SOME PROPERTIES OF AUTOMORPHIC EQUIVALENTS OF VECTORS IN A KAEHLER HYPERSURFACE

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1. INTRODUCTION

Consider a real $2n$ dimensional manifold V_{2n} of class C^r with a given covering by neighbourhoods each endowed with a co-ordinate system

$$z^a = x^a + ix^A$$
$$\bar{z}^a = x^a - ix^A$$

where $a = 1, 2, \ldots, n$, $A = n+1, \ldots, 2n$. Then we have a one to one correspondence $(z^a, \bar{z}^a) \rightleftarrows x^\alpha$, $\alpha = 1, \ldots, 2n$ and $(z^a, \bar{z}^a)$ may be considered to be co-ordinates of a point in the real $2n$ dimensional manifold V_{2n}. [1] In what follows the small Latin indices will take the values $1, 2, \ldots, n$, and the capital Latin indices will take the values $n+1, \ldots, 2n$, (here $A = n+a$, $B = n+b$, $\ldots\ldots$) and the Greek indices will take the values $1, 2, \ldots, 2n$.

Now if we can cover the manifold entirely by a system of co-ordinate neighbourhoods each endowed with complex co-ordinates $(z^a, \bar{z}^a)$ and if U_1 and U_2 being two complex co-ordinate neighbourhoods of the manifold, a point P belongs to $U_1 \cap U_2$ then the complex co-ordinates z'^a of the point P in one of these complex co-ordinate neighbourhoods are complex analytic functions with non-vanishing Jacobian of the complex co-ordinates z^a of the same point in the other complex co-ordinate neighbourhood,

i.e. $$z'^a = \psi^a(z), \bar{z}'^a = \bar{\psi}^a(\bar{z}) \qquad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

where $\bar{\psi}^a(\bar{z})$ denotes the complex conjugate of the function $\psi^a(z)$. Also if we put $\bar{z}^a = z^A$ then for $(z^a, \bar{z}^a)$ we can write z^α and for the transformation (1.1) we can write $z'^\alpha = f^\alpha(z)$. The Jacobian of (1.1) is easily seen to be $\left|\frac{\partial z'^\alpha}{\partial z^\beta}\right|$ which is real and greater than zero. Thus the manifold is always orientable.

A complex analytic manifold of complex dimensions n will be denoted by C_n. In C_n, vectors, tensors, affine connections, etc. are defined with respect to co-ordinate transformation (1.1) in the same way as in the case of real manifolds.

We define a tensor to be self-adjoint if barring and unbarring all indices simultaneously (i.e. changing the small Latin indices into capital Latin indices and the capital Latin indices into small Latin indices) of any component of the tensor changes the tensor into its complex conjugate.

Now assume that in our complex analytic manifold there is given a positive definite quadratic differential form

$$ds^2 = g_{\alpha\beta} dz^\alpha dz^\beta$$

where the symmetric tensor $g_{\alpha\beta}$ is self-adjoint.

$$g_{ab} = g_{AB} = 0$$

so that the metric form can be written as

$$ds^2 = 2g_{aB} dz^a dz^B$$

where

$$g_{aB} = g_{Ba} = \overline{g_{Ab}} = \overline{g_{bA}}.$$

This metric is called a Hermitian metric. Taking account of

$$g^{AB} = g^{ab} = 0, \; g^{aB} = g^{Ba} = \overline{g^{Ab}} = \overline{g^{bA}}, \qquad (1.2)$$

we obtain the Christoffel symbols

$$\left\{ {a \atop b\,c} \right\} = \frac{1}{2} g^{aD} \left(\frac{\partial g_{Db}}{\partial z^c} + \frac{\partial g_{Dc}}{\partial z^b} \right)$$

$$\left\{ {a \atop b\,C} \right\} = \frac{1}{2} g^{aD} \left(\frac{\partial g_{Db}}{\partial \bar{z}^c} - \frac{\partial g_{bC}}{\partial \bar{z}^d} \right)$$

$$\left\{ {a \atop B\,C} \right\} = 0$$

and the values of other components are given by symmetry and self-adjointness. If in addition we assume

$$\frac{\partial g_{Ab}}{\partial z^c} = \frac{\partial g_{Ac}}{\partial z^b}$$

or further

$$g_{aB} = \frac{\partial^2 \phi}{\partial z^a \partial z^B} \qquad (1.3)$$

then the condition (1.3) is called 'Kaehler's condition' and the metric satisfying (1.2) and (1.3) is called a 'Kaehler metric' and the manifold is called a 'Kaehler manifold'. Thus in a Kaehler manifold we have

$$\left\{ {a \atop b\,c} \right\} = g^{aD} \frac{\partial g_{Dc}}{\partial z^b},$$

$$\left\{ {a \atop b\,C} \right\} = 0,$$

$$\left\{ {a \atop B\,C} \right\} = 0,$$

the other components follow from symmetry and self-adjointness.

