ON THE NON-SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES

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1. Let $f(\theta)$ be a periodic function with period 2π and integrable (L) in $(-\pi, \pi)$. Let the Fourier series of $f(\theta)$ be

(1.1) $$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Then the conjugate series of the Fourier series is given by

(1.2) $$\sum_{n=1}^{\infty} B_n(\theta) = \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta),$$

and the conjugate function, defined as a Cauchy-integral, is given by

(1.3) $$g(\theta) = (1/2\pi) \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \psi(u) \cot (u/2)\, du,$$

where

$$\psi(u) = f(\theta+u) - f(\theta-u).$$

The Abel-limit associated with the series (1.2) is given by

(1.4) $$\lim_{x \to 1-0} \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta)\, x^n.$$

2. In what follows we use the following notations—

(2.1) $$\begin{cases} \Psi_\alpha(t) = \dfrac{1}{\Gamma\alpha} \displaystyle\int_0^t (t-u)^{\alpha-1} \psi(u)\, du, \ \alpha > 0\,; \\ \Psi_0(t) = \psi(t)\,; \\ \psi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Psi_\alpha(t), \ \alpha \geqslant 0. \end{cases}$$

(2.2) $$\begin{cases} \Delta = 1 - 2x \cos t + x^2, \\ Q'(t) = (x \sin t)/\Delta, \\ Q(t) = t/(p^2+t^2), \text{ where } p = \log x, \\ q(t) = (1/t) - Q(t). \end{cases}$$

(2.3) $$\begin{cases} Q^\rho(t) = (d/dt)^\rho \{Q(t)\}, \\ q^\rho(t) = (d/dt)^\rho \{q(t)\}. \end{cases}$$

(2.4) $$\epsilon = \text{arc} \sin (1-x).$$

3. The following theorem on the summability (A) of a conjugate series for which the conjugate function exists as a Cauchy-integral is due to Plessner [1923]:

Theorem A: Let

$$V(x, \theta) = \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta)\, x^n, \; (0 \leqslant x < 1).$$

Then if for any θ, the condition

$$(3.1) \qquad \int_0^t \psi(t)\, dt = o(t), \; (t \to 0)$$

is satisfied, then

$$\lim_{x \to 1} \left[V(x, \theta) - (1/2\pi) \int_\epsilon^\pi \psi(t) \cot (t/2)\, dt \right] = 0.$$

It follows immediately from this theorem that if condition (3.1) holds, then the divergence to $+\infty(-\infty)$ of the conjugate function (1.3) is a necessary and sufficient condition for the divergence of the Abel-limit of the conjugate series to $+\infty(-\infty)$.

Prasad [1932] obtained a more general theorem which included Plessner's theorem as a particular case. In Prasad's theorem the conjugate function was replaced by his 'Generalized Conjugate Function' and the condition (3.1) was replaced by the less stringent condition, viz.

$$\int_0^t \{\Psi(t)/t\}\, dt = o(t), \text{ as } t \to 0.$$

Generalizing Plessner's theorem in another direction, viz. by replacing the condition (3.1) by the less stringent condition

$$(3.2) \qquad \psi_\alpha(t) = o(1), \text{ as } t \to 0,$$

α being any positive integer $\geqslant 1$, the author had proved the following [Sinha, 1953]:

Theorem B: If $\psi_\alpha(t) = O(1)$, as $t \to 0$, α being a *positive integer* $\geqslant 1$, then the divergence of the generalized conjugate function

$$(1/2\pi) \int_\epsilon^\pi \psi_{\alpha-1}(t) \cot (t/2)\, dt$$

to $+\infty(-\infty)$ is a necessary and sufficient condition for the divergence to $+\infty$ $(-\infty)$ of the Abel-limit of the conjugate series of a Fourier series.

4. The object of the present paper is to extend the result of Theorem B and establish the following theorem for the more abstruse case in which $\alpha\,(>0)$ is not necessarily an integer, but may become non-integral.

Theorem 1: If $\psi_\alpha(t) = o(1)$, as $t \to 0$, where $\alpha \geqslant 1$,

$$(4.1) \qquad \left[V(x, \theta) - (1/2\pi) \int_\epsilon^\pi \psi_{\alpha-1}(t) \cot (t/2)\, dt \right] = o(1), \text{ as } x \to 1-0.$$

Theorem 2: If $\psi_\alpha(t) = O(1)$, as $t \to 0$, where $\alpha > 0$,

$$(4.2) \qquad \left[V(x, \theta) - (1/2\pi) \int_\epsilon^\pi \psi_{\alpha-1}(t) \cot (t/2)\, dt \right] = O(1), \text{ as } x \to 1-0.$$

Obviously Theorem 2 is equivalent to the statement that if $\psi_\alpha(t) \neq O(1)$, as $t \to 0$, where $\alpha > 0$, then the divergence of the generalized conjugate function

$$(1/2\pi) \int_\epsilon^\pi \psi_{\alpha-1}(t) \cot (t/2)\, dt$$

to $+\infty$ $(-\infty)$ is a necessary and sufficient condition for the divergence to $+\infty$ $(-\infty)$ of the Abel-limit of the conjugate series of a Fourier series.

