ON THE GUIDING CURVES OF A RECTILINEAR CONGRUENCE *

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1. The notations used below are those of Weatherburn (1947). We have chosen the parametric curves in such a manner that $\vec{d}$ may be expressed in terms of $\vec{n}$ and $\vec{r}_1$ only. This appears to be a new method of studying the properties of a rectilinear congruence. We will apply it for studying mainly the properties of curves on the director surface determined by the equation formed by equating to zero the second quadratic form of Sannia.

The plane of $\vec{n}$ and $\vec{d}$ (supposed different) at a point P_1 of the director surface intersects the surface along a plane curve C_1 through P_1. The plane of $\vec{n}$ and $\vec{d}$ at P_2, the consecutive point of P_1 on C_1, intersects the surface along a plane curve C_2, and so on. The envelope of the curves C_1, $C_2, \ldots$ is a curve C having the property that the ray, the normal to the surface and the tangent to the curve C at each point are coplanar.

2. Let a family of curves like C generated in the above manner be taken as the parametric curves $v = \text{const.}$, which will be referred to as the guiding curves of the congruence, and their orthogonal trajectories as the other system of parametric curves $u = \text{const.}$

Let $\vec{a}$ and $\vec{b}$ be the unit tangents to the parametric curves $v = \text{const.}$ and $u = \text{const.}$ respectively and $\vec{n}$ the unit normal to the surface at any point. Then

$$\vec{d} = \vec{n} \cos\theta + \vec{a} \sin\theta$$

where θ (the angle of incidence) is the angle between the ray and the normal to the surface at the point.

$$\therefore \quad \vec{d}_1 = \vec{n}_1 \cos\theta + \vec{a}_1 \sin\theta - \vec{n} \sin\theta\, \theta_1 + \vec{a} \cos\theta\, \theta_1$$

$$\vec{d}_2 = \vec{n}_2 \cos\theta + \vec{a}_2 \sin\theta - \vec{n} \sin\theta\, \theta_2 + \vec{a} \cos\theta\, \theta_2$$

where

$$\vec{n}_1 = -\sqrt{E}\left(k_a \vec{a} + \tau \vec{b}\right), \quad \vec{n}_2 = -\sqrt{G}\left(\tau \vec{a} + k_b \vec{b}\right)$$

$$\vec{a}_1 = \sqrt{E}\left(k_a \vec{n} + \gamma \vec{b}\right), \quad \vec{a}_2 = \sqrt{G}\left(\tau \vec{n} + \gamma' \vec{b}\right)$$

$$\gamma = -E_2/2E\sqrt{G}, \quad \gamma' = G_1/2G\sqrt{E}$$

$$\tau = M/\sqrt{EG}, \quad k_a = L/E, \quad k_b = N/G.$$

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The equation of the curves corresponding to the developables of the congruence is given by

$$A-B=0 \qquad \ldots \qquad (1)$$

where $A = \cos^2\theta[EMdu^2+(EN-GL)du\,dv-GMdv^2]/\sqrt{EG}$
$-\sqrt{G}\sin^2\theta(k_a\sqrt{E}\,du+\tau\sqrt{G}\,dv)\,dv$

and $B = \sin\theta\cos\theta\sqrt{E}\,du(\gamma\sqrt{E}\,du+\gamma'\sqrt{G}\,dv)-\sqrt{G}\,dv(\theta_1\,du+\theta_2\,dv)$.

Thus with the guiding curves and their orthogonal trajectories as the parametric curves, the developables of the congruence are given by (1).

3. If $\theta = 0$, then $\vec{d} = \vec{n}$ and the congruence is formed by normals to the director surface. In this case, the equation (1) represents the lines of curvature.

If $\theta = \pi/2$, then $\vec{d} = \vec{a}$ and the congruence is formed by the tangents to the guiding curves. In this case, one of the directions represented by (1) coincides with the guiding curve and the other direction is inclined to the guiding curve at an angle $\tan^{-1}(-k_a/\tau)$.

