AN EXACT ANALYTIC SOLUTION OF EQUATIONS FOR AN EXPLOSION WITH SPHERICAL SYMMETRY

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1. Introduction

A spherical wave of explosion expanding outwards, and with a shock surface as wave front has been studied by G. I. Taylor (1950), G. C. McVittie (1953) and J. L. Taylor (1955). These spherically symmetric solutions have been found to be appropriate for strong shocks only. Z. Kopal and others (1951) have worked similar solutions for explosions in stellar bodies. While most of the previous attempts have succeeded in reducing the non-linear partial differential equations to a set of ordinary differential equations solvable by numerical methods, J. L. Taylor succeeded in getting an analytic solution of the partial differential equations applicable for strong shocks. In the present paper it is shown that it is possible, proceeding on the line of J. L. Taylor, to obtain a set of spherically symmetric solutions of the problem which are applicable both to strong and weak shocks in terms of analytic functions. But the external medium in which such waves may expand should have a definite pressure law, namely that the pressure should fall as inverse cube of the distance from the centre of explosion. The solution is not directly applicable to terrestrial conditions but gas configurations of appropriate condition may be envisaged.

In the present solution the energy of the expanding wave has been supposed to remain constant.

2. Equations of the Problem and Boundary Conditions

As equations governing the flow behind a spherical shock we write the equation of motion and equation of continuity

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0, \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\rho\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right)=0 \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

and further assuming adiabacy

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+u\frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right)=0, \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

where u, p and ρ are velocity, pressure and density of the gas at a radial distance r from the centre at time t; γ is the ratio of the specific heats. No gravitational force is considered.

This motion will be supposed to be bounded on the outside by a shock surface at $r = R(t)$, which will move outward with velocity

$$V = \frac{dR}{dt}.$$

If ahead of the shock the undisturbed pressure and density be p_0, ρ_0 and those just behind the shock p_1, ρ_1, then we have the following Rankine-Hugoniot conditions at the shock surface:

$$\frac{\rho_1}{\rho_0} = \frac{(\gamma-1)+(\gamma+1)y}{(\gamma+1)+(\gamma-1)y}, \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\frac{V-u_1}{V} = \frac{\rho_0}{\rho_1}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

and

$$\rho V u_1 = p_1 - p_0 = (y-1)p_0, \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where

$$y = \frac{p_1}{p_0}.$$

From (4), (5) and (6) we get

$$u_1 = \frac{2V}{\gamma+1} - \frac{2\gamma p_0}{(\gamma+1)\rho_0 V}, \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

$$p_1 = \frac{2\rho_0 V^2 - (\gamma-1)p_0}{\gamma+1}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\rho_1 = \frac{(\gamma+1)\rho_0^2 V^2}{(\gamma-1)\rho_0 V^2 + 2\gamma p_0}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Equation (5) shows the well-known property that the mass particles behind the shock wave move with a velocity less than that of the shock front. Hence the mass confined within the shock front at any time t must be equal to that contained within the surface of radius R in the undisturbed state, where R is the radius of the shock wave at time t.

If we denote this mass by m we have

$$m = 4\pi \int_0^R \rho_0 r^2 \, dr = \frac{4\pi \rho_c R^{\alpha+3}}{\alpha+3},$$

where we have taken

$$\rho_0 = \rho_c r^{\alpha}, \ \rho_c \text{ being a constant.} \qquad \ldots \quad \ldots \quad \ldots \quad (10)$$

We should take $\alpha+3 > 0$, so that the limits of α are $-3 < \alpha \leqslant 0$.

Next let us seek solution of equations (1), (2) and (3) in the form

$$u = \frac{r}{t} U(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$p = r^{K+2} t^{\lambda-2} P(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\rho = r^{K} t^{\lambda} \Omega(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where

$$\eta = r^a t^b \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

The constants K, λ, a, and b are for the present kept open and are to be determined from the conditions of the problem.