5. For proving the above theorems, we shall repeatedly use a number of results, which may be stated below as lemmas, the proofs of most of them being quite straightforward:—

Lemma 1:

$$Q^{(r)}(t) = O(1/t^{r+1}), \quad (t \text{ large}),$$

$$t^r Q^{(r-1)}(t) = O[(t^2/p^2)^r], \quad (t \to 0),$$

$$q^{(r)}(t) = O(1/t^{r+1}), \quad (t \text{ large}),$$

$$t^r q^{(r)}(t) = O(p^2/t^3), \quad (t \to 0).$$

Lemma 2:

If $J(u) = \{1/\Gamma(h-\alpha+1)\} \int_u^\infty (t-u)^{h-\alpha} Q^{(h-1)}(t)\, dt$, where $\alpha \geqslant 2$ and $h = [\alpha]$, then

$$J(u) = O(u^{1-\alpha}), \quad (u \text{ large}).$$

Lemma 3:

If $J(u)$ is defined as in Lemma 2, then

$$(d/du)\{J(u)\} = \{1/\Gamma(h-\alpha+1)\} \int_u^\infty (t-u)^{h-\alpha} Q^{(h)}(t)\, dt$$

$$= O(u^{2h-\alpha+2}/p^{2h+2}), \quad (u \text{ small}).$$

Proof of Lemma 3:

Integrating by parts once, we have

$$J(u) = \{1/\Gamma(h-\alpha+1)\} \int_u^\infty (t-u)^{h-\alpha} Q^{(h-1)}(t)\, dt$$

$$= \{1/\Gamma(h-\alpha+2)\} \Big[\{Q^{(h-1)}(t) \,.\, (t-u)^{h-\alpha+1}\}_u^\infty - \int_u^\infty (t-u)^{h-\alpha+1} Q^{(h)}(t)\, dt \Big]$$

$$= \{1/\Gamma(h-\alpha+2)\} \Big[\lim_{t\to\infty} \{t^{h-\alpha+1}(1-u/t)^{h-\alpha+1} O(1/t^h)\} - \int_u^\infty (t-u)^{h-\alpha+1} Q^{(h)}(t)\, dt \Big]$$

$$= -\{1/\Gamma(h-\alpha+2)\} \int_u^\infty (t-u)^{h-\alpha+1} Q^{(h)}(t)\, dt.$$

Hence

$$(d/du)\{J(u)\} = \{1/\Gamma(h-\alpha+1)\}\int_u^\infty (t-u)^{h-\alpha}Q^{(h)}(t)\,dt.$$

Lemma 4 :

If $J(u)$ is defined as in Lemma 2, we have

$$(d/du)^2\{J(u)\} = \{1/\Gamma(h-\alpha+1)\}\int_u^\infty (t-u)^{h-\alpha}Q^{(h+1)}(t)\,dt.$$

Lemma 5 :

$$\text{If } J^*(u) = \{(-1)^h/\Gamma\alpha\Gamma(h-\alpha+1)\}\int_u^\infty (t-u)^{(h-\alpha)}q^{(h)}(t)\,dt$$

where $\alpha \geqslant 2$ and $h = [\alpha]$, then

$$J^*(u) = O(u^{-\alpha}),\ (u \text{ large}),$$
$$= O(p^2u^{-\alpha-2}),\ (u \text{ small}).$$

Lemma 6 :

If $J^*(u)$ is defined as in Lemma 5, we have

$$(d/du)\{J^*(u)\} = \{(-1)^h/\Gamma\alpha\Gamma(h-\alpha+1)\}\int_u^\infty (t-u)^{h-\alpha}q^{(h+1)}(t)\,dt.$$

Lemma 7 :

$$\text{If } J'(u) = \{1/\Gamma(2-\alpha)\}\int_u^\infty (t-u)^{1-\alpha}Q^{(1)}(t)\,dt,$$

where $1 \leqslant \alpha < 2$, then

$$J'(u) = O(u^{-\alpha}),\ (u \text{ large}).$$

Lemma 8 :

If $J'(u)$ is defined as in Lemma 7, we have

$$(d/du)\{J'(u)\} = \{1/\Gamma(2-\alpha)\}\int_u^\infty (t-u)^{1-\alpha}Q^{(2)}(t)\,dt.$$

6.1. Proof of Theorem 1.

Let

$$I = \tfrac{1}{2}\int_0^\infty \psi(t)Q(t)\,dt.$$

Then we have

$$I = \tfrac{1}{4}\int_{-\infty}^\infty \psi(t)\,.\,\{t/(p^2+t^2)\}\,dt$$

$$= \tfrac{1}{4}\left[\cdots\cdots\cdots + \int_{-3\pi}^{-\pi} + \int_{-\pi}^{\pi} + \int_{\pi}^{3\pi} + \cdots\cdots\right]\psi(t)\,.\,\{t/(p^2+t^2\}\,.\,dt$$

$$= \tfrac{1}{4}\left[\int_{-\pi}^{\pi}\psi(t)\,.\,\{t/(p^2+t^2)\}dt + \sum_{-\infty}^{\infty}{}'\int_{(2k-1)\pi}^{(2k+1)\pi}\psi(t)\,.\,\{t/(p^2+t^2)\}\,dt\right]$$

$$= \tfrac{1}{4}\left[\int_{-\pi}^{\pi} \psi(t)\,.\,\{t/(p^2+t^2)\}\,dt + \sum_{-\infty}^{\infty}{}' \int_{-\pi}^{\pi} \psi(\tau+2k\pi)\frac{\tau+2k\pi}{p^2+(\tau+2k\pi)^2}\,d\tau\right]$$

$$= \tfrac{1}{4}\int_{-\pi}^{\pi}\left[\sum_{-\infty}^{\infty}{}' \frac{\tau+2k\pi}{p^2+(\tau+2k\pi)^2}+\frac{\tau}{p^2+\tau^2}\right].\,\psi(\tau)\,d\tau\,.$$