One of the two directions determined by (1) will coincide with the guiding curve also when $\tan\theta = \tau/\gamma$. In this case,

$$\vec{d} = \left(\vec{n}\,\gamma+\vec{a}\,\tau\right)\Big/\sqrt{(\tau^2+\gamma^2)}$$

and the other direction is given by

$$\delta\theta-(k_a\sqrt{E}\,du+\tau\sqrt{G}\,dv)+\sqrt{E}\,du(k_b\cos\theta-\gamma'\sin\theta)\cos\theta = 0.$$

4. Writing $-\theta$ for θ corresponding to the reflected congruence, (1) changes into $A+B=0$.

In order that the curves in which the developables of the incident and reflected congruences cut the director surface may be the same, we must have $A \equiv 0$ or $B \equiv 0$.

Case (i)—Let us suppose that $A \equiv 0$.

Then the coefficients of du^2 and dv^2 will be zero if $M = 0$, i.e. if the parametric curves be the lines of curvature, and the coefficient of $du\,dv$ will be zero if either $L = 0$ and $N = 0$ in which case the reflecting surface is a plane

or $$[(EN-GL)/\sqrt{EG}]\cos^2\theta-(L/E)\sqrt{EG}\sin^2\theta = 0$$

or $$\cos\theta = \pm\sqrt{k_a/k_b} \text{ where } k_a < k_b \text{ and } k_ak_b > 0.$$

Hence we get

Theorem 1. The developables of the incident and the reflected congruences will cut the reflecting surface along the same pair of curves if (i) the reflecting surface be a plane, or if (ii) the guiding curves be one system of lines of curvature and the ratio of the principal curvatures at each point of incidence of the ray be equal to the square of the cosine of the angle of incidence.

Case (ii)—Let us suppose that $B \equiv 0$.

Then the coefficients of du^2, dv^2 and $du\,dv$ will be equal to zero separately if

$$\gamma\sin\theta\cos\theta = 0,\ \theta_2 = 0 \quad\text{and}\quad \sqrt{E}\gamma'\sin\theta\cos\theta-\theta_1 = 0.$$

If $\sin\theta.\cos\theta = 0$, we get $\theta = \pi/2$ or 0. Therefore, either the rays are tangents to the guiding curves or the rays are normals to the director surface. Both these results are trivial as the reflected rays will coincide with the incident rays. Hence we have

$$\gamma = 0,\quad \text{i.e.}\quad E_2 = 0,\quad \text{i.e.}\quad E = \alpha(u),$$
$$\theta_2 = 0,\quad \text{i.e.}\quad \theta = \beta(u)$$

and $$\sqrt{E}\,\gamma' \sin\theta \cos\theta - \theta_1 = 0 \quad \text{or} \quad \frac{\tan\theta}{\sqrt{G}} = f(v).$$

Hence we get

THEOREM 2. The curves on the reflecting surface giving the developables of the reflected congruence are identical with those for the incident congruence if the ratio of the tangent of the angle of incidence and the distance function of the orthogonal trajectory be constant along the guiding curves.

In order that $\tan\theta$ may be a function of u alone as $\theta = \beta(u)$, we have

$$\sqrt{G}\,f(v) = \phi(u).$$

$$\therefore \quad ds^2 = \alpha(u)du^2 + [\phi(u)/f(v)]^2\,dv^2$$

$$= dU^2 + \psi(U)dV^2$$

where $$\alpha(u)du^2 = dU^2, \quad [\phi(u)]^2 = \psi(U), \quad dv^2/[f(v)]^2 = dV^2.$$

Hence, the director surface is applicable to a surface of revolution. Therefore, we get

THEOREM 3. The curves giving the developables for the reflected congruence are identical with those for the incident congruence if the reflecting surface be applicable to a surface of revolution, and the rays cutting the parallels of latitudes at a constant angle.