The total energy Ψ of the disturbance consists of two parts, namely, the kinetic energy

$$\text{K.E.} = 4\pi \int_0^R \tfrac{1}{2}\rho u^2 r^2\, dr,$$

and the heat energy

$$\text{H.E.} = 4\pi \int_0^R \frac{p}{\gamma-1} \cdot r^2\, dr.$$

Hence the total energy

$$\Psi = 4\pi \int_0^R \left[\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right] r^2\, dr.$$

In terms of the variable η we get

$$\Psi = \frac{4\pi}{a}\int_{\infty}^{\eta_0}\left[\frac{1}{2}U^2(\eta)\Omega(\eta)\eta^{\frac{K+5}{a}-1} t^{\lambda-2-\frac{b}{a}(K+5)} + \frac{1}{\gamma-1} \cdot P(\eta)\eta^{\frac{K+3}{a}} t^{\lambda-2-\frac{b}{a}(K+5)}\right] d\eta, \quad \ldots \quad (15)$$

where η_0 is the value of η at the shock front.

We choose the shock surface to be given by $\eta_0 =$ constant. This choice fixes the velocity of the shock surface as

$$V = -\frac{b}{a} \cdot \frac{R}{t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

and exhausts all cases of spherical shock surface moving according to a law $R \propto t^{\kappa}$.

We now introduce the condition that the total energy of disturbance within the shock surface at any time t is constant. This by (15) requires

$$\lambda - 2 - \frac{b}{a}(K+5) = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Let us assume the undisturbed pressure to be given by

$$p_0 = p_c r^{\beta}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where p_c and β are constants.

Let the Mach number at the shock front be defined by

$$M^2 = \frac{V^2}{C_0^2} = \frac{b^2}{a^2} \cdot \frac{\rho_c}{\gamma p_c} \cdot \frac{R^{\alpha+2-\beta}}{t^2}. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Substituting (11), (12) and (13) in the Rankine-Hugoniot conditions (7), (8) and (9) and solving for U, P and Ω immediately behind the shock, we get

$$U(\eta_0) = -\frac{2b}{a(\gamma+1)}\left[1 - \frac{1}{M^2}\right], \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$P(\eta_0) = \frac{p_c R^{\beta-K-2} t^{-\lambda+2}}{\gamma+1} \cdot [2\gamma M^2 - (\gamma-1)], \quad \ldots \quad \ldots \quad (21)$$

and

$$\Omega(\eta_0) = \frac{(\gamma+1)\rho_c M^2}{R^{K-\alpha} t^{\lambda}[(\gamma-1)M^2+2]}. \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

The right-hand sides of equations (20), (21) and (22) are functions of η_0 alone if M, $R^{\beta-K-2}t^{-\lambda+2}$ and $R^{K-\alpha}t^{\lambda}$ depend on η_0 only.

From (19) we find that M depends on η_0, if

$$\frac{\alpha+2-\beta}{-2} = \frac{a}{b}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

and then from (21) and (22) we get

$$\frac{\beta-K-2}{-\lambda+2} = \frac{a}{b} = \frac{K-\alpha}{\lambda}. \qquad \ldots \quad \ldots \quad \ldots \quad (24)$$

From (17), (23) and (24) we have

$$\beta = -3, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

also

$$\frac{a}{b} = \frac{-(\alpha+5)}{2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

Without any loss of generality we may choose one of the four constants arbitrarily, let us put $K = \alpha$. Then from (24) we get $\lambda = 0$. Such a choice of K and λ is consistent with equations (17) and (26).

We have in fact

$$K = \alpha\,;\ \lambda = 0\,;\ a = -(\alpha+5)\,;\ b = 2. \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

Thus from equations (20), (21) and (22) and (11), (12), (13) and (14) we obtain

$$u_1 = \frac{4\eta_0^{-\frac{1}{\alpha+5}}\, t^{\frac{2}{\alpha+5}-1}}{(\alpha+5)\,(\gamma+1)} \cdot \left[1 - \frac{1}{M^2}\right], \qquad \ldots \quad \ldots \quad \ldots \quad (28)$$

$$p_1 = \frac{p_c \eta_0^{-\frac{\beta}{\alpha+5}}\, t^{\frac{2\beta}{\alpha+5}}}{\gamma+1} \cdot [2\gamma M^2 - (\gamma-1)], \qquad \ldots \quad \ldots \quad (29)$$

$$\rho_1 = \frac{(\gamma+1)\rho_c M^2 \eta^{-\frac{\alpha}{\alpha+5}}\, t^{\frac{2\alpha}{\alpha+5}}}{(\gamma-1)M^2+2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