Now from $-\pi<\tau<\pi$,

$$\sum_{-\infty}^{\infty}{}' \frac{\tau+2k\pi}{p^2+(\tau+2k\pi)^2}+\frac{\tau}{p^2+\tau^2}$$

$$= \sum_{-\infty}^{\infty}{}' \frac{1}{2i}\left[\frac{1}{p-i(\tau+2k\pi)}-\frac{1}{p+i(\tau+2k\pi)}\right]+\frac{1}{2i}\left[\frac{1}{p-i\tau}-\frac{1}{p+i\tau}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\tau+ip}+\sum_{-\infty}^{\infty}{}' \frac{1}{(\tau+ip)+2k\pi}\right]+\frac{1}{2}\left[\frac{1}{\tau-ip}+\sum_{-\infty}^{\infty}{}' \frac{1}{(\tau-ip)+2k\pi}\right]$$

$$= \frac{1}{4}\left[\frac{1}{\frac{\tau+ip}{2}}+2\left(\frac{\tau+ip}{2}\right)\sum_{1}^{\infty}\frac{1}{\left(\frac{\tau+ip}{2}\right)^2-\pi^2k^2}\right]$$

$$+\frac{1}{4}\left[\frac{1}{\frac{\tau-ip}{2}}+2\left(\frac{\tau-ip}{2}\right)\sum_{1}^{\infty}\frac{1}{\left(\frac{\tau-ip}{2}\right)^2-\pi^2k^2}\right]$$

$= \tfrac{1}{4}\,[\cot \tfrac{1}{2}(\tau+ip)+\cot \tfrac{1}{2}(\tau-ip)]$ [Titchmarsh, 1939.]

$= \tfrac{1}{2}\,[\sin \tau/(\cos ip-\cos \tau)]$

$= x \sin \tau/(1-2x\cos\tau+x^2)$

$= Q'(\tau)$.

Thus finally

$$I = \tfrac{1}{4}\int_{-\pi}^{\pi}\psi(\tau)Q'(\tau)d\tau = \tfrac{1}{2}\int_{0}^{\pi}\psi(t)Q'(t)\,dt. \tag{6.11}$$

6.2. We have

$$V(x,\theta) = \sum_{1}^{\infty} B_n(\theta)x^n$$

$$= (1/\pi)\int_{-\pi}^{\pi} f(\theta+t)Q'(t)\,dt$$

$$= (1/\pi)\int_{0}^{\pi} \psi(t)Q'(t)\,dt$$

$$= (1/\pi)\int_{0}^{\infty} \psi(t)Q(t)\,dt.$$

Hence

$$V(x,\theta)-(1/\pi)\int_\epsilon^\infty \{\psi_{\alpha-1}(t)/t\}\,dt$$

$$= (1/\pi)\left[\int_\epsilon^\infty \psi(t)Q(t)\,dt-\int_\epsilon^\infty \{\psi_{\alpha-1}(t)/t\}\,dt\right]$$

$$= (1/\pi)(I-I'), \text{ say.} \tag{6.21}$$

Thus in order to establish Theorem 1, we have only to prove that if $\psi_\alpha(t) = o(1)$, then

$$I-I' = o(1).$$

6.3. We first consider the case in which $\alpha \geqslant 2$.

Integrating by parts $(h-1)$ times, where h is the greatest integer not greater than α, we have

$$I = \int_0^\infty \psi(t)Q(t)\,dt$$

$$= \left[\sum_{\rho=1}^{h-1}(-1)^{\rho-1}\Psi_\rho(t)Q^{(\rho-1)}(t)\right]_0^\infty +(-1)^{h-1}\int_0^\infty \Psi_{h-1}(t)Q^{(h-1)}(t)\,dt$$

$$= \left[\sum_{\rho=1}^{h-1} o(t^\rho).\,O(t^{-\rho})\right]_{t\to\infty} +(-1)^{h-1}I_1, \quad \text{(by Lemma 1)},$$

$$= o(1)+(-1)^{h-1}I_1.$$

Now

$$I_1 = \int_0^\infty \Psi_{h-1}(t)\,Q^{(h-1)}(t)\,dt$$

$$= \{1/\Gamma(h-\alpha+1)\}\int_0^\infty Q^{(h-1)}(t)\,dt\int_0^t (t-u)^{h-\alpha}\,d\Psi_{\alpha-1}(u)$$

$$= \{1/\Gamma(h-\alpha+1)\}\int_0^\infty d\Psi_{\alpha-1}(u)\int_u^\infty (t-u)^{h-\alpha}\,Q^{(h-1)}(t)\,dt$$

$$= \int_0^\infty J(u)\,d\Psi_{\alpha-1}(u).$$

Integrating by parts once, we get

$$I_1 = \Big[\Psi_{\alpha-1}(u)\,J(u)\Big]_0^\infty - \int_0^\infty \Psi_{\alpha-1}(u).\,(d/du)\,\{J(u)\}\,du$$

$$= [o(u^{\alpha-1}).\quad O(u^{1-\alpha})]_{u\to\infty} - I_2, \quad \text{(by Lemma 2)},$$

$$= o(1)-I_2.$$

Now

$$I_2 = \int_0^\infty \Psi_{\alpha-1}\ (u).\ (d/du)\ \{J\ (u)\ \}\ du$$

$$= (1/\Gamma\,\alpha)\int_0^\infty u^{\alpha-1}\,\psi_{\alpha-1}(u).\ (d/du)\ \{J\ (u)\ \}\ du$$

$$= (1/\Gamma\,\alpha)\left(\int_0^\epsilon + \int_\epsilon^\infty\right) u^{\alpha-1}\,\psi_{\alpha-1}(u).\ (d/du)\ \{J(u)\}\ du$$

$$= (1/\Gamma\,\alpha)\ (\,I_3+I_4\,).$$

We have now integrating once by parts

$$I_3 = \int_0^\epsilon u^{\alpha-1}\,\psi_{\alpha-1}(u).\ (d/du)\ \{J(u)\ \}\ du$$

$$= \left[u^\alpha\,\psi_\alpha\,(u).\ (d/du)\ \{J(u)\}\right]_0^\epsilon - (\alpha-1)\int_0^\epsilon u^{\alpha-1}\,\psi_\alpha(u)\ (d/du)\{J(u)\}\ du$$

$$-\int_0^\epsilon u^\alpha\psi_\alpha(u).\ (d/du)^2\ \{J(u)\}\ du$$

$$= \epsilon^\alpha\,\psi_\alpha\,(\epsilon).\quad O(\epsilon^{2h-\alpha+2}/\,p^{2h+2}) - (\alpha-1)\,I_5 - I_6,\ \text{(by Lemma 3)}.$$

$$= o(1) - (\alpha-1)\ I_5 - I_6,\ [\text{since } \epsilon^2/p^2 = O(1),\ \text{as } \epsilon\to 0.]$$

