THE INTERNAL BALLISTICS OF A TAPERED-BORE GUN

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1. Introduction

The Internal Ballistics of a Tapered-Bore Gun were first discussed by Gehrilich (1929), Schmitz (1935) and Justrow (1936). Recently Corner (1950) has discussed the modifications in the equations of Internal Ballistics due to the variation of the bore cross-section area. By simple substitutions in the equations of Internal Ballistics for a tapered-bore gun for an Isothermal Model, he is able to eliminate the varying cross-section area from four of the five equations and to show that these equations remain of the same form. From this he deduces that the solution of the equations of Internal Ballistics for a gun for which the bore cross-section area varies in any manner corresponds, point by point, with the solution for the orthodox problem differing only by having a constant bore area, and further that the velocities, fractions burnt and pressures are equal when the volumes behind the projectile are equal. He concludes that the tapered-bore gun introduces nothing new into the central problem of Internal Ballistics except, of course, many interesting theoretical problems connected with the rapid deformation of the projectile, as well as the almost unexplored hydrodynamical problem of Lagrange's correction and of the strong bore resistance—problems which are not well understood even for the orthodox gun.

We find, however, that since the varying cross-section area cannot be altogether eliminated from the equations of Internal Ballistics by any transformation, the conclusions drawn by Corner need thorough reconsideration. To deduce the solution of the tapered-bore gun from that for the orthodox gun, the law of burning for the tapered-bore gun would have to be modified from point to point and the velocities, fractions burnt and pressures for the two cases would not be equal, but would just correspond when the volumes behind the projectiles are equal. Moreover, the modification in the law of burning from point to point would imply that the analytic solutions of the equations of Internal Ballistics for the orthodox gun cannot be used for the tapered-bore gun. Accordingly, it is necessary to search for new solutions.

In the present paper, two methods of solving the equations for the tapered-bore gun are proposed. The methods are of general application, as they have been developed for the non-isothermal model and for the pressure-index law $[B(p) = \beta p^{\alpha}]$ or the linear law $[B(p) = \beta(p+p_1)]$. As usual, however, we neglect the covolume-correction terms.

The case of an orthodox gun follows as a particular case. In this case our methods yield two new methods for solving the equations of Internal Ballistics for each of the two laws of burning for an orthodox gun.

In the still more particular case when the pressure-index α is unity or when p_1 is zero, i.e. when the rate of burning is proportional to pressure, we get two new methods of solving the equations of Internal Ballistics for an orthodox gun. However, the new methods are not very much simpler than the many standard methods available for this case.

In the most general case, our solutions are expressed in terms of the solutions of ordinary non-linear differential equations of the second order. The integration of these equations requires elaborate calculations and is likely to be a time-consuming process. Therefore, in order to study the effect of variation of bore cross-section area on Internal Ballistics, we study here an Isothermal Model for the specific case of a constant burning surface for three different tapered-bore guns including the conical-bore gun. The level of accuracy is the same as that of Crow's method of Internal Ballistics for the orthodox gun or of the Corner's theory (1948, 1950) for the High-Low pressure gun. The equations of this model can be integrated in finite terms, or for guns with moderate tapering, as series solutions and in each case the results can be usefully compared with the results for the corresponding orthodox gun.

In particular we have made the following three comparisons:—

(a) We compare the position, pressure and velocity at all-burnt for two guns—one of constant bore area and the other tapered but with same charge, chamber capacity, and same initial area of cross-section of the bore.

(b) In this case the tapered-bore gun has a larger initial area of cross-section and a smaller bore length and the emergent calibres are equal.

(c) In the third case we modify the web-size or the central ballistic parameter so that the maximum pressure remains the same and then compare the muzzle velocities for the orthodox and the tapered-bore gun.

We also extend the theory developed for a single charge to the case of the use of composite charges in a tapered-bore gun.

2. Basic Equations of Isothermal Model

The basic equations of Internal Ballistics for the Isothermal Model, for constant cross-section area of the bore and for linear law of burning are

$$\left.\begin{aligned} w_1 v\frac{dv}{dx} &= \frac{A_0 p}{1+\dfrac{C}{2w}} && \quad (1)\\ p\left[K_0+Ax-\frac{C}{\delta}\right] &= C\lambda z\left(1+\frac{C}{6w}\right) && \quad (2)\\ z &= (1-f)(1+\theta f) && \quad (3)\\ D\frac{df}{dt} &= -\beta p && \quad (4)\\ \frac{dx}{dt} &= v && \quad (5)\end{aligned}\right\}\quad [A]$$

Making substitutions,

$$X = K_0+A_0x-\frac{C}{\delta} \qquad (6)$$

$$T = A_0 t \qquad (7)$$

these become

$$w_1 v \frac{dv}{dX} = \frac{p}{1+\frac{C}{2w}} \quad \ldots \quad (8)$$

$$p X = C\lambda z \left(1+\frac{C}{2w}\right) \quad \ldots \quad (9)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (10) \qquad [B]$$

$$D \frac{df}{dT} = -\frac{\beta p}{A_0} \quad \ldots \quad (11)$$

$$\frac{dX}{dT} = v \quad \ldots \quad (12)$$

For varying cross-section area (1) and (2) are modified to

$$w_1 v \frac{dv}{dx} = \frac{A(x)\, p}{1+\frac{C}{2w}} \quad \ldots \quad (13)$$

$$p\left[K_0 + \int A\, dx - \frac{C}{\delta}\right] = Cz\lambda\left(1+\frac{C}{6w}\right). \quad \ldots \quad (14)$$

Making the substitutions

$$K_0 + \int A\, dx - \frac{C}{\delta} = L(x) \quad \ldots \quad (15)$$

$$\tau = \int A\, dt \quad \ldots \quad (16)$$

the equations for the varying cross-section area are:

$$w_1 v \frac{dv}{dL} = \frac{p}{1+\frac{C}{2w}} \quad \ldots \quad (17)$$

$$pL = Cz\lambda\left(1+\frac{C}{6w}\right) \quad \ldots \quad (18)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (19) \qquad [C]$$

$$D \frac{df}{d\tau} = -\frac{\beta p}{A} \quad \ldots \quad (20)$$

$$\frac{dL}{d\tau} = v \quad \ldots \quad (21)$$

Comparing systems (B) and (C), we find that in the two sets, four of the five equations are the same, but (11) and (20) differ, since in (11), A_0 is constant and in (20), A is varying with shot-travel. If we leave (20) out of account, then from (17), (18),

(19) and (21), we cannot solve even theoretically for p, v, z, f as functions of L, and thus Corner's conclusion needs modification.

The same result is seen more effectively by writing the set (C) as

$$w_1 v \frac{dv}{dL} = \frac{p}{1+\dfrac{C}{2w}}$$

$$pL = C\lambda(1-f)(1+\theta f)\left(1+\frac{C}{6w}\right)$$

$$D\frac{df}{dL} = -\frac{\beta p}{Av}$$

From these three equations, we can solve for v, p, f as functions of L, *but not independently of* A; and thus velocity, pressure and fraction burnt are not equal for the tapered-bore and the orthodox gun when the volumes behind the shot are equal in the two cases.