in which

$$M^2 = \frac{4\rho_c}{(\alpha+5)^2 \gamma p_c} \cdot \frac{1}{\eta_0} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

is a constant. We may now calculate the velocity of the shock

$$V = C_0 M = \sqrt{\left\{\frac{4}{(\alpha+5)^2} \cdot \frac{p_0}{p_c} \cdot \frac{\rho_c}{\rho_0} \cdot \frac{1}{\eta_0}\right\}}$$

$$= \frac{2}{(\alpha+5)\sqrt{\eta_0}} \cdot R^{-\frac{\alpha+3}{2}}$$

3. Solution of Equations (1), (2) and of (3)

The condition inside the wave will be obtained from the solution of the equations (1), (2) and (3).

From equation (13) we get by differentiation

$$\frac{\partial\rho}{\partial t} = \frac{\lambda\rho}{t} + \frac{b}{t} \cdot r^K t^\lambda \eta \Omega'(\eta) \quad \ldots \quad (32)$$

and

$$\frac{\partial\rho}{\partial r} = \frac{K\rho}{r} + \frac{a}{r} \cdot r^K t^\lambda \eta \Omega'(\eta). \quad \ldots \quad (33)$$

Eliminating $\Omega'(\eta)$ and t from (32) and (33) by using (16) we get

$$\frac{\partial\rho}{\partial t} = -V \cdot \frac{r}{R} \cdot \frac{\partial\rho}{\partial r} + \alpha\rho \cdot \frac{V}{R}. \quad \ldots \quad (34)$$

Treating equations (12) and (16) exactly in the same way we get

$$\frac{\partial p}{\partial t} = -V \cdot \frac{r}{R} \cdot \frac{\partial p}{\partial r} - 3p \cdot \frac{V}{R}. \quad \ldots \quad (35)$$

From equations (1), (2) and (3) it can be shown that

$$\frac{\partial E}{\partial t} + \frac{1}{r^2} \cdot \frac{\partial}{\partial r}(r^2 uI) = 0, \quad \ldots \quad (36)$$

where

$$E = \tfrac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} \quad \ldots \quad (37)$$

and

$$I = \tfrac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1}. \quad \ldots \quad (38)$$

Now

$$E = t^{-\frac{6}{\alpha+5}} \Phi(\eta). \quad \ldots \quad (39)$$

From equation (39) we get

$$\frac{\partial E}{\partial t} = -\frac{6}{\alpha+5} \cdot \frac{E}{t} - \frac{2}{\alpha+5} \cdot \frac{r}{t} \cdot \frac{\partial E}{\partial r}. \quad \ldots \quad (40)$$

Equations (36) and (40) give

$$\frac{\partial}{\partial r}(r^2 uI) = \frac{\partial}{\partial r}\left[\frac{2}{\alpha+5} \cdot \frac{E}{t} \cdot r^3\right].$$

From this by integration one has

$$r^2 uI - \frac{2}{\alpha+5} \cdot \frac{E}{t} \cdot r^3 = F(t).$$

We choose $F(t)$ to be equal to zero, and obtain

$$\frac{E}{I} = \frac{\alpha+5}{2} \cdot \frac{ut}{r}.$$

Combining this last equation with equations (11) and (26) we get

$$\frac{u}{V}=\frac{E}{I}\cdot\frac{r}{R}. \qquad \cdots \qquad (41)$$

Let

$$\frac{u}{V}=u', \text{ and } \frac{r}{R}=r',$$

then from equation (41), we have

$$u' = r'.\frac{\frac{1}{2}u^2+\frac{p}{(\gamma-1)\rho}}{\frac{1}{2}u^2+\frac{\gamma p}{(\gamma-1)\rho}};$$

solving for $\frac{p}{\rho}$ we get

$$\frac{p}{\rho}=\frac{\gamma-1}{2}.V^2.u'^2.\frac{r'-u'}{\gamma u'-r'},$$

$$=C.u'^2.\frac{r'-u'}{\gamma u'-r'}, \qquad \cdots \qquad (42)$$

where we have put $C=\frac{\gamma-1}{2}.V^2$, which, however, is not constant.

