \,.\, dy\,; \qquad \ldots \quad \ldots \quad (9)$$

where for the paths of integration we take the circles $x = e^{-\lambda} = e^{-\lambda_0 - i\xi}$, $y = e^{-\mu} = e^{-\mu_0 - i\eta}$ with $-\pi \leqslant \xi \leqslant \pi$, $-\pi \leqslant \eta \leqslant \pi$; λ_0, μ_0 are the real positive values of λ and μ satisfying the equations

$$\sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \frac{r}{1+e^{r\lambda+s\mu}} = m_1\,, \quad \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \frac{s}{1+e^{r\lambda+s\mu}} = n_1 \qquad \ldots \quad \ldots \quad (10)$$

or from (7)
$$m_1 = \frac{3}{4}\frac{\zeta(3)}{\lambda^2\mu} + \frac{1}{4}\frac{\zeta(2)}{\lambda^2} + \frac{1}{12}\left(\frac{\mu}{\lambda^2} - \frac{1}{\mu}\right)\log 2,$$

and
$$n_1 = \frac{3}{4}\frac{\zeta(3)}{\lambda\mu^2} + \frac{1}{4}\frac{\zeta(2)}{\mu^2} + \frac{1}{12}\left(\frac{\lambda}{\mu^2} - \frac{1}{\lambda}\right)\log 2.$$

Therefore,

$$\lambda_0 = \frac{c n_1^{\frac{1}{3}}}{m_1^{\frac{2}{3}}} + \frac{d}{(m_1 n_1)^{\frac{1}{3}}}\left(2 - \frac{n_1}{m_1}\right) + \frac{1}{m_1}\left(\frac{m_1}{n_1} - \frac{n_1}{m_1}\right)\left(\log 2 - \frac{3d^2}{c}\right)$$

and

$$\mu_0 = \frac{c m_1^{\frac{1}{3}}}{n_1^{\frac{2}{3}}} + \frac{d}{(m_1 n_1)^{\frac{1}{3}}}\left(2 - \frac{m_1}{n_1}\right) + \frac{1}{n_1}\left(\frac{n_1}{m_1} - \frac{m_1}{n_1}\right)\left(\log 2 - \frac{3d^2}{c}\right) \qquad \ldots \quad (11)$$

where

$$c = \left[\frac{3}{4}\zeta(3)\right]^{\frac{1}{3}} \text{ and } d = \frac{\zeta(2)}{12c}\,.$$

We note that in this particular case when m_1 and n_1 tend to infinity

$$\frac{\lambda_0}{\mu_0} = O(1), \quad \frac{\mu_0}{\lambda_0} = o(1)\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Writing $Z(x, y) \equiv F(\lambda, \mu)$ in (10), and following a similar procedure and argument as given in later part of §4 in Auluck's (1953) Partitions of Bipartite Numbers. The series for $\log F(\lambda, \mu)$ is convergent, and we can expand it as follows

$$\log F(\lambda, \mu) = \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} (-1)^{t-1} \frac{e^{-t(r\lambda+s\mu)}}{t}$$

$$= \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t} e^{-t(r\lambda+s\mu)} - \sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} \frac{1}{t} e^{-2t(r\lambda+s\mu)}$$

and

$$\frac{\partial^3}{\partial\lambda^3} \log F(\lambda, \mu) = -\sum_{\substack{r=0 \\ r+s>0}}^{\infty} \sum_{s=0}^{\infty} \sum_{t=1}^{\infty} t^2 r^3 e^{-t(r\lambda+s\mu)}$$

$$+\sum_{\substack{r=0 \\ r+s>0}}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} 8t^2r^3e^{-2t(r\lambda+s\mu)}$$

the term by term differentiation being justified by the uniform convergence of the series in $R(\lambda) > \delta$, $R(\mu) > \delta$, for fixed $\delta > 0$. Further

$$\left|\frac{\partial^3}{\partial\lambda^3}\log F(\lambda,\mu)\right| \leqslant \sum_{\substack{r=0 \\ r+s>0}}^{\infty}\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} t^2r^3\left[e^{-t(r\lambda+s\mu)}+8e^{-2t(r\lambda+s\mu)}\right]$$

$$= -\frac{\partial^3}{\partial\lambda_0^3}[\log F(\lambda_0,\mu_0)+\log F(2\lambda_0+2\mu_0)]$$

If λ, μ lie in the stolz angles,

$$\log F(\lambda,\mu) \sim \frac{3}{4}\frac{\xi(3)}{\lambda\mu}$$

In order to get the asymptotic expression for its partial derivates, say

$$\frac{\partial}{\partial\lambda}\log F(\lambda,\mu) \sim \frac{3}{4}\frac{\xi(3)}{\lambda^2\mu}$$

we use the equation

$$X'(\lambda) = \frac{1}{2\pi i}\frac{X(\zeta)\,d\zeta}{(\zeta-\lambda)^2},$$

where

$$X'(\lambda) = \log F(\lambda,\mu) - \frac{3}{4}\frac{\zeta(3)}{\lambda\mu},$$

and c_1, a circle with centre λ and radius $\rho = \delta|\lambda|$, where δ is a positive constant small enough to ensure that c_1 lies entirely in a stolz angle $G_1(\delta)$, whenever λ lies in G. Similar arguments can be used for deriving the asymptotic expressions for the other partial derivates of $\log F(\lambda,\mu)$. It follows, therefore, that

$$A \sim \frac{3}{2}\zeta(3)\lambda_0^{-3}\mu_0^{-1}, \qquad B \sim \frac{3}{4}\zeta(3)\lambda_0^{-2}\mu_0^{-2},$$

$$C \sim \frac{3}{2}\zeta(3)\lambda_0^{-1}\mu_0^{-3} \quad\text{and}\quad AC-B^2 \sim \frac{27}{16}\zeta^2(3)\lambda_0^{-4}\mu_0^{-4}.$$