If we do not make the transformation for time, the two sets of equations are:

$$\left.\begin{aligned}
w_1 v \frac{dv}{dx} &= \frac{p}{1+\dfrac{C}{2w}} = \frac{w_1}{A_0}\frac{dv}{dt} = \frac{w_1}{A_0}\frac{d^2x}{dt^2} && \quad (22)\\
pX &= Cz\lambda\left(1+\frac{C}{6w}\right) && \quad (23)\\
z &= (1-f)(1+\theta f) && \quad (24)\\
D\frac{df}{dt} &= -\beta p && \quad (25)\\
\frac{dX}{dt} &= A_0 v && \quad (26)
\end{aligned}\right\} \quad [D]$$

and

$$\left.\begin{aligned}
w_1 v \frac{dv}{dL} &= \frac{p}{1+\dfrac{C}{2w}} = \frac{w_1}{A}\frac{dv}{dt} = \frac{w_1}{A}\frac{d}{dt}\left(\frac{1}{A}\frac{dL}{dt}\right) && \quad (27)\\
pL &= Cz\lambda\left(1+\frac{C}{6w}\right) && \quad (28)\\
z &= (1-f)(1+\theta f) && \quad (29)\\
D\frac{df}{dt} &= -\beta p && \quad (30)\\
\frac{dL}{dt} &= Av && \quad (31)
\end{aligned}\right\} \quad [E]$$

Again we note that (26) and (31) are different. If we leave these two equations out of account, we get in each case only 4 equations to connect 6 quantities, viz. v, x, p, z, f, t or v, L, p, z, f, t and so the relation between v and x is not the same as that between v and L, a conclusion implied in Corner's statement.

3. Solution of the Equations of Internal Ballistics for a Tapered-Bore Gun for the Non-Isothermal Model

Let V be the volume of the air-space behind the projectile at any instant, then the equations of Internal Ballistics are:

$$FCz = pV+\tfrac{1}{2}(\gamma-1)w_1v^2 \qquad (32)$$

$$w_1\frac{dv}{dt} = w_1v\frac{dv}{dx} = pA(x) = p\frac{dV}{dx} \qquad (33)$$

$$D\frac{df}{dt} = -\beta p^{\alpha} \qquad (34)$$

$$z = (1-f)(1+\theta f) \qquad (35)$$

Let V_0 be the initial air-space at shot-start and let A_0 be the corresponding area of cross-section. Also let

$$V_0 = A_0 l. \qquad (36)$$

To reduce equations (32)–(35) in terms of non-dimensional variables and constants, we make the substitutions:

$$\xi = \frac{V}{V_0} \qquad (37)$$

$$\zeta = \frac{A_0 l}{FC}p = \frac{V_0}{FC}p \qquad (38)$$

$$\eta = \frac{A_0 D}{FC\beta}\left(\frac{FC}{A_0 l}\right)^{1-\alpha} v \qquad (39)$$

$$M = \frac{A_0^2 D^2}{FC\beta^2 w_1}\left(\frac{FC}{A_0 l}\right)^{2-2\alpha} \qquad (40)$$

The transformed equations are:

$$z = \zeta\xi+\tfrac{1}{2}(\gamma-1)\frac{\eta^2}{M} \qquad (41)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \qquad (42)$$

$$\eta\frac{df}{d\xi} = -\frac{A_0}{A}\zeta^{\alpha} \qquad (43)$$

$$z = (1-f)(1+\theta f) \qquad (44)$$

If A is constant, these reduce to the corresponding equations of Clemmow (1951) for the orthodox gun.

From (41) and (42)

$$dz = \xi^{1-\gamma}dY \quad \text{or} \quad \frac{dY}{dz} = \xi^{\gamma-1}, \qquad (45)$$

where

$$Y = \zeta\xi^{\gamma}, \qquad (46)$$

so that

$$\xi = \left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}} \quad \ldots \quad (47)$$

and

$$\frac{d\xi}{dz} = \frac{1}{\gamma\zeta^2}\left(\zeta - Y\frac{d\zeta}{dY}\right). \quad \ldots \quad (48)$$

From (42) and (43)

$$\frac{dz}{df}\frac{d}{dz}\left[-\frac{A_0}{A}\zeta^{\alpha}\frac{d\xi}{dz}\frac{dz}{df}\right] = -\frac{MA}{A_0}\zeta^{1-\alpha} \quad \ldots \quad (49)$$

Again from (44)

$$\frac{dz}{df} = -\nu\sqrt{1-qz}, \quad \ldots \quad (50)$$

where

$$\nu = 1+\theta \quad \ldots \quad (51a)$$

$$q = \frac{4\theta}{(1+\theta)^2} \quad \ldots \quad (51b)$$

Substituting from (45), (46), (48) and (50) in (49), we get

$$\nu\sqrt{1-qz}\frac{d}{dz}\left[\frac{A_0}{A}\zeta^{\alpha}\frac{\zeta - Y\frac{d\zeta}{dY}}{\gamma\zeta^2}\nu\sqrt{1-qz}\right] = \frac{MA}{A_0}\zeta^{1-\alpha}$$

or

$$(1-qz)\left(\frac{Y}{\xi}\right)^{\gamma-1}\frac{d}{dY}\left[\frac{1}{A}\zeta^{\alpha-2}(\zeta - Y\zeta')\right] - \tfrac{1}{2}\frac{q}{A}\zeta^{\alpha-2}(\zeta - Y\zeta') = \frac{MA\gamma}{A_0^2\nu^2}\zeta^{1-\alpha}, \quad \ldots \quad (52)$$

where dashes denote differentiation with respect to Y

To find z we eliminate η from (41) and (43)

$$z = \zeta\xi + \tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}\zeta^{2\alpha}\left(\frac{d\xi}{dz}\right)^2\left(\frac{dz}{df}\right)^2$$

$$= \zeta\left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}} + \tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}\zeta^{2\alpha}\frac{1}{\gamma^2\zeta^4}(\zeta - Y\zeta')^2\nu^2(1-qz)$$

or

$$1-qz = \frac{1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}}{1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\zeta^{2\alpha-4}(\zeta - Y\zeta')^2} \quad \ldots \quad (53)$$

Finally substituting in (52) we get

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}\right]\left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\zeta^{\alpha-2}(\zeta - Y\zeta')\right] = \left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\zeta^{2\alpha-4}(\zeta - Y\zeta')^2\right]$$

$$\times\left[\tfrac{1}{2}q\frac{A_0}{A}\zeta^{\alpha-2}(\zeta - Y\zeta') + \frac{MA\gamma}{A_0\nu^2}\zeta^{1-\alpha}\right] \quad \ldots \quad (54)$$

This is the fundamental differential equation of this method. For a tapered-bore gun of known shape, A is a known function of x or of v or of ξ or of $\left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}}$. Substituting this function in (54), we get the non-linear differential equation with ζ as dependent and Y as independent variable, the numerical integration of which will give the solution of the system of equations of Internal Ballistics.