Thus we have

(6.31) $$I = o(1) + \left[(-1)^h/\pi\Gamma\alpha\right] I_4 - (\alpha-1)\ I_5 - I_6.$$

We first show that $I_5 = o(1)$ and also $I_6 = o(1)$.

We have

$$I_5 = \int_0^\epsilon u^{\alpha-1}\,\psi_\alpha(u).\ (d/du)\ \{J(u)\}\ du$$

$$= \{1/\Gamma(h-\alpha+1)\}\int_0^\epsilon u^{\alpha-1}\,\psi_\alpha\,(u)\ du\int_u^\infty (t-u)^{h-\alpha}\,Q^{(h)}\,(t)\ dt,$$

(by Lemma 3),

$$= \{1/\Gamma(h-\alpha+1)\}\int_\epsilon^\infty Q^{(h)}\,(t)\ dt\int_0^\epsilon u^{\alpha-1}\,\psi_\alpha(u)\ (t-u)^{h-\alpha}\,du$$

$$+\ \{1/\Gamma(h-\alpha+1)\}\int_0^\epsilon u^{\alpha-1}\,\psi_\alpha(u)\ du\int_u^\epsilon (t-u)^{h-\alpha}\,Q^{(h)}\,(t)\ dt$$

$$= (I_{5,1}+I_{5,2})\ /\Gamma\ (h-\alpha+1).$$

Now

$$I_{5,1} = \int_{\epsilon}^{\infty} Q^{(h)}(t)\, dt \int_{0}^{\epsilon} u^{\alpha-1} \psi_{\alpha}(u)\, (t-u)^{h-\alpha}\, du$$

$$= O\left[\int_{\epsilon}^{\infty} t^{-h-1}\, dt \int_{0}^{\epsilon} o(1).\, u^{\alpha-1}\, (t-u)^{h-\alpha}\, du\right], \text{ (by Lemma 1)},$$

$$= o(1).\ O\left[\int_{\epsilon}^{\infty} t^{-h-1}\, \epsilon^{\alpha-1}\, dt \int_{0}^{\epsilon} (t-u)^{h-\alpha}\, du\right]$$

$$= o(1).\ O\left[\int_{\epsilon}^{\infty} \epsilon^{\alpha-1}\, t^{-h-1}\, t^{h-\alpha+1}\, dt\right]$$

$$= o(1).\ O\left[\epsilon^{\alpha-1} / t^{\alpha-1}\right]_{\epsilon}^{\infty}$$

$$= o(1).$$

$$I_{5,2} = \int_{0}^{\epsilon} Q^{(h)}(t)\, dt \int_{0}^{t} u^{\alpha-1} \psi_{\alpha}(u)\, (t-u)^{h-\alpha}\, du$$

$$= o(1). \int_{0}^{\epsilon} t^{h}\, Q^{(h)}(t)\, dt \int_{0}^{1} x^{\alpha-1}\, (1-x)^{h-\alpha}\, dx$$

$$= o(1).\, O\left[\int_{0}^{\epsilon} (t^{2h+1}/p^{2h+2})\, dt\right]$$

$$= o(1).\, O(\epsilon^{2h+2}/p^{2h+2}), \left[\text{since} \int_{0}^{1} x^{\alpha-1}\, (1-x)^{h-\alpha}\, dx < \infty\right],$$

$$= o(1).$$

Next

$$I_{6} = \int_{0}^{\epsilon} u^{\alpha}\, \psi_{\alpha}(u)\, (d/du)^{2}\, \{J(u)\}\, du$$

$$= \{1/\Gamma(h-\alpha+1)\} \int_{0}^{\epsilon} u^{\alpha}\, \psi_{\alpha}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha}\, Q^{(h+1)}(t)\, dt, \quad \text{(by Lemma 4)},$$

$$= \{1/\Gamma(h-\alpha+1)\} \left[\int_{\epsilon}^{\infty} Q^{(h+1)}(t)\, dt \int_{0}^{\epsilon} u^{\alpha} \psi_{\alpha}(u) (t-u)^{h-\alpha}\, du + \int_{0}^{\epsilon} u^{\alpha} \psi_{\alpha}(u)\, du \int_{u}^{\epsilon} (t-u)^{h-\alpha} Q^{(h+1)}(t)\, dt\right]$$

$$= (I_{6,1} + I_{6,2})/\Gamma(h-\alpha+1).$$

Now

$$
\begin{aligned}
I_{6,1} &= \int_{\epsilon}^{\infty} Q^{(h+1)}(t)\, dt \int_{0}^{\epsilon} u^{\alpha}\psi_{\alpha}(u)(t-u)^{h-\alpha}\, du \\
&= O\left[\int_{\epsilon}^{\infty} t^{-h-2}\, dt \int_{0}^{\epsilon} o(1).u^{\alpha}(t-u)^{h-\alpha}\, du\right], \text{ (by Lemma 1)}, \\
&= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} t^{-h-2}\epsilon^{\alpha}\, dt \int_{0}^{\epsilon} (t-u)^{h-\alpha}\, du\right] \\
&= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} \epsilon^{\alpha} t^{-h-2} t^{h-\alpha+1}\, dt\right] \\
&= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} (\epsilon^{\alpha}/t^{\alpha+1})\, dt\right] \\
&= o(1)\,.\,O\left[\epsilon^{\alpha}/t^{\alpha}\right]_{\epsilon}^{\infty} \\
&= o(1)\,.
\end{aligned}
$$