For

$$|\xi| \leqslant \lambda_0^{\frac{7}{4}},\quad |\eta| \leqslant \mu_0^{\frac{7}{4}},\quad |R_4(\xi,\eta)| = o\left[\frac{1}{\lambda^5}(|\xi|^2+|\eta|^2)\right] = o\left(\lambda_0^{\frac{1}{4}}\right).$$

Hence the contribution to the double integral I in (13) from the above ranges is asymptotic to

$$\frac{1}{(AC)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(x^2+\frac{2B}{(AC)^{\frac{1}{2}}}xy+y^2\right)\right]dx\,dy = \frac{2\pi}{(AC-B^2)^{\frac{1}{2}}}.$$

$(AC-B^2)$ is positive. For λ_0, μ_0 sufficiently small this follows from the asymptotic formula for $AC-B^2$ given above.

Again, for the range $\lambda_0^{\frac{7}{4}} \leqslant \xi \leqslant \gamma_1\lambda$, $-\gamma_0\mu_0 \leqslant \eta \leqslant \gamma_2\mu_0$, following the method of Auluck (1953), we get $A\xi^2+2B\xi\eta+C\eta^2 = o(AC-B^2)^{\frac{1}{2}}$, k_2, k_3 are positive constants. We now consider the contribution I_1 to I from the ranges

$$\gamma_1\,\lambda_0 \leqslant \xi \leqslant \pi, \qquad -\pi \leqslant \eta \leqslant \pi\,.$$

We have

$$I_1 = \int_{\gamma_1\lambda_0}^{\pi}\int_{-\pi}^{\pi} \exp\left\{\log F(\lambda,\,\mu)-\log F(\lambda_0,\,\mu_0)+m_1 i\xi+n_1 i\eta\right\} d\xi\,.\,d\eta$$

or
$$I_1 = \int_{\gamma_1\lambda_0}^{\pi}\int_{-\pi}^{\pi} \frac{F(\lambda,\,\mu)}{F(\lambda_0\mu_0)}\, e^{m_1 i\xi+n_1 i\eta}\, d\xi\,.\,d\eta.$$

Now

$$\left|\frac{1+e^{-r\lambda-s\mu}}{1+e^{-r\lambda_0-s\mu_0}}\right| < 1 \text{ for } r \geqslant 0 \text{ and } s \geqslant 1.$$

Therefore

$$I_1 < \int_{\gamma_1\,\lambda_0}^{\pi} \left|\prod_{r=1}^{\infty} \frac{1+e^{-r\lambda}}{1+e^{-r\lambda_0}}\right| d\xi.$$

$$< \int_{\gamma_1\lambda_0}^{\pi} \exp\frac{\pi^2}{12\lambda_0}\left(\frac{1}{1+\gamma_1^2}-1\right) d\xi.$$

$$= O\left(-\frac{\pi^2\gamma_1^2}{12\lambda_0(1+\gamma_1^2)}\right)$$

$$= O(\lambda_0^4) = O[\sqrt{AC-B^2}]$$

similarly, it can be shown that the contributions to I from all the other ranges is for the order $O\left\{(AC-B^2)^{-\frac{1}{2}}\right\}$.
Therefore,

$$q(m,\ n) \sim \frac{F(\lambda_0,\,\mu_0)}{2\pi(AC-B^2)^{\frac{1}{2}}} \exp\,(m_1\lambda_0+n_1\mu_0) \qquad \ldots \qquad \ldots \quad (14)$$

substituting the values of λ_0 and μ from (11) in this expression, we finally have the asymptotic formula
Because

$$(m_1\lambda_0+n_1\mu_0)+\log F(\lambda_0,\,\mu_0)$$

$$= 3C(m_1 n_1^{\frac{1}{3}})+\frac{3d}{(m_1n_1)^{\frac{1}{3}}}(m_1+n_1)+\frac{d^2}{C}\left(11-\frac{8n_1}{m_1}-\frac{8m_1}{n_1}\right)$$

$$-\frac{3}{4}\log 2+\frac{\log 2}{12}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)$$

and because

$$AC-B^2 \sim \frac{27}{16}\zeta^2(3)\lambda_0^{-4}\mu_0^{-4} = 3C^6\lambda_0^{-4}\mu_0^{-4} = \frac{3}{C^2}(m_1n_1)^{\frac{4}{3}}\,.$$

Therefore,

$$q(m, n) \sim \frac{C}{2\pi\sqrt{3}\,(m_1n_1)^{\frac{2}{3}}}\left\{3C(m_1n_1)^{\frac{1}{3}}+\frac{3d(m_1+n_1)}{(m_1n_1)^{\frac{1}{3}}}\right\}$$

$$-\frac{3}{4}\log 2+\frac{\log 2}{12}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)+\frac{11d^2}{C}-\frac{8d^2}{C}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)$$

or

$$q(m, n) \sim \frac{C}{2\pi\sqrt{3}}\exp\left\{3C(m_1n_1)^{\frac{1}{3}}+\frac{3d(m_1+n_1)}{(m_1n_1)^{\frac{1}{3}}}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)\right.$$

$$-\frac{3}{4}\log 2+\frac{1}{12}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)\log 2+\frac{11d^2}{C}$$

$$\left.-\frac{8d^2}{C}\left(\frac{m_1}{n_1}+\frac{n_1}{m_1}\right)-\frac{2}{3}\log(m_1n_1),\right. \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

where m_1 and n_1 are of the same order.

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