Particular Cases

We examine now the following cases:—

(i) *Conical bore gun:*

Here

$$\frac{A}{A_0} = \frac{\pi y^2}{\pi a^2} = \left(\frac{y}{a}\right)^2 = \left(1-\frac{x}{b}\right)^2.$$

Also

$$V = K_0+\int_0^x \pi y^2\, dx = K_0+\tfrac{1}{3}\,\frac{A_0}{b^2}\,[b^3-(b-x)^3]$$

$$\therefore \quad \xi = 1+\tfrac{1}{3}\,\frac{A_0 b}{K}\left[1-\left(1-\frac{x}{b}\right)^3\right]$$

$$= 1+\mu-\mu\left(1-\frac{x}{b}\right)^3$$

where μ is the ratio of the volume of the 'complete' cone to the volume of the chamber

$$\therefore \quad \frac{A}{A_0} = \left(\frac{1+\mu-\xi}{\mu}\right)^{\frac{2}{3}} = \left[\frac{1+\mu}{\mu}-\frac{1}{\mu}\left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}}\right]^{\frac{2}{3}} \qquad \ldots \quad \ldots \quad (55)$$

(ii) *Orthodox gun of constant bore area:*

Here $A = A_0$ and (54) gives

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}\right]\frac{d}{dY}\,[\zeta^{\alpha-2}(\zeta-Y\zeta')] = \left[1+\tfrac{1}{2}\,\frac{\gamma-1}{M}\,q\nu^2\zeta^{2\alpha-4}(\zeta-Y\zeta')^2\right]\times$$
$$\times\left[\tfrac{1}{2}q\zeta^{\alpha-2}(\zeta-Y\zeta')+\frac{M\gamma}{\nu^2}\,\zeta^{1-\alpha}\right] \qquad \ldots \quad (56)$$

(iii) *Constant burning surface area: conical bore:*

Here

$$q=0, \quad \nu=1, \quad \frac{A}{A_0} = \left[\frac{1+\mu}{\mu}-\frac{1}{\mu}\left(\frac{Y}{\xi}\right)^{\frac{1}{\gamma}}\right]^{\frac{2}{3}}$$

$$\therefore \quad \left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left[\left(\frac{1+\mu}{\mu}-\frac{1}{\mu}\left(\frac{Y}{\zeta}\right)^{1/\gamma}\right)^{-2/3}\zeta^{\alpha-2}(\zeta-Y\zeta')\right]$$

$$= M\gamma\zeta^{1-\alpha}\left[\frac{1+\mu}{\mu}-\frac{1}{\mu}\left(\frac{Y}{\zeta}\right)^{1/\gamma}\right]^{2/3} \qquad \ldots \quad \ldots \quad \ldots \quad (57)$$

(iv) *Constant burning surface area: orthodox gun:*

Here

$$q = 0, \quad \nu = 1, \quad A = A_0$$

$$\left(\frac{Y}{\zeta}\right)^{\gamma-1} \frac{d}{dY}\left[\zeta^{\alpha-2}(\zeta - Y\zeta')\right] = M\zeta^{1-\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

(v) *Linear law: constant burning surface area: isothermal model: conical bore:*

In (57) put $\gamma = 1$, $\alpha = 1$ and replace M by its modified value $\overline{M}$

$$\frac{d}{dY}\left[\left(\frac{1+\mu-\frac{Y}{\zeta}}{\mu}\right)^{-2/3}\left(1-\frac{Y\zeta'}{\zeta}\right)\right] = \overline{M}\left[\frac{1+\mu-\frac{Y}{\zeta}}{\mu}\right]^{2/3} \quad \ldots \quad (59)$$

Initial Conditions

We represent the initial conditions of band engraving by means of a shot-start pressure p_0 so that initially

$$\xi = 1, \quad \zeta = \zeta_0 = \frac{A_0 l}{FC} p_0, \quad Y = \zeta_0, \quad \ldots \quad \ldots \quad \ldots \quad (60a)$$

$$\frac{d\zeta}{dY} = \frac{d\zeta}{\xi^{\gamma} d\zeta + \gamma\xi^{\gamma-1}\zeta\, d\xi} = \frac{d\zeta}{\xi^{\gamma} d\zeta + \gamma\xi^{\gamma-1}\zeta \frac{1}{V_0} Av\, dt.}$$

Initially $v = 0$,

$$\therefore \quad \frac{d\zeta}{dY} = 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60b)$$

Solution of the Main Problems of Internal Ballistics

By solving (54) or any of its particular cases, subject to initial conditions (6), we get ζ and $\frac{d\zeta}{dY}$ as tabulated functions of Y, so that, let

$$\zeta \equiv I(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

$$\frac{d\zeta}{dY} \equiv J(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

then

$$\xi = \left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}} = \left[\frac{Y}{I(Y)}\right]^{\frac{1}{\gamma}} \equiv K(Y) \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

From (45)

$$dz = \frac{dY}{\xi^{\gamma-1}} = \frac{dY}{[K(Y)]^{\gamma-1}}$$

$$\therefore \quad z - z_0 = \int_{\zeta_0}^{Y} \frac{dY}{[K(Y)]^{\gamma-1}} \equiv L(Y) \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

From (41), then

$$\eta^2 = \frac{2M}{\gamma-1}(z-\zeta\xi)$$

$$= \frac{2M}{\gamma-1}[z_0+L(Y)-I(Y)K(Y)]$$

$$= [M(Y)]^2 \quad \ldots \quad (65)$$

At shot-start

$$z_0 = \zeta_0 = \frac{Al}{FC}p_0 \quad \ldots \quad (66)$$

This determines z_0 in terms of the shot-start pressure.

Now for any given value of f, (3) determines z, (64) determines Y, (63) and (37) determine shot-travel, (38) and (61) determine pressure, (39) and (65) determine velocity.

Maximum pressure:

For maximum pressure $\frac{d\zeta}{dY}$ must vanish and since Y is a monotonic increasing variable $\frac{d\zeta}{dY}$ should change sign from positive to negative.

Let Y_1 be the solution of $J(Y) = 0$, then from (38) and (61) maximum pressure is given by

$$\frac{A_0 l}{FC}p_1 = \zeta_1 = I(Y_1) \quad \ldots \quad (67)$$

$$\xi_1 = \frac{V_1}{V_0} = K(Y_1) \quad \ldots \quad (68)$$

determine V_1 and therefore shot-travel up to the instant of maximum pressure.

Again from (39) and (65)

$$\eta_1^2 = \frac{A_0^2 D^2}{FC\beta^2}\left(\frac{FC}{A_0 l}\right)^{2-2\alpha} v_1^2 = [M(Y_1)]^2 \quad \ldots \quad (69)$$

All-burnt position:

Let suffix 2 correspond to this position, then

$L(Y_2) = 1 - z_0$ determines Y_2 and

$$\xi_2 = \frac{V_2}{V_0} = \frac{\int_0^x A\,dx}{V_0} = K(Y_2) \quad \ldots \quad (70)$$

$$\frac{A_0 l}{FC}p_2 = \zeta_2 = I(Y_2) \quad \ldots \quad (71)$$

and

$$\eta_2^2 = \frac{A_0^2 D^2}{FC\beta^2}\left(\frac{FC}{A_0 l}\right)^{2-2\alpha} v_2^2 = [M(Y_2)]^2 \quad \ldots \quad (72)$$

determine shot-travel, pressure and velocity at all-burnt.

Motion after all-burnt: muzzle velocity:

After all-burnt $z = 1$, $dz = 0$.

$\therefore$ from (45), $dY = 0$ or $d(\zeta\xi^{\gamma}) = 0$.

$\therefore$ $\zeta\,\xi^{\gamma} = \zeta_2\xi_2^{\gamma}$.

Let suffix 3 denote the muzzle-position.

$$\zeta_3\xi_3^{\gamma} = \zeta_2\xi_2^{\gamma}, \qquad \ldots \quad (73)$$

where

$$\xi_3 = \frac{V_3}{V_0}.$$

(73) determines the pressure at the muzzle.
From (41)

$$\eta_3^2 = \frac{2M}{\gamma-1}\,[1-\zeta_3\xi_3]$$

$$= \frac{2M}{\gamma-1}\left[1-\zeta_2\xi_2^{\gamma}\xi_3^{1-\gamma}\right] \qquad \ldots \quad (74)$$

(74) determines the muzzle velocity.