The equation (1) will represent orthogonal curves if

$$GE \cos\theta\,(\tau\cos\theta - \gamma\sin\theta) + E\big(\theta_2\sqrt{G} - G\tau\big) = 0$$

i.e. $$\theta_2/\sqrt{G} - \sin\theta\,(\tau\sin\theta + \gamma\cos\theta) = 0 \quad .. \quad .. \quad .. \quad (2)$$

The equation (2) is evidently satisfied if

$$\tan\theta = -\gamma/\tau = f(u)$$

From equation (2), we also get

THEOREM 4. If the rectilinear congruence has any two of the following properties, it will also have the third :

(i) the curves representing the developables of the congruence are orthogonal, (ii) the angle of incidence is constant along the orthogonal trajectories of the guiding curves, (iii) the parametric curves are plane lines of curvature.

As (2) remains unchanged on writing $-\theta$ for θ only when $M = 0$, we get

THEOREM 5. If the guiding curves form one system of lines of curvature and the curves giving the developables of the incident congruence be orthogonal, then the curves giving the developables of the reflected congruence are also orthogonal.

If $M = 0$, (2) gives

$$\theta_2 + \sin\theta\cos\theta\,E_2/2E = 0, \quad \text{i.e.}\ \sqrt{E}\tan\theta = f(u).$$

Hence we get

THEOREM 6. The curves giving the developables of the congruence are orthogonal, if the guiding curves form a system of lines of curvature and the product of the tangent of the angle of incidence and the distance function of the guiding curves be constant along the other system of lines of curvature.

If the guiding curves be geodesics, then $E_2 = 0$, i.e. $\gamma = 0$. Now (2) gives

$$\theta_2/\sqrt{G} - \tau\sin^2\theta = 0, \quad \text{i.e.}\ \frac{1}{\sqrt{G}}\cdot\frac{\partial}{\partial v}\cot\theta = -\tau.$$

Hence we get

THEOREM 7. If the arc-rate of change of the cotangent of the angle of incidence of the ray along the orthogonal trajectory of the guiding curves, which are geodesics, is equal to the torsion of the geodesic, then the developables of the congruence cut the reflecting surface along orthogonal curves.

The curves represented by (1) will be conjugate if

$$(k_a\theta_2/\sqrt{G}-\tau\theta_1/\sqrt{E})+\sin\theta\cos\theta\,(\tau\gamma'-k_b\gamma)=0 \quad \ldots \quad \ldots \quad (3)$$

The condition (3) is identically satisfied if

(i) $L = M = N = 0$ or (ii) $\theta = 0$ or (iii) $\theta = \pi/2$.

In the first case, the director surface becomes a plane and hence all pairs of curves on it are conjugate. In the second case, the curves (1) become the lines of curvature and hence are conjugate. In the third case, we get

THEOREM 8. The developables of the congruence formed by the tangents to a family of curves on a surface cut it in conjugate lines.

The condition (3) remains unchanged if we write $-\theta$ for θ.

Hence we get

THEOREM 9. If the developables of the incident congruence cut the reflecting surface in conjugate curves, the developables of the reflected congruence will also cut it in conjugate curves.

Now we shall find the condition or conditions required for the developables of the incident and the reflected congruences to meet the director surface in the same pair of conjugate curves.

If the director surface is not a plane, the developables of the incident and the reflected congruences will cut the director surface in the same pair of curves if

(i) $M = 0$, $\cos^2\theta = k_a/k_b$,

or if (ii) $\gamma = 0$, $\theta_2 = 0$ and $\sqrt{E}\,\gamma'\sin\theta\cos\theta-\theta_1 = 0$.

In the first case, (3) gives $\dfrac{\partial}{\partial v}\log\sqrt{E}\sin\theta = 0$, i.e. the congruence is normal, as will be seen later. In the second case, (3) is satisfied.

Hence we get

THEOREM 10. If the developables of the incident and the reflected congruences cut the director surface along the same pair of curves, then either they cut the director surface along conjugate curves or they cut the director surface along conjugate curves if the rays of any normal congruence cut the director surface obliquely.