$$
\begin{aligned}
I_{6,2} &= \int_{0}^{\epsilon} Q^{(h+1)}(t)\, dt \int_{0}^{t} u^{\alpha}\psi_{\alpha}(u)(t-u)^{h-\alpha}\, du \\
&= o(1).\int_{0}^{\epsilon} t^{h+1} Q^{(h+1)}(t)\, dt \int_{0}^{1} x^{\alpha}(1-x)^{h-\alpha}\, dx \\
&= o(1)\,.\,O\left[\int_{0}^{\epsilon} (t^{2h+3}/p^{2h+4})\, dt\right], \text{ (by Lemma 1)}, \\
&= o(1)\,.\,O\left(\epsilon^{2h+4}/p^{2h+4}\right) \\
&= o(1)\,.
\end{aligned}
$$

Hence now

(6.32) $$I = o(1) + \{(-1)^{h}/\pi\Gamma\alpha\} I_4,$$

so that combining (6.21) and (6.32), we have

$$
\begin{aligned}
&V(x,\theta) - (1/\pi)\int_{\epsilon}^{\infty} \{\psi_{\alpha-1}(u)/u\}\, du \\
&\quad = o(1) + \{(-1)^{h}/\pi\Gamma\alpha\}\,.\,I_4 - (1/\pi)I' \\
&\quad = o(1) + K\,.\int_{\epsilon}^{\infty} u^{\alpha-1}\psi_{\alpha-1}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha} Q^{(h)}(t)\, dt - (1/\pi)I', \\
&\qquad\qquad [\text{where } K = \{(-1)^{h}/\pi\Gamma\alpha\Gamma(h-\alpha+1)\}], \\
&\quad = o(1) + K\,.\left[\int_{\epsilon}^{\infty} u^{\alpha-1}\psi_{\alpha-1}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha}\left\{\frac{(-1)^{h}\Gamma(h+1)}{t^{h+1}} - q^{(h)}(t)\right\} dt\right] - (1/\pi)I'
\end{aligned}
$$

$$= o(1) . + K . (-1)^h \Gamma(h+1) . \int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha-1}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha} t^{-h-1}\, dt$$
$$-(1/\pi) I' - I_7$$

$$= o(1) + 2(-1)^h K . \Gamma(h+1) \int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha-1}(u)\, du . \int_{0}^{\pi/2} u^{-\alpha} (\sin\theta)^{2\alpha-1}$$
$$\times (\cos\theta)^{2h-2\alpha+1} d\theta - I_7 - (1/\pi) I'$$

$$= o(1) + (1/\pi) \int_{\epsilon}^{\infty} \{\psi_{\alpha-1}(u)/u\}\, du - I_7 - (1/\pi) I'$$
$$= o(1) - I_7 .$$

It now remains to show that $I_7 = o(1)$.

Now

$$I_7 = K. \left[\int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha-1}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha} g^{(h)}(t)\, dt \right]$$
$$= \int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha-1}(u)\, J^*(u)\, du.$$

Integrating by parts once, we get

$$I_7 = \left[u^{\alpha-1} J^*(u).\quad u\psi_{\alpha}(u) \right]_{\epsilon}^{\infty}$$
$$- \int_{\epsilon}^{\infty} u\psi_{\alpha}(u) \left[(\alpha-1)\, u^{\alpha-2} J^*(u) + u^{\alpha-1} (d/du) \{J^*(u)\}\, du \right]$$
$$= O\left[u^{\alpha} \psi_{\alpha}(u) . u^{-\alpha} \right]_{u\to\infty} + O\left[u^{\alpha} \psi_{\alpha}(u) . (p^2 u^{-\alpha-2}) \right]_{u=\epsilon}$$
$$-(\alpha-1)\, I_8 - I_9$$
$$= o(1) - (\alpha-1)\, I_8 - I_9.$$

We now proceed to show that $I_8 = o(1)$ and $I_9 = o(1)$.

Now

$$I_8 = \int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha}(u)\, J^*(u)\, du$$
$$= K. \int_{\epsilon}^{\infty} u^{\alpha-1} \psi_{\alpha}(u)\, du \int_{u}^{\infty} (t-u)^{h-\alpha} g^{(h)}(t)\, dt$$
$$= O\left[p^2 \int_{\epsilon}^{\infty} u^{\alpha-1} | \psi_{\alpha}(u) | . u^{-\alpha-2}\, du \int_{0}^{1} x^{\alpha+1} (1-x)^{h-\alpha}\, dx \right],$$

(by Lemma 1),

$$= O\left[p^2\left(\int_\epsilon^\delta + \int_\delta^\infty\right) u^{-3} |\psi_\alpha(u)|\, du\right]$$
$$= O\,(I_{8,1}+I_{8,2}).$$

$$I_{8,1} = p^2 \int_\epsilon^\delta u^{-3} |\psi_\alpha(u)|\, du$$
$$= o\left(p^2 \int_\epsilon^\delta u^{-3}\, du\right)$$
$$= o(p^2/\epsilon^2)$$
$$= o(1), \quad \text{since } p = o(1).$$

$$I_{8,2} = p^2 \int_\delta^\infty u^{-3} |\psi_\alpha(u)|\, du$$
$$= p^2 \int_\delta^\infty u^{-3}.\, O(1).\, du$$
$$= O\left[p^2 \int_\delta^\infty u^{-3}\, du\right]$$
$$= o(1).$$