4. Second Method of Solving the Equations

In this method, we use ξ as dependent and Y as independent variables.
From (49)

$$\nu\sqrt{1-qz}\,\xi^{\gamma-1}\,\frac{d}{dY}\left[\frac{A_0}{A}\left(\frac{Y}{\xi^{\gamma}}\right)^{\alpha}\frac{d\xi}{dY}\,\xi^{\gamma-1}\,\nu\sqrt{1-qz}\right] = \frac{MA}{A_0}\left(\frac{Y}{\xi^{\gamma}}\right)^{1-\alpha}$$

or

$$(1-qz)\xi^{\gamma-1}\,\frac{d}{dY}\left[\frac{A_0}{A}\,Y^{\alpha}\xi^{\gamma-1-\alpha\gamma}\xi'\right] - \tfrac{1}{2}q\left[\frac{A_0}{A}\,Y^{\alpha}\xi^{\gamma-1-\alpha\gamma}\xi'\right] = \frac{MA}{A_0\nu^2}\,Y^{1-\alpha}\xi^{-\gamma(1-\alpha)}.$$

From (53)

$$1-qz = \frac{1-qY\xi^{1-\gamma}}{1+\frac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2Y^{2\alpha}\xi^{2\gamma-2-2\alpha\gamma}\xi'^2}.$$

Substituting, we get

$$[1-qY\xi^{1-\gamma}]\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\,Y^{\alpha}\xi^{\gamma-1-\alpha\gamma}\xi'\right] = \left[\frac{MA}{A_0\nu^2}\,Y^{1-\alpha}\xi^{\gamma\alpha-\gamma}+\tfrac{1}{2}q\frac{A_0}{A}\,Y^{\alpha}\xi^{\gamma-1-\alpha\gamma}\xi'\right]$$

$$\times\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2Y^{2\alpha}\xi^{2\gamma-2-2\alpha\gamma}\xi'^2\right] \quad (75)$$

Particular Cases

(i) *Orthodox gun :*

Putting $A = A_0$, we get

$$[1-qY\xi^{1-\gamma}]\,[Y\xi\xi''+(\gamma-1-\alpha\gamma)Y\xi'^2+\alpha\xi\xi'] = \left[1+\tfrac{1}{2}\,\frac{\gamma-1}{M}\,q\nu^2Y^{3\alpha}\xi^{2\gamma-2-2\alpha\gamma}\xi'^2\right] \times \left[\frac{M}{\nu^2}\,Y^{2-2\alpha}\xi^{2\gamma\alpha-3\gamma+3}+\tfrac{1}{2}qY\xi^{2-\gamma}\xi'\right] \quad (76)$$

(ii) *Orthodox gun : constant-burning surface :*

Here $q = 0$, $\nu = 1$, and (76) gives

$$Y\xi\xi''+(\gamma-1-\alpha\gamma)Y\xi'^2+\alpha\xi\xi' = MY^{2-2\alpha}\xi^{2\gamma\alpha-3\gamma+3} \quad \ldots \quad \ldots \quad (77)$$

(iii) *Orthodox gun : isothermal model : constant-burning surface :*

Putting $\gamma = 1$ and replacing M by adjuster $\overline{M}$, we have

$$Y\xi\xi''-\alpha Y\xi'^2+\alpha\xi\xi' = \overline{M}Y^{2-2\alpha}\xi^{2\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

Substituting in (77)

$$\xi = U^{\frac{2n}{\gamma-n}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79a)$$

$$Y = \frac{(1+n)V}{n(\gamma-n)M} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79b)$$

$$\alpha = \tfrac{1}{2}\left(3-\frac{1}{n}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79c)$$

We get after some simplification, Clemmow's (1951) equation

$$\frac{2UV}{1+n}\frac{d^2U}{dV^2}-\frac{2n(\gamma-1)}{(1+n)(\gamma-n)}\,V\left(\frac{dU}{dV}\right)^2+U\frac{dU}{dV} = 1 \quad \ldots \quad \ldots \quad (80)$$

(iv) *Conical-bore gun :*

The equation is obtained by replacing $\dfrac{A}{A_0}$ by $\left(\dfrac{1+\mu-\xi}{\mu}\right)^2$ in (75).

Initial Conditions

The initial conditions for the integration of (75) or (76) are easily seen to be

$$\xi = 1, \quad Y = \zeta_0, \quad \xi' = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (81)$$

Solution of the Equations

Subject to initial conditions (81), (75) can be integrated numerically and let the solution be

$$\xi \equiv P(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

$$\xi' \equiv Q(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83)$$

then

$$\zeta = \frac{Y}{\xi^{\gamma}} = Y[P(Y)]^{-\gamma} \equiv R(Y) \quad \ldots \quad \ldots \quad \ldots \quad (84)$$

From (45)

$$z - z_0 = \int_{\zeta_0}^{Y} [P(Y)]^{1-\gamma} dY \equiv S(Y) \quad \ldots \quad \ldots \quad \ldots \quad (85)$$

then (41) gives

$$\eta^2 = \frac{2M}{\gamma-1}[z_0 + S(Y) - P(Y)R(Y)] \equiv [T(Y)]^2 \quad \ldots \quad \ldots \quad (86)$$

Maximum Pressure

For maximum pressure $\frac{d\zeta}{dY} = 0$.

$\therefore$ from (84)

$$\frac{1}{Y} - \gamma \frac{P'(Y)}{P(Y)} = 0$$

or

$$\frac{1}{Y} - \gamma \frac{Q(Y)}{P(Y)} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

Let Y_1 be solution of (87) at which $\frac{d\zeta}{dY}$ changes its sign from positive to negative, then $P(Y_1)$, $R(Y_1)$, $T(Y_1)$ determine the shot-travel, pressure and velocity at the instant of maximum pressure.

All-Burnt Position

Let Y_2 be the solution of (85) when $z = 1$, then $P(Y_2)$, $R(Y_2)$, $T(Y_2)$ determine the shot-travel, pressure and velocity at all-burnt position.

Motion after All-Burnt. Muzzle Velocity

These are discussed in exactly the same way as in Method I.

5. Solution for the Linear Law of Burning

We have given above the solution of the equations of Internal Ballistics for the pressure-index law of burning. Another law of burning, which has been found more plausible both as theoretical and experimental grounds, is the Linear Law [Mansell (1907), Murour (1931), Dederick (1947), Corner (1950), pages 42, 72, 206).

$$D\frac{df}{dt} = -\beta(p + p_1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (88)$$

where β and p_1 are positive constants, depending on the composition of the propellant and the initial temperature of firing. The solution of the equations of Internal Ballistics for law (88) for the orthodox gun has been discussed by Kapur (1956*d*). In the present section we show how the methods developed in the last two sections can be adapted to this law.