5. We have

$$a = \vec{r}_1\,.\,\vec{d}_1 = \sqrt{E}\,\vec{a}\,.\,\vec{d}_1 = \sqrt{E}\,(-\sqrt{E}\,k_a\cos\theta+\cos\theta\,\theta_1)$$

$$c = \vec{r}_2\,.\,\vec{d}_2 = \sqrt{G}\,\vec{b}\,.\,\vec{d}_2 = \sqrt{G}\,(-\sqrt{G}\,k_b\cos\theta+\sqrt{G}\,\gamma'\sin\theta)$$

$$b' = \vec{r}_1\,.\,\vec{d}_2 = \sqrt{E}\,\vec{a}\,.\,\vec{d}_2 = \sqrt{E}\,(-\sqrt{G}\,\tau\cos\theta+\cos\theta\,\theta_2)$$

$$b = \vec{r}_2\,.\,\vec{d}_1 = \sqrt{G}\,\vec{b}\,.\,\vec{d}_1 = \sqrt{G}\,(-\sqrt{E}\,\tau\cos\theta+\sqrt{E}\,\gamma\sin\theta).$$

$$\therefore\quad b'-b = \sqrt{E}\cos\theta\,\theta_2-\sqrt{EG}\,\gamma\sin\theta \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Hence we get

Theorem 11. If the rectilinear congruence has any two of the following properties, it will also have the third:

(i) It is normal, (ii) guiding curves are geodesics, (iii) angle of incidence is constant along the orthogonal trajectories of the guiding curves.

The equation (4) can be written as

$$(b'-b)/\sqrt{E}\sin\theta = \cot\theta\,\theta_2+E_2/2E = \frac{\partial}{\partial v}\log\sqrt{E}\sin\theta.$$

Hence we get

Theorem 12. For a normal congruence, the product of the sine of the angle of incidence and the distance function of the guiding curve is constant along its orthogonal trajectory, and conversely.

We have

$$\overrightarrow{dr}\,.\,\overrightarrow{dd} = -\cos\theta\,(L\,du^2+2M\,du\,dv+N\,dv^2)$$
$$+\sqrt{E}\,du\,dv(\cos\theta\;\theta_2+\gamma\sqrt{G}\sin\theta)+G\gamma'\,dv^2.$$

Hence we immediately get the well-known result (Mishra, 1951):

If the congruence is formed by normals to a surface, then the surfaces of the rectilinear congruence whose lines of striction lie on it cut it in asymptotic lines.

Let $\vec{D}$ denote unit vector along the reflected ray. Then

$$\vec{D}+\vec{d} = 2\vec{n}\cos\theta.$$

$$\therefore\quad \overrightarrow{dr}\,.\,\overrightarrow{dD}+\overrightarrow{dr}\,.\,\overrightarrow{dd} = -2\cos\theta\,(L\,du^2+2M\,du\,dv+N\,dv^2)$$

$$\therefore\quad \overrightarrow{dr}\,.\,\overrightarrow{dD} = -\cos\theta\,(L\,du^2+2M\,du\,dv+N\,dv^2)$$
$$-\sqrt{E}\,du\,dv\,(\cos\theta\;\theta_2+\gamma\sqrt{G}\sin\theta)+G\gamma'\,dv^2.$$

Hence we get

Theorem 13. If the curves giving the surfaces of the incident congruence whose lines of striction lie on the director surface be asymptotic lines, those for the reflected congruence will also be so.

Now the curves $\overrightarrow{dr}.\,\overrightarrow{dd} = 0$ and $\overrightarrow{dr}.\,\overrightarrow{dD} = 0$ will become the asymptotic lines if $\gamma' = 0$ (i.e. $G_1 = 0$, i.e. the curves $u =$ const. are geodesics and hence the curves $v =$ const. are geodesic parallels)

and $\qquad \cot\theta\;\theta_2-E_2/2E = 0, \quad \text{i.e.} \quad \sin\theta = \sqrt{E}f(u).$

Hence we get

Theorem 14. The curves giving the surfaces of the incident and reflected congruences whose lines of striction lie on the director surface will be asymptotic lines, if the sine of the angle of incidence varies as the distance function of the guiding curve along its orthogonal trajectory.