Next

$$I_9 = \int_\epsilon^\infty u^\alpha \psi_\alpha(u)(d/du)\,\{J^*(u)\}\, du$$
$$= K.\int_\epsilon^\infty u^\alpha \psi_\alpha(u)\, du \int_u^\infty (t-u)^{h-\alpha}\, q^{(h+1)}(t)\, dt$$
$$= O\left[p^2 \int_\epsilon^\infty u^\alpha |\psi_\alpha(u)|\, u^{-\alpha-3}\, du \int_0^1 x^{\alpha+2}\,(1-x)^{h-\alpha}\, dx\right],$$

(by Lemma 1),

$$= O\left[p^2\left(\int_\epsilon^\delta + \int_\delta^\infty\right) u^{-3} |\psi_\alpha(u)|\, du\right]$$
$$= O(I_{9,1}+I_{9,2}).$$

Now

$$I_{9,1} = p^2 \int_\epsilon^\delta u^{-3} |\psi_\alpha(u)|\, du$$
$$= o\left(p^2 \int_\epsilon^\delta u^{-3}\, du\right)$$
$$= o(p^2/\epsilon^2)$$
$$= o(1).$$

$$I_{9,2} = p^2 \int_\delta^\infty u^{-3} \, |\psi_\alpha(u)| \, du$$
$$= p^2 \int_\delta^\infty u^{-3}. \, O(1). \, du$$
$$= O\left(p^2 \int_\delta^\infty u^{-3} \, du\right)$$
$$= o(1).$$

This completes the proof of Theorem 1 when $\alpha \geqslant 2$.

6.4. We now proceed to give the proof for the case in which $1 \leqslant \alpha < 2$.

From (6.21) we know that in order to establish the theorem we have to prove that if $\psi_\alpha(t) = o(1)$, $1 \leqslant \alpha < 2$, then

$$I - I' = o(1).$$

Integrating once by parts, we get

$$I = \int_0^\infty \psi(t) \, Q(t) \, dt$$
$$= \left[Q(t)\Psi_1(t)\right]_0^\infty - \int_0^\infty \Psi_1(t) Q^{(1)}(t) \, dt$$
$$= o(1) - \{1/\Gamma(2-\alpha)\} \int_0^\infty Q^{(1)}(t) \, dt \int_0^t (t-u)^{1-\alpha} \, d\,\Psi_\alpha(u)$$
$$= o(1) - \int_0^\infty J'(u) d\Psi_\alpha(u)$$
$$= o(1) + I_1.$$

Now integrating by parts once we get

$$I_1 = -\int_0^\infty J'(u) d\Psi_\alpha(u)$$
$$= -\left[J'(u)\Psi_\alpha(u)\right]_0^\infty + \int_0^\infty \Psi_\alpha(u)(d/du) \, \{J'(u)\} \, du$$
$$= \left[O(u^{-\alpha}). \, o(u^\alpha)\right]_{u\to\infty} + I_2$$
$$= o(1) + I_2.$$

Now

$$I_2 = \int_0^\infty \Psi_\alpha(u)(d/du) \, \{J'(u)\} \, du$$
$$= \{1/\Gamma(\alpha+1)\} \left(\int_0^\epsilon + \int_\epsilon^\infty\right) u^\alpha \psi_\alpha(u). \, (d/du) \, \{J'(u)\} \, du$$
$$= (I_3 + I_4)/\Gamma(\alpha+1).$$

We now show that $I_3 = o(1)$.

Now

$$I_3 = \int_0^{\epsilon} u^{\alpha}\psi_{\alpha}(u)\,(d/du)\{J'(u)\}\,du$$

$$= \{1/\Gamma(2-\alpha)\} \int_0^{\epsilon} u^{\alpha}\psi_{\alpha}(u)\,du \int_u^{\infty} (t-u)^{1-\alpha} Q^{(2)}(t)\,dt$$

$$= \{1/\Gamma(2-\alpha)\}\left[\int_0^{\epsilon} u^{\alpha}\psi_{\alpha}(u)\,du \int_u^{\epsilon} (t-u)^{1-\alpha} Q^{(2)}(t)\,dt \right.$$

$$\left. + \int_{\epsilon}^{\infty} Q^{(2)}(t)\,dt \int_0^{\epsilon} u^{\alpha}\psi_{\alpha}(u)(t-u)^{1-\alpha}\,du\right]$$

$$= (I_{3,1}+I_{3,2})/\Gamma(2-\alpha).$$

$$I_{3,1} = \int_0^{\epsilon} u^{\alpha}\psi_{\alpha}(u)\,du \int_u^{\epsilon} (t-u)^{1-\alpha} Q^{(2)}(t)\,dt$$

$$= \int_0^{\epsilon} Q^{(2)}(t)\,dt \int_0^{t} u^{\alpha}\psi_{\alpha}(u)\,(t-u)^{1-\alpha}\,du$$

$$= o(1)\,.\int_0^{\epsilon} t^2 Q^{(2)}(t)\,dt \int_0^{1} x^{\alpha}(1-x)^{1-\alpha}\,dx$$

$$= o(1)\,.\,O\left[\int_0^{\epsilon} (t^5/p^6)\,dt\right]$$

$$= o(1)\,.\,O(\epsilon^6/p^6)$$

$$= o(1).$$

$$I_{3,2} = \int_{\epsilon}^{\infty} Q^{(2)}(t)\,dt \int_0^{\gamma} u^{\alpha}\psi_{\alpha}(u)\,(t-u)^{1-\alpha}\,du$$