2. Automorphic Equivalence

Two tensors $T^{\alpha\beta\cdots}_{\delta\epsilon\cdots}$ and $L^{\alpha\beta\cdots}_{\delta\epsilon\cdots}$ of the same order $(2l+1)$ and the same kind, are said to be automorphically equivalent if each has components which are holomorphic functions of co-ordinates (i.e. locally expressible as a convergent power series) and one is the transform of the other under the automorphism (i.e. a non-singular linear transformation of co-ordinates which maps a space on to itself)

$$z'^a = ez^a, \; z'^A = \bar{e}z^A$$

where $e = \pm i$ and $\bar{e}$ is the conjugate of e, or $\bar{e} = -e$ and $e\bar{e} = 1$. Incidentally $(eu^a, \bar{e}u^A)$ is the automorphic equivalent of (u^a, u^A).

Consider two contravariant vector fields (u^a, u^A) and (v^a, v^A) each of which is self-adjoint. Let

$$g_{aB} u^a v^B = re^{i\theta}$$

then because of self-adjoint property it follows that

$$g_{Ab}u^A v^b = re^{-i\theta}$$

$$\therefore \quad 2r\cos\theta = g_{aB}u^a v^B + g_{Ab}u^A v^b$$

where θ may be taken to represent the angle between the two vectors symbolically. A vector (u^a, u^A) can now be defined to be orthogonal to (v^a, v^A) if

$$g_{aB}u^a v^B + g_{Ab}u^A v^b = 0.$$

This definition can be extended as follows:

Let there be a self-adjoint vector $u^\alpha = (u^a, u^A)$ and let $(u^a, 0)$ and $(0, u^A)$ be two vectors which can be formed from the components of (u^a, u^A). Let

$$\xi^a = \lambda u^a,\ \xi^A = \mu u^A,\ \left[\xi^\alpha = \lambda u^\alpha_{(a,0)} + \mu u^\alpha_{(0,A)}\right]$$

where λ and μ are parameters. Hence these vectors $(u^a, 0)$ and $(0, u^A)$ define a real geodesic V_2 as in the case of Riemannian manifolds. For $\lambda = \mu = 1$ we get the original vector (u^a, u^A). Therefore this vector (u^a, u^A) lies on the geodesic V_2^u formed by $(u^a, 0)$ and $(0, u^A)$. Also for $\lambda = e$, $\mu = \bar{e}$ we find that $(eu^a, \bar{e}u^A)$ lies on this geodesic surface. The orthogonal property is identically satisfied for a vector and its automorphic equivalent. Let (v^a, v^A) be another vector not lying on the geodesic V_2^u. Let (v^a, v^A) and $(ev^a, \bar{e}v^A)$ define another geodesic V_2^v. We define (u^a, u^A) to be orthogonal to (v^a, v^A) if every vector lying in V_2^u is orthogonal to every vector lying in V_2^v. That is, if

$$g_{aB}u^a v^B = 0$$

$$g_{Ab}u^A v^b = 0.$$

3. A Complex Hypersurface of a Kaehler C_n

While defining a complex hypersurface of a Kaehler manifold, we shall quote a result for Algebraic manifolds on 'product manifolds' [5]: 'If a complex analytic manifold C_m of class u and complex dimension m, and another complex analytic manifold C_n of class v and complex dimension n exist, then the product manifold $C_{m+n} = C_m \times C_n$ is of class u or v whichever is lesser of the two and complex dimension $(m+n)$.' Hence if we define a hypersurface of a compact orientable Kaehler C_{n+1} of complex dimension $(n+1)$ and real dimension $2n+2$ and of class c^r to be a subspace of complex dimension n and real dimension $2n$, then the class of this hypersurface is at least c^r.

Note: In what follows the small Greek indices $\delta, \gamma, \ldots$ will take the values $1, 2, \ldots, n+1$, and the capital Greek indices will take the values $n+2, \ldots, 2n+2$.