6. We shall now study the rectilinear congruences for which the angle of incidence is constant.

The vector position of any point on a ray of any rectilinear congruence can be expressed as

$$\vec{R} = \vec{r}+\lambda\vec{d}.$$

Let $\vec{R}$ be a point on another surface $\bar{S}$ cutting the ray at a distance λ from the director surface S.

$$\therefore \quad \vec{R}_1 = \vec{r}_1+\lambda_1\vec{d}+\lambda\vec{d}_1 \text{ and } \vec{R}_2 = \vec{r}_2+\lambda_2\vec{d}+\lambda\vec{d}_2.$$

Let p, q, r be the components of the vector $\vec{R}_1\times\vec{R}_2$ along $\vec{d}$, $\vec{d}_1$, $\vec{d}_2$. Then

$$p = H\vec{n}\,.\,\vec{d}+\lambda(ag+ce)/\sqrt{eg}+\lambda^2\sqrt{eg}$$

$$q = H\vec{n}\,.\,\vec{d}_1-h\lambda\vec{d}\,.\,\vec{r}_1-\lambda\lambda_1 h+(\lambda_2 b'-\lambda_1 c)\sqrt{e/g}$$

$$r = H\vec{n}\,.\,\vec{d}_2-h\lambda\vec{d}\,.\,\vec{r}_2-\lambda\lambda_2 h-(\lambda_2 a+\lambda_1 b)\sqrt{g/e}.$$

Also $$\vec{R}_1\times\vec{R}_2 = p\vec{d}+q\vec{d}_1/e+r\vec{d}_2/g.$$

$$\therefore \quad \bar{H}^2 = p^2+q^2/e+r^2/g.$$

If the surface $\bar{S}$ cuts the ray at a constant angle,

$$\vec{\bar{n}}\,.\,\vec{d} = \cos\theta = \vec{R}_1\times\vec{R}_2\,.\,\vec{d}/\bar{H} = p/\bar{H}, \text{ i.e. } \bar{H} = p\sec\theta.$$

Hence we have

$$p^2\sec^2\theta = p^2+q^2/e+r^2/g$$

$$\text{i.e.} \quad p^2\tan^2\theta = q^2/e+r^2/g.$$

This is a partial differential equation of the first order and second degree in λ. Hence we get

THEOREM 15. There is a single infinite family of surfaces which cut a given congruence at a given constant angle (different from zero).

If the director surface be a member of the family, i.e. if θ = const., then $\theta_1 = 0$ and $\theta_2 = 0$.

$$\therefore \quad \vec{d}_1 = A\sqrt{E}\,\vec{b}-\sqrt{E}\cos\theta\, k_a\vec{a}+\sqrt{E}\sin\theta\, k_a\vec{n}$$

$$\therefore \quad \vec{d}_2 = B\sqrt{G}\,\vec{b}-\sqrt{G}\cos\theta\,\tau\vec{a}+\sqrt{G}\sin\theta\,\tau\vec{n}$$

where $A = \gamma\sin\theta-\tau\cos\theta$ and $B = \gamma'\sin\theta-k_b\cos\theta$.

$$\therefore \quad e = E(A^2+k_a^2),\ f = \sqrt{EG}(AB+k_a\tau),\ g = G(B^2+\tau^2),$$

$$h = \sqrt{EG}\,(A\tau-k_aB)$$

and $$d\sigma^2 = (A\sqrt{E}\,du+B\sqrt{G}\,dv)^2+(k_a\sqrt{E}\,du+\tau\sqrt{G}\,dv)^2.$$

Also $a = -Ek_a\cos\theta$, $\quad b = A\sqrt{EG}$, $\quad b' = -\sqrt{EG}\,\tau\cos\theta$, $\quad c = BG$.