$$= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} t^{-3}\epsilon^{\alpha}\,dt \int_0^{\epsilon} (t-u)^{1-\alpha}\,du\right]$$

$$= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} \epsilon^{\alpha} t^{-3} t^{2-\alpha}\,dt\right]$$

$$= o(1)\,.\,O\left[\int_{\epsilon}^{\infty} (\epsilon^{\alpha}/t^{\alpha+1})\,dt\right]$$

$$= o(1)\,.\,O\big[(\epsilon^{\alpha}/t^{\alpha})\big]_{\epsilon}^{\infty}$$

$$= o(1).$$

(6.41) Thus $$I = o(1)+[1/\pi\Gamma(\alpha+1)]\,.\,I_4$$

Now we have

$$V(x,\theta)-(1/\pi)\int_\epsilon^\infty \{\psi_{\alpha-1}(u)/u\}\,du$$

$$= o(1)+[1/\pi\Gamma(\alpha+1)]\,.\,I_4-(1/\pi)I'$$

$$\doteq o(1)+K.\int_\epsilon^\infty u^\alpha\psi_\alpha(u)\,du\int_u^\infty (t-u)^{1-\alpha}Q^{(2)}(t)\,dt-(1/\pi)I',$$

where

$$[K = \{1/\pi\Gamma(\alpha+1)\Gamma(2-\alpha)\}],$$

$$= o(1)+K.\int_\epsilon^\infty u^\alpha\psi_\alpha(u)\,du\int_u^\infty (t-u)^{1-\alpha}\{(2/t^3)-q^{(2)}(t)\}\,dt-(1/\pi)I'$$

$$= o(1)+2K\int_\epsilon^\infty u^\alpha\psi_\alpha(u)\,du\int_u^\infty (t-u)^{1-\alpha}t^{-3}dt-I_5-(1/\pi)I'$$

$$= o(1)+4K\int_\epsilon^\infty u^{-1}\psi_\alpha(u)\,du\int_0^{\pi/2} (\sin\theta)^{3-2\alpha}(\cos\theta)^{2\alpha+1}d\theta$$
$$-I_5-(1/\pi)I'$$

$$= o(1)+(1/\pi)\int_\epsilon^\infty u^{-1}\psi_\alpha(u)\,du-I_5-(1/\pi)\int_\epsilon^\infty u^{-1}\psi_{\alpha-1}(u)\,du$$

$$= o(1)+(1/\pi)\int_\epsilon^\infty u^{-1}\psi_\alpha(u)\,du-I_5-(1/\pi)\Big[\psi_\alpha(u)\Big]_\epsilon^\infty$$
$$-(1/\pi)\int_\epsilon^\infty u^{-1}\psi_\alpha(u)\,du$$

$$= o(1)-I_5.$$

It now remains to show that $I_5 = o(1)$.

$$I_5 = K\int_\epsilon^\infty u^\alpha\psi_\alpha(u)\,du\int_u^\infty (t-u)^{1-\alpha}q^{(2)}(t)\,dt$$

$$= O\left[p^2\int_\epsilon^\infty u^\alpha|\psi_\alpha(u)|\,.\,u^{-\alpha-3}\,du\int_0^1 x^{\alpha+2}(1-x)^{1-\alpha}\,dx\right]$$

$$= O\left[p^2\left(\int_\epsilon^\delta+\int_\delta^\infty\right)u^{-3}|\psi_\alpha(u)|\,du\right]$$

$$= O(I_{5,1}+I_{5,2}).$$

Now

$$I_{5,1} = p^2 \int_{\epsilon}^{\delta} u^{-3} |\psi_\alpha(u)| \, du$$

$$= o\left(p^2 \int_{\epsilon}^{\delta} u^{-3} \, du\right)$$

$$= o(1).$$

$$I_{5,2} = p^2 \int_{\delta}^{\infty} u^{-3} |\psi_\alpha(u)| \, du$$

$$= O\left(p^2 \int_{\delta}^{\infty} u^{-3} \, du\right)$$

$$= o(1).$$

This completes the proof of Theorem 1.

7. *Proof of Theorem 2*—Theorem 2 can be proved by arguments parallel to those used in the proof of Theorem 1 with the only difference that we have to use the condition

$$\psi_\alpha(t) = O(1)$$

instead of

$$\psi_\alpha(t) = o(1)$$

so long as $\alpha \geqslant 1$. But when $0 < \alpha < 1$, we prove the theorem as follows:

$$I = \int_{0}^{\infty} \psi(t) \, Q(t) \, dt$$

$$= \{1/\Gamma(1-\alpha)\} \int_{0}^{\infty} Q(t) \, dt \int_{0}^{t} (t-u)^{-\alpha} d\Psi_\alpha(u)$$

$$= \{1/\Gamma(1-\alpha)\} \int_{0}^{\infty} d\Psi_\alpha(u) \int_{u}^{\infty} (t-u)^{-\alpha} Q(t) \, dt$$

$$= \int_{0}^{\infty} J_1(u) d\Psi_\alpha(u),$$

where

$$J_1(u) = \frac{1}{\Gamma(1-\alpha)} \int_{u}^{\infty} (t-u)^{-\alpha} Q(t) \, dt = O(u^{-\alpha}), \text{ as } u \to \infty.$$