The equation (43) in this case is replaced by

$$\eta \frac{df}{d\xi} = -\frac{A_0}{A}(\zeta+\zeta_1) \quad \ldots \quad (89)$$

where

$$\zeta_1 = \frac{A_0 l}{FC} p_1 = \frac{V_0}{FC} p_1 \quad \ldots \quad (90)$$

From (42) and (89)

$$\frac{dz}{df}\frac{d}{dz}\left\{-\frac{A_0}{A}(\zeta+\zeta_1)\frac{d\xi}{dz}\frac{dz}{df}\right\} = -\frac{MA}{A_0}\frac{\zeta}{\zeta+\zeta_1} \quad \ldots \quad (91)$$

Substituting from (45), (46), (48) and (50) in (91), we get

$$\nu\sqrt{1-qz}\,\frac{d}{dz}\left\{\frac{A_0}{A}\,(\zeta+\zeta_1)\,\frac{\zeta-Y\dfrac{d\zeta}{dY}}{\gamma\zeta^2}\,\nu\sqrt{1-qz}\right\} = \frac{MA}{A_0}\frac{\zeta}{\zeta+\zeta_1}$$

or

$$(1-qz)\left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left\{\frac{A_0}{A}\frac{\zeta+\zeta_1}{\zeta^2}(\zeta-Y\zeta')\right\}-\tfrac{1}{2}q\frac{A_0}{A}\frac{\zeta+\zeta_1}{\zeta^2}(\zeta-Y\zeta') = \frac{MA\gamma}{A_0\nu^2}\frac{\zeta}{\zeta+\zeta_1} \quad (92)$$

To find z, we eliminate η from (41) and (89)

$$z = \zeta\xi+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}(\zeta+\zeta_1)^2\left(\frac{d\xi}{dz}\right)^2\left(\frac{dz}{df}\right)^2$$

$$= \zeta\left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}}+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}\frac{(\zeta+\zeta_1)^2}{\gamma^2\zeta^4}(\zeta-Y\zeta')^2\nu^2(1-qz)$$

or

$$1-qz = \frac{1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}}{1+\tfrac{1}{2}\dfrac{\gamma-1}{M}\dfrac{A_0^2}{A^2}q\nu^2\dfrac{(\zeta+\zeta_1)^2}{\zeta^4}(\zeta-Y\zeta')^2} \quad \ldots \quad (93)$$

Substituting in (92), we get

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}\right]\left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\frac{(\zeta+\zeta_1)(\zeta-Y\zeta')}{\zeta^2}\right]$$

$$= \left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\frac{(\zeta+\zeta_1)^2}{\zeta^4}(\zeta-Y\zeta')^2\right]$$

$$\times\left[\tfrac{1}{2}q\frac{A_0}{A}\frac{\zeta+\zeta_1}{\zeta^2}(\zeta-Y\zeta')+\frac{MA\gamma}{A_0\nu^2}\frac{\zeta}{\zeta+\zeta_1}\right] \quad \ldots \quad (94)$$

If $(\zeta+\zeta_1)$ in (94) is replaced by ζ^α, it reduces to equation (54).
For the law of burning

$$D\frac{df}{dt} = -\beta(p+p_1)^\alpha \quad \ldots \quad (95)$$

the fundamental equation becomes

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}\right]\left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\frac{(\zeta+\zeta_1)^{\alpha}(\zeta-Y\zeta')}{\zeta^2}\right]$$

$$=\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\frac{(\zeta+\zeta_1)^{2\alpha}}{\zeta^4}(\zeta-Y\zeta')^2\right]$$

$$\times\left[\tfrac{1}{2}q\frac{A_0}{A}\frac{(\zeta+\zeta_1)^{\alpha}}{\zeta^2}(\zeta-Y\zeta')+\frac{MA\gamma}{A_0\nu^2}\frac{\zeta}{(\zeta+\zeta_1)^{\alpha}}\right] \quad .. \quad (96)$$

The law (95) has the advantage that it includes both the pressure-index law and the linear law as particular cases. Thus when $\alpha = 1$, it gives the linear law and (96) reduces to (94). When $\zeta_1 = 0$, it gives the pressure-index law and (96) reduces to (54). When $\alpha = 1$, $\xi_1 = 0$ it gives the standard law when the rate of burning is proportional to pressure.

If $A = A_0$, (94) reduces to the equation (25) for the orthodox gun deduced by Kapur (1956*d*).

As a matter of fact our present approach enables us to obtain the fundamental equation for any law of burning. Thus for both tapered-bore and orthodox guns, for the law of burning,

$$D\frac{df}{dt} = -\beta\Phi(p), \quad .. \quad .. \quad .. \quad .. \quad (97)$$

the fundamental equation is

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}}\right]\left(\frac{Y}{\zeta}\right)^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\frac{\Phi(\zeta)(\zeta-Y\zeta')}{\zeta^2}\right]$$

$$=\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\frac{[\Phi(\zeta)]^2}{\zeta^4}\quad(\zeta-Y\zeta')\right]$$

$$\times\left[\tfrac{1}{2}q\frac{A_0}{A}\frac{\Phi(\zeta)}{\zeta^2}(\zeta-Y\zeta')+\frac{M\gamma}{\nu^2}\frac{A}{A_0}\frac{\zeta}{\Phi(\zeta)}\right] \quad .. \quad .. \quad (98)$$

Second Method

From (91)

$$\nu\sqrt{1-qz}\,\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)\frac{d\xi}{dY}\xi^{\gamma-1}\nu\sqrt{1-qz}\right]=M\frac{A}{A_0}\frac{Y}{Y+\zeta_1\xi^{\gamma}}$$

or

$$(1-qz)\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\xi^{\gamma-1}\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)\xi'\right]-\tfrac{1}{2}q\left[\frac{A_0}{A}\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)\xi'\xi^{\gamma-1}\right]=\frac{MA}{A_0\nu^2}\frac{Y}{Y+\zeta_1\xi^{\gamma}}.$$

From (93)

$$1-qz=\frac{1-qY\xi^{1-\gamma}}{1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)^2\xi^{2\gamma-2}\xi'^2}$$

$\therefore$ the fundamental equation becomes

$$[1-qY\xi^{1-\gamma}]\,\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}(Y+\zeta_1\xi^{\gamma})\,\xi^{-1}\xi'\right]=\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2(Y+\zeta_1\xi^{\gamma})^2\xi^{-2}\xi'^2\right]$$
$$\times\left[\frac{MA}{A_0\nu^2}\frac{Y}{Y+\zeta_1\xi^{\gamma}}+\tfrac{1}{2}q\frac{A_0}{A}(Y+\zeta_1\xi^{\gamma})\,\xi^{-1}\xi'\right] \quad .. \quad (99)$$

For the law (95) it becomes

$$[1-qY\xi^{1-\gamma}]\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)^{\alpha}\xi^{\gamma-1}\xi'\right]$$
$$=\left[\frac{MA}{A_0\nu^2}\frac{\frac{Y}{\xi^{\gamma}}}{\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)^{\alpha}}+\tfrac{1}{2}q\frac{A_0}{A}\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)^{\alpha}\xi^{\gamma-1}\xi'\right]$$
$$\times\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\left(\frac{Y}{\xi^{\gamma}}+\zeta_1\right)^{2\alpha}\xi^{2\gamma-2}\xi'^2\right] \quad .. \quad (100)$$

This reduces to (99) when $\alpha = 1$, to (75) when $\zeta_1 = 0$ and to equation (64) of Kapur (1956*d*) when $\alpha = 1$, $A = A_0$.