Let the enveloping space be a Kaehler C_{n+1} which is given by the metric $ds^2 = 2g_{\Delta\gamma}dZ^\Delta\, dZ^\gamma$. The metric tensor for the hypersurface will be denoted by g'_{aB}. Let (Z^γ, Z^Γ) be the co-ordinates of a point in C_{n+1} and let (z^a, z^A) be the co-ordinates of a point in the hypersurface C_n. For points in C_n the Z's are expressible as functions of the z's. Let us consider those hypersurfaces which express Z's as complex analytic functions of the z's, i.e. which satisfy the following conditions:—

$$Z^\gamma = \psi^\gamma(z^1, \ldots, z^n), \quad Z^\Gamma = \bar{\psi}^\gamma(z^{n+1}, \ldots, z^{2n}) \qquad .. \quad (3.1)$$

Here $\psi^\gamma(z^1, \ldots z^n)$ which is a function of z^a only, and $\bar{\psi}^\gamma(z^{n+1}, \ldots, z^{2n})$ which is a function of z^A only, are complex conjugate. As in Riemannian manifolds of real dimensions we find that $(Z^{\gamma},_{a}, Z^{\Gamma},_{A})$ is the vector tangential to the curve of parameter (z^{a}, z^{A}) in C_n.

Let (N^{γ}, N^{Γ}) be the unit vector in C_{n+1} normal to the hypersurface C_n. From our definition of orthogonality of two vectors which are not automorphic equivalents we have

$$g_{\gamma\Delta} Z^{\gamma},_{a} N^{\Delta} = 0 \quad \ldots \quad (3.2)$$

$$g_{\Gamma\delta} Z^{\Gamma},_{A} N^{\delta} = 0 \quad \ldots \quad (3.3)$$

$$2g_{\gamma\Delta} N^{\gamma} N^{\Delta} = 1 \quad \ldots \quad (3.4)$$

From (3.2) we get n equations, from (3.3) we get n equations and from (3.4) we get one equation. Hence we get $(2n+1)$ equations to determine $(2n+2)$ components. Hence (N^{γ}, N^{Γ}) cannot be determined uniquely. Also if we put $M^{\gamma} = eN^{\gamma}$, $M^{\Gamma} = \bar{e}N^{\Gamma}$ where $e = \pm i$ and $\bar{e}$ is the conjugate of e, we find that (M^{γ}, M^{Γ}) also satisfies (3.2), (3.3) and (3.4). Hence (M^{γ}, M^{Γ}), i.e. the automorphic equivalent of (N^{γ}, N^{Γ}) is also a unit vector normal to the hypersurface C_n, and so is any vector $(\xi^{\gamma}, \xi^{\Gamma})$ lying on the real geodesic V_2 spanned by (N^{γ}, N^{Γ}) and its automorphic equivalent (M^{γ}, M^{Γ}) such that

$$\xi^{\gamma} = \lambda N^{\gamma} + \mu M^{\gamma} = (\lambda + e\mu) N^{\gamma}$$

$$\xi^{\Gamma} = \bar{\lambda} N^{\Gamma} + \bar{\mu} M^{\Gamma} = (\bar{\lambda} + \bar{e}\bar{\mu}) N^{\Gamma}$$

where λ, μ are parameters. If we put $\lambda + e\mu = K$ then if $|K| = 1$, then $(\xi^{\gamma}, \xi^{\Gamma})$ is a unit vector in C_{n+1} normal to the hypersurface C_n.

Here the metric tensors of C_{n+1} and those of C_n are connected by

$$g'_{aB} = g_{\gamma\Delta} Z^{\gamma},_{a} Z^{\Delta},_{B}.$$

Also the enveloping manifold C_{n+1} being a Kaehler space, $g_{\gamma\Delta} = \dfrac{\partial^2 \phi}{\partial Z^{\gamma} \partial Z^{\Delta}}$ where ϕ is real. Now

$$g'_{aB} = g_{\gamma\Delta} \frac{\partial Z^{\gamma}}{\partial z^{a}} \cdot \frac{\partial Z^{\Delta}}{\partial z^{B}} = \frac{\partial^2 \phi}{\partial Z^{\gamma} \partial Z^{\Delta}} \cdot \frac{\partial Z^{\gamma}}{\partial z^{a}} \cdot \frac{\partial Z^{\Delta}}{\partial z^{B}}$$

$$= \left\{ \frac{\partial}{\partial z^{a}} \left(\frac{\partial \phi}{\partial Z^{\Delta}} \right) \right\} \frac{\partial Z^{\Delta}}{\partial z^{B}} \quad \ldots \quad (3.5)$$

Since $\dfrac{\partial^2 Z^{\Delta}}{\partial z^{a} \partial z^{B}} = 0$, we can write (3.5) as

$$g'_{aB} = \frac{\partial}{\partial z^{a}} \left(\frac{\partial \phi}{\partial Z^{\Delta}} \cdot \frac{\partial Z^{\Delta}}{\partial z^{B}} \right) = \frac{\partial^2 \phi}{\partial z^{a} \partial z^{B}}.$$

Therefore the hypersurface is also a Kaehler space and the same function ϕ determines the metric tensors of C_{n+1} as well as of C_n.