Integrating by parts once, we get

$$I = \left[\Psi_\alpha(u) J_1(u)\right]_0^\infty - \int_{0}^{\infty} \Psi_\alpha(u) (d/du) \{J_1(u)\} \, du,$$

where

$$(d/du)\{J_1(u)\} = \frac{1}{\Gamma(1-\alpha)}\int_u^\infty (t-u)^{-\alpha}Q^{(1)}(t)\,dt.$$

$$= [O(u^\alpha)\,.\,O(u^{-\alpha})]_{u\to\infty} - I_1$$
$$= O(1) - I_1.$$

Now

$$I_1 = \int_0^\infty \Psi_\alpha(u)(d/du)\{J_1(u)\}\,du$$
$$= \{1/\Gamma(\alpha+1)\}\left(\int_0^\epsilon + \int_\epsilon^\infty\right) u^\alpha\psi_\alpha(u)\,(d/du)\,\{J_1(u)\}\,du$$
$$= (I_2+I_3)/(\alpha+1).$$

Now

$$I_2 = \int_0^\epsilon u^\alpha\psi_\alpha(u)(d/du)\,\{J_1(u)\}\,du$$
$$= \{1/\Gamma(1-\alpha)\}\left[\int_0^\epsilon u^\alpha\psi_\alpha(u)\,du\int_u^\infty (t-u)^{-\alpha}Q^{(1)}(t)\,dt\right]$$
$$= \frac{1}{\Gamma(1-\alpha)}\left[\int_0^\epsilon Q^{(1)}(t)\,dt\int_0^t u^\alpha.\,\psi_\alpha(u).(t-u)^{-\alpha}\,du\right.$$
$$\left.+\int_\epsilon Q^{(1)}(t)\,dt\int_0^\epsilon u^\alpha\psi_\alpha(u)\,.\,(t-u)^{-\alpha}\,du\right]$$
$$= (I_{2,1}+I_{2,2})/\Gamma(1-\alpha)$$

$$I_{2,1} = \int_0^\epsilon Q^{(1)}(t)\,dt\int_0^t u^\alpha\psi_\alpha(u)(t-u)^{-\alpha}du$$
$$= O(1)\,.\int_0^\epsilon tQ^{(1)}(t)\,dt\int_0^1 x^\alpha(1-x)^{-\alpha}dx$$
$$= O(1)\,.\,O\left[\int_0^\epsilon (t^3/p^4)\,dt\right]$$
$$= O(1).$$

$$I_{2,2} = \int_\epsilon^\infty Q^{(1)}(t)\,dt\int_0^\epsilon u^\alpha\psi_\alpha(u)\,(t-u)^{-\alpha}du$$
$$= O(1)\,.\,O\left[\int_\epsilon^\infty t^{-2}\epsilon^\alpha dt\int_0^\epsilon (t-u)^{-\alpha}du\right]$$
$$= O(1)\,.\,O\left[\int_\epsilon^\infty \epsilon^\alpha t^{-2}t^{1-\alpha}dt\right]$$
$$= O(1)\,.\,O\,[(\epsilon^\alpha/t^\alpha)]_\epsilon^\infty$$
$$= O(1).$$

(7.1) Thus $I = O(1) - \{1/\pi\Gamma(\alpha+1)\} I_3$

so that

$$V(x,\theta) - (1/\pi)\int_\epsilon^\infty \{\psi_{\alpha-1}(u)/u\} du$$

$$= O(1) - [1/\pi\Gamma(\alpha+1)] I_3 - (1/\pi) I'$$

$$= O(1) - \left[1/\pi\Gamma(\alpha+1)\right] \int_\epsilon^\infty u^\alpha \psi_\alpha(u)\, du \int_u^\infty (t-u)^{-\alpha} Q^{(1)}(t)\, dt - (1/\pi) I'$$

$$= O(1) - K \int_\epsilon^\infty u^\alpha \psi_\alpha(u)\, du \int_u^\infty (t-u)^{-\alpha} \{(-1/t^2) - q^{(1)}(t)\}\, dt - (1/\pi) I',$$

where $K = \{1/\pi\Gamma(1-\alpha)\Gamma(1+\alpha)\}$

$$= O(1) + K \int_\epsilon^\infty u^\alpha \psi_\alpha(u)\, du \int_u^\infty (t-u)^{-\alpha} . t^{-2}\, dt + I_4 - (1/\pi) I'$$

$$= O(1) + 2K \int_\epsilon^\infty u^{-1} \psi_\alpha(u)\, du \int_0^{\pi/2} (\sin\theta)^{1-2\alpha} (\cos\theta)^{1+2\alpha} d\theta + I_4 - (1/\pi) I'$$

$$= O(1) + (1/\pi)\int_\epsilon^\infty u^{-1}\psi_\alpha(u)\, du + I_4 - (1/\pi)\int_\epsilon^\infty u^{-1}\psi_{\alpha-1}(u)\, du$$

$$= O(1) + I_4.$$

It now remains to show that $I_4 = O(1)$.

Now

$$I_4 = K\int_\epsilon^\infty u^\alpha \psi_\alpha(u)\, du \int_u^\infty (t-u)^{-\alpha} q^{(1)}(t)\, dt$$

$$= O\left[p^2 \int_\epsilon^\infty u^\alpha |\psi_\alpha(u)| . u^{-\alpha-3} du \int_0^1 x^{\alpha+2}(1-x)^{-\alpha} dx\right]$$

$$= O\left[p^2 \left(\int_\epsilon^\delta + \int_\delta^\infty\right) u^{-3} |\psi_\alpha(u)|\, du\right]$$

$$= O[I_{4,1} + I_{4,2}].$$

$$I_{4,1} = p^2 \int_\epsilon^\delta u^{-3} |\psi_\alpha(u)|\, du$$

$$= O(1).$$

$$I_{4,2} = p^2 \int_\delta^\infty u^{-3} |\psi_\alpha(u)|\, du$$

$$= O(1).$$

This completes the proof of Theorem 2.

It is obvious that when α ($\geqslant 1$) is a positive integer, i.e. when $h = \alpha$, there will arise an automatic simplification in the proof given above; a straightforward treatment of this case has already been given in the author's earlier paper [1953].

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