For the most general law of burning (97), the fundamental equation of this method is

$$[1-qY\xi^{1-\gamma}]\,\xi^{\gamma-1}\frac{d}{dY}\left[\frac{A_0}{A}\Phi\left(\frac{Y}{\xi^{\gamma}}\right)\xi^{\gamma-1}\xi'\right]=\left[\frac{MA}{A_0\nu^2}\frac{\frac{Y}{\xi^{\gamma}}}{\Phi\left(\frac{Y}{\xi^{\gamma}}\right)}+\tfrac{1}{2}q\frac{A_0}{A}\Phi\left(\frac{Y}{\xi^{\gamma}}\right)\xi^{\gamma-1}\xi'\right]$$
$$\times\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{A_0^2}{A^2}q\nu^2\left(\Phi\left(\frac{Y}{\xi^{\gamma}}\right)\right)^2\xi^{2\gamma-2}\xi'^2\right] \quad .. \quad (101)$$

(98) and (101) are the most general equations, for they apply to (i) all laws of burning, (ii) all types of tapering.

6. Discussion of the Isothermal Model

In this section, we discuss the Internal Ballistics of a tapered-bore gun under the following assumptions:—

(i) We adopt the Isothermal approximation in which a mean temperature of the propellant gases is assumed throughout the period of burning of the charge. Also we represent the initial resistance to the motion of the shot by a change of effective rate of burning. Let $\overline{M}$ be the adjusted central ballistic parameter.

(ii) We assume the rate of burning proportional to pressure, i.e. we take $\alpha = 1$, $\zeta_1 = 0$.

(iii) We assume the form-function $z = 1-f$, i.e. we take $\theta = 0$. The common shapes, tube, ribbon and multitube have effective form-factors θ sufficiently close to zero to make a successful practical analysis possible with $\theta = 0$.

We may point out that these are precisely the assumptions under which Corner (1948, 1950) discussed his theory of the H/L gun and thus our theory is of the same level of accuracy as his in that case.

The equation for this case is obtained from (75) or (99) or (100) by putting $\zeta_1 = 0$, $\alpha = 1$, $q = 0$, $\nu = 1$, $\gamma = 1$ and replacing M by $\overline{M}$, so that we get

$$\frac{d}{dY}\left[\frac{A_0}{A}\, Y \frac{\xi'}{\xi}\right] = \overline{M}\,\frac{A}{A_0} \qquad \ldots \quad (102)$$

Since $\dfrac{A}{A_0}$ is a known function of ξ, this equation can always be integrated numerically to give ξ as a function of Y. In this case, Y is, of course, the same as z and so we can write (102) as

$$\frac{d}{dz}\left[z\,\frac{\xi'}{\xi}\,\frac{A_0}{A}\right] = \overline{M}\,\frac{A}{A_0} \qquad \ldots \quad (103)$$

The initial conditions for the integration of (103) are:

$$z = 0, \quad \xi = 1, \quad \xi' = \overline{M} \qquad \ldots \quad (104)$$

We shall now discuss the integration of (103) for a gun for which

$$\frac{A}{A_0} = \frac{1}{\xi} \qquad \ldots \quad (105)$$

In this case (103) becomes

$$\frac{d}{dz}[z\xi'] = \frac{\overline{M}}{\xi} \qquad \ldots \quad (106)$$

or

$$\frac{d}{dz}\left[z\,\frac{dX}{dz}\right] = \frac{1}{X} \qquad \ldots \quad (107)$$

where

$$\xi = \sqrt{\overline{M}}\,X \qquad \ldots \quad (108)$$

The initial conditions for the integration of (107) are:

$$z = 0, \quad X = X_0 = \frac{1}{\sqrt{\overline{M}}}, \quad \frac{dX}{dz} = \sqrt{\overline{M}} \qquad \ldots \quad (109)$$

Fortunately, equation (107) is the same equation as was obtained by Corner (1948) for the H/L gun, when in his equation, we put $b = 0$, $\nu = 0$. He integrated his equation numerically for $X_0 \leq 5$, i.e. for $\overline{M} \geq \frac{1}{25}$. Thus his numerical solution covers all the cases of interest to us.

Now

$$\zeta = \frac{z}{\xi} = \frac{z}{\sqrt{\overline{M}}\,X}.$$

Corner found in the course of the calculation that $\dfrac{X}{z}$ generally decreased steadily as z approached unity. The maximum pressure, therefore, occurs at all-burnt or so near it that the value of the pressure at burnt is a sufficiently good approximation. We are, therefore, interested in the pressure, velocity and shot-travel at all-burnt only and these can be obtained easily from Corner's (1948) tables for the H/L gun.

Making use of his solution, we get for the volume behind the projectile at all-burnt

TABLE I

$\bar{M}$	0·49	1	1·96	2·89	4
ξ_B	1·51	1·83	2·47	2·97	3·5

However, if the bore area is constant and equal to A_0, we get by putting $\frac{A}{A_0} = 1$ in (103) and integrating subject to (104)

$$\xi = e^{\bar{M}z}, \qquad \text{.. .. .. ..} \quad (110)$$

so that at all-burnt

$$\xi_B = e^{\bar{M}},$$

and we get

TABLE II

$\bar{M}$	0·49	1	1·96	2·89	4
ξ_B	1·6323	2·7180	7·0094	17·9922	54·60

Thus we see that if the charge and chamber capacity are the same, the volume behind the shot at burnt is much smaller in a tapered-bore gun than in an orthodox gun of constant bore area A_0.

For finding the pressure at burnt, we note that $\zeta = \frac{z}{\xi}$, for both the models and therefore at burnt, we get:

TABLE III

$\bar{M}$	0·49	1	1·96	2·89	4
ξ_B [Tapered-bore]	0·5525	0·5464	0·4049	0·3367	0·2857
ξ_B [Constant-bore area A_0]	0·6126	0·3679	0·1409	0·0556	0·0183

Thus we see that for same charge and chamber capacity, the pressure at burnt is much higher in this tapered-bore gun than in an orthodox gun of cross-section A_0 *of the bore.*

For finding the velocity, (43) gives

$$\eta = -\frac{A_0}{A}\zeta^{\alpha}\frac{d\xi}{df}.$$

In our case

$$z = 1-f, \quad \alpha = 1, \quad \frac{A}{A_0} = \frac{1}{\xi}$$

$$\therefore \quad \eta = \zeta\xi\frac{d\xi}{dz} = z\frac{d\xi}{dz}$$

$\therefore$ at all-burnt

$$\eta_B = \left(\frac{d\xi}{dz}\right)_B = \sqrt{\bar{M}}\left(\frac{dX}{dz}\right)_B \qquad \text{.. .. ..} \quad (111)$$

For a gun of constant cross-section area bore

$$\eta = -\zeta^{\alpha}\frac{d\xi}{df} = \zeta\frac{d\xi}{dz} = \frac{z}{\xi}\frac{d\xi}{dz}$$

$\therefore$ at all-burnt, using (110)

$$\eta_B = \frac{1}{e^{\overline{M}}} \overline{M} e^{\overline{M}} = \overline{M} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (112)$$

Equations (111) and (112) give, on making use of Corner's tables:

TABLE IV

$\overline{M}$	0·49	1	1·96	2·39	4
η_B [Tapered-bore]	0·394	0·715	1·212	1·698	1·848
η_B [Constant-bore cross-section area A_0]	0·490	1·000	1·960	2·890	4·000

Thus for same charge and chamber capacity, the velocity at burnt is smaller in the case of this tapered-bore gun than in the case of an orthodox gun with same A_0 and the difference becomes more pronounced as $\overline{M}$ increases.