Combining (2) and (3) we have

$$\frac{1}{p}\cdot\frac{\partial p}{\partial r}-\frac{\gamma-1}{\rho}\cdot\frac{\partial\rho}{\partial r}=-\frac{1}{pu}\cdot\frac{\partial p}{\partial t}+\frac{\gamma-1}{u\rho}\cdot\frac{\partial\rho}{\partial t}-\frac{1}{u}\cdot\frac{\partial u}{\partial r}-\frac{2}{r}. \qquad \cdots \qquad (43)$$

Replacing $\frac{\partial p}{\partial t}$ and $\frac{\partial\rho}{\partial t}$ on the right-hand side by means of equations (34) and (35) we get

$$\frac{1}{p}\cdot\frac{\partial p}{\partial r}-\frac{\gamma-1}{\rho}\cdot\frac{\partial\rho}{\partial r}=-\frac{2}{r}-\frac{\frac{\alpha(\gamma-1)+1}{R}-\frac{1}{V}\cdot\frac{\partial u}{\partial r}}{\frac{r}{R}-\frac{u}{V}}. \qquad \cdots \qquad (44)$$

We integrate (44) with respect to r and get

$$\frac{p}{\rho^{\gamma-1}}=C'r'^{-2}(r'-u')^{-1}[f(r')]^{-1}, \qquad \cdots \qquad (45)$$

where C' is a function of time, and

$$\log f(r')=\alpha(\gamma-1)\int_1^{r'}\frac{dr'}{r'-u'}.$$

Solving for ρ and p from (42) and (45) we get

$$\frac{\rho}{\rho_1}=C_1(r'-u')^{-\frac{2}{2-\gamma}}(u'r')^{-\frac{2}{2-\gamma}}(\gamma u'-r')^{\frac{1}{2-\gamma}}[f(r')]^{-\frac{1}{2-\gamma}} \qquad \cdots \qquad (46)$$

and

$$\frac{p}{p_1}=C_2(r'-u')^{-\frac{\gamma}{2-\gamma}}(\gamma u'-r')^{\frac{\gamma-1}{2-\gamma}}r'^{-\frac{2}{2-\gamma}}u'^{\frac{2-2\gamma}{2-\gamma}}[f(r')]^{-\frac{1}{2-\gamma}} \qquad \cdots \qquad (47)$$

where C_1 and C_2 are determined from the value of u_1 as given by equation (28). We have in fact

$$\left.\begin{aligned} C_1 &= (1-u_1')^{\frac{2}{2-\gamma}} (u_1')^{\frac{2}{2-\gamma}} (\gamma u_1'-1)^{-\frac{1}{2-\gamma}} \\ C_2 &= (1-u_1')^{\frac{\gamma}{2-\gamma}} (\gamma u_1'-1)^{-\frac{\gamma-1}{2-\gamma}} u_1'^{-\frac{2-2\gamma}{2-\gamma}}, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (48)$$

and

where $u_1' = \dfrac{u_1}{V}$.

From equations (2) and (34) we get

$$\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial r'} = \frac{1}{r'-u'}\cdot\frac{\partial u'}{\partial r'} + \frac{2}{r'-u'}\cdot\frac{u'}{r'} + \frac{\alpha}{r'-u'}.$$

Using the value for ρ from equation (46) in this equation we get

$$-\frac{2}{2-\gamma}\cdot\frac{1}{r'-u'}\cdot\left[1-\frac{\partial u'}{\partial r'}\right] - \frac{2}{2-\gamma}\cdot\left[\frac{1}{u'}\cdot\frac{\partial u'}{\partial r'}+\frac{1}{r'}\right]$$
$$+\frac{1}{2-\gamma}\cdot\frac{1}{\gamma u'-r'}\cdot\left[\gamma\frac{\partial u'}{\partial r'}-1\right]$$
$$= \frac{1}{r'-u'}\cdot\frac{\partial u'}{\partial r'} - \frac{2}{r'} + \frac{2}{r'-u'} + \frac{\alpha}{2-\gamma}\cdot\frac{1}{r'-u'} \quad \ldots \quad \ldots \quad (49)$$