If the principal surfaces will correspond to the parametric curves on the unit sphere,

$$k_a\tau+AB = 0 \text{ and } A = \tau\cos\theta,$$

i.e. $$\tau\,(B\cos\theta+k_a) = 0 \qquad \ldots \quad (5)$$

and $$A = \tau\cos\theta \quad \text{or} \quad \gamma\sin\theta = 2\tau\cos\theta \qquad \ldots \quad (6)$$

Hence we get the well-known result that the principal surfaces of a normal congruence correspond to the parametric curves on the unit sphere.

If $\gamma = 0$ and $\tau = 0$, then (5) and (6) are simultaneously satisfied. Hence we get

THEOREM 16. The principal surfaces of a rectilinear congruence cutting the director surface at a constant angle will correspond to the parametric curves on the unit sphere if the guiding curves be both geodesics and lines of curvature.

Now $$ag+ce-f(b+b') = Hh(B\cos\theta-k_a).$$

Therefore, the director surface will be the middle surface of the congruence if

$$B\cos\theta-k_a = 0,$$

i.e. $$k_a\tan^2\theta-\gamma'\tan\theta+J = 0 \qquad \text{(7)}$$

Hence we immediately get the well-known result that the director surface of a normal congruence coincides with the middle surface when and only when it is minimal.

As θ is constant, the roots of the equation (7) should be constants and hence the sum of the roots $= \gamma'/k_a =$ const. and the product of the roots $= J/k_a =$ const. gives $k_b/k_a =$ const. Hence the ratio of these ratios γ'/k_b will also be a constant. Clearly, these conditions are necessary but not sufficient. Hence we get

THEOREM 17. If the director surface of a rectilinear congruence cutting it at a constant angle coincides with the middle surface, the ratio of the curvature of the guiding curve to the geodesic curvature of its orthogonal trajectory and the ratio of their curvatures at each point will be constants.

THEOREM 18. If the director surface of a rectilinear congruence cutting it at a constant angle coincides with the middle surface, the ratio of the curvature and the geodesic curvature of the orthogonal trajectory of the guiding curve at each point is constant.

The limits are the roots of the equation

$$h^2r^2+Hh(B\cos\theta-k_a)r-H^2\left[k_aB\cos\theta+\tfrac{1}{4}(A-\tau\cos\theta)^2\right] = 0 \qquad \text{(8)}$$

The focal distances are the roots of the equation

$$h\rho^2+(B\cos\theta-k_a)H\rho+H\cos\theta = 0 \qquad \text{(9)}$$

The parameter of distribution β is given by

$$\beta(e\,du^2+2f\,du\,dv+g\,dv^2) = AE\cos\theta\,du^2+\sqrt{EG}(B\cos\theta+k_a)\,du\,dv+G\tau\,dv^2.$$

Therefore, the developables of the congruence are given by

$$AE\cos\theta\,du^2+\sqrt{EG}(B\cos\theta+k_a)\,du\,dv+G\tau\,dv^2 = 0.$$

The principal parameters are the roots of the equation

$$h^2\beta^2-Hh(A-\tau\cos\theta)\beta+H^2\left[A\tau\cos\theta-\tfrac{1}{4}(B\cos\theta+k_a)^2\right] = 0$$

Therefore, the product of the principal parameters

$$= (H^2/h^2)\left[A\tau\cos\theta-\tfrac{1}{4}(B\cos\theta+k_a)^2\right].$$

Also, the square of the focal distance

$$= \text{square of the difference of the roots of (9)}$$

$$= (H^2/h^2)(B\cos\theta-k_a)^2-4(H/h)\cos\theta$$

$$= -4(H^2/h^2)\left[A\tau\cos\theta-(B\cos\theta+k_a)^2\right].$$

Hence we get the result that four times the product of the principal parameters is equal to the negative of the square of the focal distance, which is known to be true for all rectilinear congruences.

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