From Tables II and III, we find that if we use the same $\overline{M}$ for both the guns, the volume behind the projectile is very much smaller and pressure very much larger for the tapered-bore gun. A more useful comparison between two types of guns can be made if we adjust $\overline{M}$ for the tapered-bore gun, so that the maximum pressure is the same for both the guns and then compare the ballistics of the two guns.

For an orthodox gun with parameter $\overline{M}'$,

$$\zeta = ze^{-\overline{M}'z}$$

ζ is maximum when $z = \frac{1}{\overline{M}'}$, provided $\overline{M}' > 1$, and then

$$\zeta_{\max.} = \frac{1}{e\overline{M}'} = \frac{\cdot 3679}{\overline{M}'} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (113)$$

If $\overline{M}' \leqslant 1$, the maximum pressure occurs at all-burnt and is given by

$$\zeta_{\max.} = \frac{1}{e^{\overline{M}'}} = e^{-\overline{M}'} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (114)$$

From (113), (114) and Table III, we get for same maximum pressure

TABLE V

$\overline{M}$	0·49	1	1·96	2·89	4	6·25	8
$\overline{M}'$	0·365	0·60	0·91	1·1	1·3	1·68	1·91

and then for the position at all-burnt, we have

TABLE VI

$\overline{M}$	0·49	1	1·96	2·89	4	6·25	8
ξ_B [The present tapered-bore gun]	1·51	1·83	2·47	2·97	3·50	4·48	5·19
ξ_B [Orthodox with adjusted $\overline{M}'$]	1·51	1·82	2·48	2·99	3·67	5·36	6·74

For $\overline{M} \leqslant 2{\cdot}718$, the maximum pressure occurs at all-burnt and thus ζ_B (and therefore ξ_B) would be the same in both the cases. After that we find ξ_B greater for the orthodox gun. Thus on using adjusted central ballistic parameter, *we find that for same maximum pressure and for same chamber volume, the tapered-bore would give slightly better external ballistics.*

Now

$$\overline{M} = \frac{A_0^2 D^2}{FC\beta^2 w_1}, \quad v^2 = \frac{F^2C^2\beta^2}{A_0^2D^2}\eta^2 = \frac{FC}{w_1}\frac{\eta^2}{\overline{M}}, \quad \zeta = \frac{V_0}{FC}p \quad \ldots \quad \ldots (115)$$

In our above discussion, we have taken same chamber-capacity, same force-constant and same charge mass for both the tapered-bore and the orthodox gun, so that $\frac{\zeta}{p}$ has the same value for both. Therefore the most convenient way of adjusting $\overline{M}$ is to adjust the web-size and obviously we have to use a larger web-size for the tapered-bore gun to get the same maximum pressure.

Now for motion after all-burnt, we get for the muzzle velocity

$$v_3^2 = \frac{FC}{w_1}\frac{\eta_3^2}{\overline{M}} = \frac{2}{\gamma-1}\frac{FC}{w_1}\left[1-\left(\frac{\xi_B}{\xi_3}\right)^{\gamma-1}\right] \quad \ldots \quad \ldots (116)$$

Thus if we take same ξ_3, since ξ_B is in general smaller for the tapered-bore gun, the muzzle velocity will be slightly larger for it.

In practice, however, while V_0 may be the same for tapered-bore and orthodox gun, A_0 would in general be larger for the tapered-bore gun so that the barrel may not have to be too long. Actually A_0 is so adjusted that the emergent calibres of the tapered-bore and the orthodox gun are the same and then the ballistic size D is adjusted to adjust for $\overline{M}$. In this way we can have ξ_3 for the tapered-bore gun even larger than that for the corresponding orthodox gun and this would imply a further increase in muzzle velocity.

Another way to keep the same maximum pressure in the two guns is to use a smaller V_0 and a smaller charge C in the tapered-bore gun, but this would mean a smaller muzzle velocity.

The tapered-bore gun we have discussed in this section is a steeply tapered one and the reasons for discussing it here are:

(i) availability of the solution of the corresponding equation;

(ii) this gun exaggerates the effects of tapering and though in other tapered-bore guns the effects may be quantitatively less pronounced, yet qualitatively they are likely to present the same trend.

In the next section, we discuss the Isothermal Model for a moderately tapered-bore gun. The discussion of this case is important, as it contains the solution for the conical-bore gun as a particular case.

7. The Isothermal Model for a Moderately Tapered-Bore Gun

In this section, we discuss the particular tapered-bore gun for which

$$\frac{A}{A_0} = a - b\xi \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (117)$$

Since, however, when $\xi = 1$, $\frac{A}{A_0} = 1$, this can be written as

$$\frac{A}{A_0} = 1 + b - b\xi \quad \ldots \quad \ldots \quad \ldots \quad \ldots (118)$$

We shall see that when b is small, this gun would be a good approximation to a conical bore gun. Also b, in practice, would lie between 0·02 and 0·1, and thus, as an approximation, we shall be justified in neglecting squares and higher powers of b.

For this gun (103) becomes

$$\frac{d}{dz}\left[z\frac{\xi'}{\xi}\frac{1}{1+b-b\xi}\right] = \overline{M}\,(1+b-b\xi) \quad \ldots \quad \ldots \quad \ldots (119)$$

or

$$z\xi(1+b-b\xi)\xi''-z(1+b-2b\xi)\xi'^2+\xi\xi'(1+b-b\xi)-\overline{M}\xi^2(1+b-b\xi)^3 = 0 \quad \ldots (120)$$

When $b = 0$, it gives

$$z\xi\xi''-z\xi'^2+\xi\xi'-\overline{M}\xi^2 = 0 \quad \ldots \quad \ldots \quad \ldots (121)$$

of which the solution subject to initial conditions (104) is

$$\xi = e^{\overline{M}z} \equiv P(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots (122)$$

Let the solution of (120) be

$$\xi = P(z)+bV_1(z)+b^2V_2(z)+\ \ldots\ +b^rV_r(z)+\ \ldots \quad \ldots (123)$$

where

$$V_r(0) = 0, \quad V_r'(0) = 0 \quad [r = 1, 2, \ldots] \quad \ldots \quad \ldots \quad \ldots (124)$$

Substituting from (123) in (120) and equating the coefficients of the various powers of b, we get

$$zPP''-zP'^2+PP'-\overline{M}P^2 = 0 \quad \ldots \quad \ldots \quad \ldots (125)$$

$$zPV_1''+V_1'[P-2P'z]+V_1[P''z+P'-2P\overline{M}]$$
$$= -z(1-P)PP''+z(1-2P)P'^2-PP'(1-P)+3\overline{M}P^2(1-P)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (126)$$

$$zPV_r''+V'[P-2P'z]+V_r[P''z+P'-2P\overline{M}]$$
$$= \phi_r(z, P, P', P'', V_1, V_2, \ldots\ V_{r-1})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (127)$$

Here ϕ_r is some function of the variables indicated. (125) is the same as (121) and its solution has already been found. To solve (126), we note that if D denotes the operator $\frac{d}{dz}$, (126) becomes

$$[zD^2+(1-2\overline{M}z)D+(\overline{M}^2z-\overline{M})]V_1 = e^{\overline{M}z}\overline{M}[2-2e^{\overline{M}z}-z\overline{M}e^{\overline{M}z}]$$

or

$$[zD-\overline{M}z][zD-\overline{M}z]V_1 = \overline{M}ze^{\overline{M}z}[2-e^{\overline{M}z}(2+\overline{M}z)] \quad \ldots \quad \ldots (128)$$