The solution of equation (49) can be written as

$$2\log u' = \chi_1 \log r' + \chi_2 \log\left(r' - \frac{3\gamma-1}{\alpha+5}\cdot u'\right) + \chi_3 \log(\gamma u'-r')$$
$$+ 2\log u_1' - \chi_2 \log\left(1-\frac{3\gamma-1}{\alpha+5}\cdot u_1'\right) - \chi_3 \log\left(\gamma u_1' - 1\right), \quad \ldots \quad (50)$$

where

$$\chi_1 = \frac{-2(\alpha+5)(\gamma-1)}{3\gamma-1}$$
$$\chi_2 = \frac{\gamma^2(-4\alpha-13)+\gamma(-6\alpha-2\alpha^2+7)-2\alpha-12}{(3\gamma-1)(2\gamma+\alpha\gamma+1)}$$
$$\chi_3 = \frac{(\alpha+5)(\gamma-1)}{2\gamma+\alpha\gamma+1}.$$

Equations (46), (47) and (50) give the solution of our problem. They constitute a rigorous solution of the non-linear equations (1), (2) and (3) in which the shock front moves characterized by the constant Mach number M. The value of this Mach number is unrestricted and may be large or small. But the unsatisfactory feature of our solution is that it is only applicable to the case in which the pressure in the undisturbed region varies as the inverse cube of the distance from the centre of explosion; α being undetermined the density law is open but there is a range for the value of α, namely $-3 < \alpha \leqslant 0$.

4. Numerical Integration of Equations

Integration has been effected in a particular case to get a numerical solution. The undisturbed gas has been taken for purpose of integration to be a mass which can be in equilibrium under gravitational force, though we have not considered gravitation for propagation of the wave. Substituting in the equation for equilibrium

$$\frac{1}{r^2}\cdot\frac{d}{dr}\left(\frac{r^2}{\rho}\cdot\frac{dp}{dr}\right) = -4\pi G\rho,$$

the law of variation of p (inverse third power of r), one finds that ρ should vary as inverse $\frac{5}{2}$ power $\left(\alpha = -\frac{5}{2}\right)$, so that the undisturbed gas mass behaves as a polytrope 5. Integration has been performed for a strong shock taking $M^2 = 100$, and $\gamma = \frac{5}{3}$. The variations of the velocity, pressure and density have been given in Table 1.

TABLE 1

r'	1	0·9	0·8	0·7	0·6	0·5	0·4
u'	0·7425	0·6788	0·6149	0·5507	0·486	0·4208	0·3548
$\frac{p}{p_1}$	1	0·6735	0·3955	0·1861	0·0588	0·0078	....
$\frac{\rho}{\rho_1}$	1	0·9121	0·7586	0·5352	0·2743	0·0670	....

5. CASE OF CYLINDRICAL SYMMETRY

The problem of explosion along a line in a gas cloud may be treated in the same manner. As in the case of an explosion with spherical symmetry the gas cloud has for this solution a definite pressure law, and it is found that in the case of a cylindrical shock wave in a gas medium the pressure in the undisturbed medium should fall off as the inverse square of the distance from the line of explosion. As in the case of spherical shock wave the density in the undisturbed medium is of the form (10) but with restriction on the value of α, namely $-2 < \alpha \leqslant 0$.

In the case of cylindrical shock wave, the equation corresponding to (2) is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\rho\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right) = 0 \quad .. \quad .. \quad .. \quad (2a)$$

the corresponding total energy within the shock wave per unit length is

$$2\pi\int_0^R\left[\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}\right]r\,dr = \frac{2\pi}{b}\int_\infty^{\eta_0}\left[\frac{1}{2}U^2\Omega\eta^{\frac{K+4}{b}-1}t^{\lambda-2-\frac{b}{a}(K+4)}+\right.$$
$$\left.+\frac{1}{\gamma-1}\cdot P\eta^{\frac{K+2}{b}}t^{\lambda-2-\frac{b}{a}(K+4)}\right]d\eta,$$

the equation corresponding to (17) is

$$\lambda-2-\frac{b}{a}(K+4) = 0. \quad .. \quad .. \quad .. \quad .. \quad (17a)$$