Hence a complex hypersurface of a Kaehler manifold satisfying conditions (3.1) *is necessarily another Kaehler manifold.*

By means of tensor differentiation of $g'_{aB} = g_{\gamma\Delta} Z^{\gamma},_{a} Z^{\Delta},_{B}$ with respect to z^{c} we get

$$g_{\gamma\Delta} Z^{\gamma};_{ac} Z^{\Delta},_{B} + g_{\gamma\Delta} Z^{\gamma},_{a} Z^{\Delta};_{Bc} = 0.$$

But

$$Z^{\Delta}{}_{;Bc} = Z^{\Delta}{}_{,Bc} + \left\{ {\Delta \atop \alpha\Gamma} \right\} Z^{\alpha}{}_{,c} Z^{\Gamma}{}_{,B} = 0$$

$$\therefore \quad g_{\gamma\Delta} Z^{\gamma}{}_{;ac} Z^{\Delta}{}_{,B} = 0, \qquad g_{\Gamma\delta} Z^{\Gamma}{}_{;AC} Z^{\delta}{}_{,b} = 0$$

This shows that $\begin{pmatrix} Z^{\gamma}{}_{;ac} & 0 \\ 0 & Z^{\Gamma}{}_{;AC} \end{pmatrix}$ is a vector in C_{n+1} normal to the hypersurface C_n and a tensor of the second order in C_n. Hence we may write

$$Z^{\gamma}{}_{;ac} = \Omega_{ac} N^{\gamma}$$

$$Z^{\Gamma}{}_{;AC} = \Omega_{AC} N^{\Gamma}$$

since $Z^{\gamma}{}_{;aC} = 0$, $Z^{\Gamma}{}_{;Ac} = 0$ and (N^{γ}, N^{Γ}) is not a zero vector, it follows that $\Omega_{aC} = 0$, $\Omega_{Ac} = 0$. In analogy with Riemannian manifolds we shall call this tensor $\begin{pmatrix} \Omega_{ac} & 0 \\ 0 & \Omega_{AC} \end{pmatrix}$ the second fundamental tensor of C_n.

Note: In what follows the small Greek indices $\alpha, \beta, \gamma, \ldots$ will take the values $1, 2, \ldots, 2n$.

Conjugate directions.

In analogy with Riemannian spaces we shall formally define two directions (u^a, u^A) and (v^a, v^A) to be conjugate if

$$\Omega_{\alpha\beta} u^{\alpha} v^{\beta} = 0$$

i.e.

$$\Omega_{ab} u^a v^b + \Omega_{AB} u^A v^B = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3.6)$$

In (3.6) if we replace (u^a, u^A) and (v^a, v^A) by their automorphic equivalents $(eu^a, \bar{e}u^A)$ and $(ev^a, \bar{e}v^A)$ we find that (3.6) is identically satisfied.

Hence if two vectors are conjugate, their automorphic equivalents also define conjugate directions. In particular if (v^a, v^A) coincides with (u^a, u^A) that is if the vector (u^a, u^A) is self-conjugate, then its automorphic equivalent is also self-conjugate. Hence if self-conjugacy is formally defined to be the condition for asymptotic lines in a Kaehler manifold, then it follows that if a vector field defines an asymptotic line then so does its automorphic equivalent.

Lines of curvature.

In our discussion of lines of curvature we shall consider a Kaehler hypersurface C_n whose second fundamental tensor is

$$\Omega_{\alpha\beta} = \begin{pmatrix} \Omega_{ab} & 0 \\ 0 & \Omega_{AB} \end{pmatrix}$$

Consider an orthogonal ennuple in C_n, in which $p^{\alpha}_{h|}$, $h = 1, \ldots, n$ denote a set of n mutually orthogonal self-adjoint unit vectors no two of which are automorphic equivalents, and $(q^{a}_{h|}, q^{A}_{h|})$ are the automorphic equivalents of $(p^{a}_{h|}, p^{A}_{h|})$. According to our definition of orthogonality we can choose $(p^{a}_{h|}, p^{A}_{h|})$ arbitrarily, but the other

n vectors completing the ennuple must be the automorphic equivalents of $(p^a_{h|}, p^A_{h|})$, $h = 1, \ldots, n$.