The equations corresponding to (25) and (26) are

$$\beta = -2 \quad \ldots \quad (25a)$$

and

$$\frac{a}{b} = \frac{-(\alpha+4)}{2} \quad \ldots \quad (26a)$$

Corresponding to equation (27) we have

$$K = \alpha\,;\ \lambda = 0\,;\ \beta = -2\,;\ a = -(\alpha+4)\,;\ b = 2. \quad \ldots \quad (27a)$$

The equations corresponding to (36) and (39) are

$$\frac{\partial E}{\partial t} + \frac{1}{r}\cdot\frac{\partial}{\partial r}(ruI) = 0, \quad \ldots \quad (36a)$$

and

$$E = t^{-\frac{4}{\alpha+4}}\Phi(\eta) \quad \ldots \quad (39a)$$

whereas the equation corresponding to (44) is

$$\frac{1}{p}\cdot\frac{\partial p}{\partial r} - \frac{\gamma-1}{\rho}\cdot\frac{\partial \rho}{\partial r} = -\frac{1}{r} - \frac{\dfrac{\alpha(\gamma-1)+1}{R} - \dfrac{1}{V}\cdot\dfrac{\partial u}{\partial r}}{\dfrac{r}{R} - \dfrac{u}{V}}. \quad \ldots \quad (44a)$$

Finally we get

$$\frac{\rho}{\rho_1} = C_1(r'-u')^{-\frac{2}{2-\gamma}}\,u'^{-\frac{2}{2-\gamma}}\,r'^{-\frac{1}{2-\gamma}}\,(\gamma u'-r')^{\frac{1}{2-\gamma}}\,[f(r')]^{-\frac{1}{2-\gamma}}. \quad \ldots \quad (46a)$$

$$\frac{p}{p_1} = C_2(r'-u')^{-\frac{\gamma}{2-\gamma}}\,u'^{\frac{2-2\gamma}{2-\gamma}}\,r'^{-\frac{1}{2-\gamma}}\,(\gamma u'-r')^{\frac{\gamma-1}{2-\gamma}}\,[f(r')]^{-\frac{1}{2-\gamma}} \quad \ldots \quad (47a)$$

where

$$\log f(r') = \alpha(\gamma-1)\int_1^{r'}\frac{dr'}{r'-u'}.$$

and

$$\left.\begin{aligned} C_1 &= (1-u_1')^{\frac{2}{2-\gamma}}\,u_1'^{\frac{2}{2-\gamma}}\,(\gamma u_1'-1)^{-\frac{1}{2-\gamma}} \\ C_2 &= (1-u_1')^{\frac{\gamma}{2-\gamma}}\,u_1'^{-\frac{2-2\gamma}{2-\gamma}}\,(\gamma u_1'-1)^{-\frac{\gamma-1}{2-\gamma}} \end{aligned}\right\} \quad \ldots \quad (48a)$$

where $u_1' = \dfrac{u_1}{V}$.

and

$$2\log u' = \chi_1 \log r' + \chi_2 \log\left(r' - \frac{2\gamma-1}{\alpha+4}\,.\,u'\right) + \chi_3 \log(\gamma u'-r')$$
$$+ 2\log u_1 - \chi_2 \log\left(1 - \frac{2\gamma-1}{\alpha+4}\,.\,u_1'\right) - \chi_3\log(\gamma u_1'-1), \quad \ldots \quad (50a)$$

where

$$\chi_1 = \frac{-(\alpha+4)(\gamma-1)}{2\gamma-1}$$

$$\chi_2 = \frac{\gamma^2(-\alpha^2-4\alpha-8)+8\gamma-\alpha-6}{(2\gamma-1)(\alpha\gamma+2\gamma+1)}.$$

$$\chi_3 = \frac{(\alpha+4)(\gamma-1)}{2\gamma+\alpha\gamma+1}.$$

Equations (46*a*), (47*a*) and (50*a*) give the solution of the problem of a line explosion with cylindrical symmetry.

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Abstract

A case of explosion with spherical symmetry has been worked out by solving directly the partial differential equations, the solution being applicable both to weak and strong expanding shock surface, the Mach number of which remains constant. Constancy of the total energy of explosion has been assumed. The solution, however, is only applicable to a gaseous medium where the undisturbed pressure falls as the inverse cube of the distance from the centre.

Analogous treatment of a cylindrical wave is also shown to be possible.

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