Hence if a set of n vectors constitute n of the lines of curvature through a point in C_n then the other n lines of curvature through that point must be the automorphic equivalents of these n vectors.

We shall define lines of curvature in the same manner as in a real Riemannian manifold. Let $(p^a_{h|}, p^A_{h|})$, $(q^a_{h|}, q^A_{h|})$, $h = 1, \ldots; n$, form an orthogonal ennuple in a Kaehlerian C_n; $(q^a_{h|}, q^A_{h|})$ being the automorphic equivalent of $(p^a_{h|}, p^A_{h|})$. These will form the lines of curvature through P if

$$(\Omega_{\alpha\beta} - k^p_{h|} g'_{\alpha\beta})\, p^\alpha_{h|} = 0 \quad \ldots \quad (3.7)$$

$$(\Omega_{\alpha\beta} - k^q_{h|} g'_{\alpha\beta})\, q^\alpha_{h|} = 0 \quad \ldots \quad (3.8)$$

where $k^p_{h|}, k^q_{h|}$, are scalars.

Multiplying (3.7) by $p^\beta_{h|}$ and summing with respect to β we have

$$\Omega_{ab} p^a_{h|} p^b_{h|} + \Omega_{AB} p^A_{h|} p^B_{h|} = k^p_{h|} \quad \ldots \quad (3.9)$$

Again multiplying (3.7) by $p^\beta_{l|}$ and summing with respect to β we have

$$\Omega_{ab} p^a_{h|} p^b_{l|} + \Omega_{AB} p^A_{h|} p^B_{l|} = 0 \quad \ldots \quad (3.10)$$

This shows that the lines of curvature are conjugate.

Multiplying (3.7) by $q^\beta_{l|}$ and putting $q^a_{l|} = e p^a_{l|}$, $q^A_{l|} = \bar{e} p^A_{l|}$ we get

$$\Omega_{ab} p^a_{h|} p^b_{l|} - \Omega_{AB} p^A_{h|} p^B_{l|} = 0 \quad \ldots \quad (3.11)$$

From (3.10) and (3.11) we find that in this case the condition of conjugacy reduces to

$$\Omega_{ab} p^a_{h|} p^b_{l|} = 0, \; \Omega_{AB} p^A_{h|} p^B_{l|} = 0, \; h \neq l.$$

Now multiplying (3.8) by $q^\beta_{h|}$ and substituting $q^\beta_{h|} = \pm e p^\beta_{h|}$ (positive sign being taken when $\beta = 1, \ldots, n$, and negative sign being taken when $\beta = n+1, \ldots, 2n$) we have

$$\Omega_{ab}(e p^a_{h|})(e p^b_{h|}) + \Omega_{AB}(\bar{e} p^A_{h|})(\bar{e} p^B_{h|}) = k^q_{h|}$$

$$\therefore \quad -\Omega_{ab} p^a_{h|} p^b_{h|} - \Omega_{AB} p^A_{h|} p^B_{h|} = k^q_h = -k^p_{h|}.$$

Therefore the number of independent principal curvatures defined in analogy with Riemannian manifolds is n.

We denote

$$k^p_{h|} = -k^q_{h|} = k_{h|} \quad h = 1, \ldots, n.$$

Again multiplying (3.7) by $q^\beta_{h|}$ and summing with respect to β we get

$$\Omega_{ab} p^a_{h|} (e p^b_{h|}) + \Omega_{AB} p^A_{h|} (\bar{e} p^B_{h|}) = 0$$

$$\therefore \quad \Omega_{ab} p^a_{h|} p^b_{h|} = \Omega_{AB} p^A_{h|} p^B_{h|} \quad \ldots \quad (3.12)$$

Since $(p^a_{h|}, p^A_{h|})$ is a self-adjoint vector, it follows from (3.12) that the tensor $\begin{pmatrix}\Omega_{ab} & 0 \\ 0 & \Omega_{AB}\end{pmatrix}$ is self-adjoint and $k_{h|}$ is a real scalar.

In C_n we find the mean curvature M is given by

$$M = \sum_1^n k^p_{h|} + \sum_1^n k^q_{h|} = 0.$$

Hence every hypersurface C_n of a Kaehlerian C_{n+1} defined with respect to (3.1) *is of minimal variety.*

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