Let

$$[zD-\overline{M}z]V_1 = W_1, \quad \ldots \quad \ldots \quad \ldots \quad \ldots (129)$$

then

$$\frac{dW_1}{dz}-\overline{M}W_1 = \overline{M}e^{\overline{M}z}[2-2e^{\overline{M}z}-\overline{M}ze^{\overline{M}z}]$$

or

$$W_1 e^{-\overline{M}z} = \overline{M}\left[2z - \frac{2e^{\overline{M}z}}{\overline{M}} - \overline{M}\left(\frac{e^{\overline{M}z}\,z}{\overline{M}} - \frac{e^{\overline{M}z}}{\overline{M}^2}\right)\right] + \text{const.}$$

When $z = 0$, $W_1 = 0$.

$$\therefore \quad W_1 = \overline{M}e^{\overline{M}z}\left[2z - \frac{1}{\overline{M}}(e^{\overline{M}z} - 1) - ze^{\overline{M}z}\right] \quad \ldots \quad \ldots (130)$$

From (129) and (130)

$$\frac{dV_1}{dz} - \overline{M}V_1 = \overline{M}e^{\overline{M}z}\left[2 - e^{\overline{M}z} - \frac{e^{\overline{M}z} - 1}{\overline{M}z}\right]$$

$$\therefore V_1(z) = \overline{M}\left\{2z - \frac{e^{\overline{M}z} - 1}{\overline{M}} - \frac{1}{\overline{M}}\left[\overline{M}z + \frac{(\overline{M}z)^2}{2\lfloor 2} + \frac{(\overline{M}z)^3}{3\lfloor 3} + \ldots\right]\right\} \quad \ldots (131)$$

Similarly to solve (127), we get on substituting for $P(z)$

$$(zD - \overline{M}z)(zD - \overline{M}z)V_r = ze^{-\overline{M}z}\phi_r(z, P, P', P'', V_1, V_2, \ldots, V_{r-1}) \quad \ldots (132)$$

Since from the integration of the earlier equations $P, P', P'', V_1, V_2, \ldots, V_{r-1}$ are known functions of z, the R.H.S. of (132) is a known function of z, $\psi_r(z)$ (say) then (132) can be integrated to give

$$W_r(z) = e^{\overline{M}z}\int_0^z \frac{e^{\overline{M}z}}{z}\psi_r(z)dz$$

and

$$V_r(z) = e^{\overline{M}z}\int_0^z \frac{e^{\overline{M}z}}{z}\int_0^z \frac{e^{\overline{M}z}}{z}\psi_r(z)dz \quad \ldots \quad \ldots \quad \ldots (133)$$

Thus we see that we can find the coefficients of the various powers of b in (123).

In the rest of the section, we neglect squares and higher powers of b, so that we get

$$\xi = e^{\overline{M}z} - b\left[\sum_{n=2}^{\infty}\frac{(\overline{M}z)^n}{\lfloor n}\,\frac{n+1}{n}\right] \quad \ldots \quad \ldots \quad \ldots (134)$$

Thus we see that up to this order of approximation, for any given z, the volume behind the projectile for this tapered-bore gun is less than that for the corresponding gun of same A_0. In particular this holds for the all-burnt position.

Again up to the first power of b,

$$\zeta = \frac{z}{\xi} = \frac{z}{e^{\overline{M}} - b\left[\sum_2^{\infty}\frac{(\overline{M}z)^n}{\lfloor n}\,\frac{n+1}{n}\right]}$$

$$= \frac{z}{e^{\overline{M}z}}\left[1 + be^{-\overline{M}z}\sum_2^{\infty}\frac{(\overline{M}z)^n}{\lfloor n}\,\frac{n+1}{n}\right] \quad \ldots \quad \ldots \quad \ldots (135)$$

Thus up to this order of approximation, for any given z, the pressure in this tapered-bore gun is greater than in the corresponding orthodox gun. In particular the pressure at all-burnt is greater for the tapered-bore gun.

Again

$$\eta = \zeta \frac{d\xi}{dz}$$

$$= \frac{z}{e^{\overline{M}z}} \left[1 + be^{-\overline{M}z} \sum_{n=2}^{\infty} \frac{(\overline{M}z)^n}{\lfloor n+1} \frac{n+1}{n} \right] \left[\overline{M}e^{\overline{M}z} - b\overline{M} \sum_{n=2}^{\infty} \frac{(\overline{M}z)^{n-1}}{\lfloor n-1} \frac{n+1}{n} \right]$$

$$= \overline{M}z \left\{ 1 - be^{-\overline{M}z} \left[\sum_{n=2}^{\infty} \frac{n+1}{n} \frac{(\overline{M}z)^{n-1}}{\lfloor n-1} \left(1 - \frac{\overline{M}z}{n} \right) \right] \right\} \quad \text{.. .. .. (136)}$$

Thus, in general, to this approximation, for any given z, the velocity is less than the velocity in the corresponding orthodox gun. In particular the velocity at all-burnt is smaller for the tapered-bore gun.

Before proceeding further, we note that the infinite series

$$\overline{M}z + \frac{(\overline{M}z)^2}{2\lfloor 2} + \frac{(\overline{M}z)^3}{3\lfloor 3} + \ldots = \sum_{1}^{\infty} \frac{(\overline{M}z)^n}{n\lfloor n} \quad \text{.. .. (137)}$$

occurring in the expression for $V_1(z)$ in the equation (131) is the expression for the definite integral

$$\int_0^z \frac{e^{\overline{M}z}-1}{z} dz = \int_0^{\overline{M}z} \frac{e^t-1}{t} dt. \quad \text{.. .. .. (138)}$$

Fortunately, this integral is available as a tabulated function. In fact, the exponential integral is defined by

$$\overline{E_i}(x) = \log_e \gamma x + \int_0^x \frac{e^t-1}{t} dt, \quad \text{.. .. .. (139)}$$

where γ is a constant defined by

$$\log_e \gamma = \int_0^1 \frac{1-e^t}{t} dt - \int_1^{\infty} \frac{e^{-t}}{t} dt = 0{\cdot}577215665 \quad \text{.. .. (140)}$$

From (131), (137), (138) and (139), we find that

$$V_1(z) = 2\overline{M}z - (e^{\overline{M}z}-1) - \int_0^{\overline{M}z} \frac{e^t-1}{t} dt$$

or

$$V_1(z) = 2\overline{M}z - (e^{\overline{M}z}-1) - \overline{E_i}(\overline{M}z) + \log_e \gamma + \log_e (\overline{M}z) \quad \text{.. (141)}$$

Making use of the tables of $\bar{E}_i(x)$ (Jahmke and Emde (1945)), we get the following table:—

TABLE VII

	$\bar{M} = 1$		$\bar{M} = 2$	
z	$P(z)$	$-V_1(z)$	$P(z)$	$-V_1(z)$
0·1	1·1052	0·0078	1·2214	0·0319
0·2	1·2214	0·0319	1·4918	0·1357
0·3	1·3499	0·0738	1·8220	0·3255
0·4	1·4918	0·1357	2·2260	0·6194
0·5	1·6487	0·2289	2·7180	1·0359
0·6	1·8220	0·3255	3·3198	1·6014
0·7	2·0140	0·4584	4·0537	2·3475
0·8	2·4600	0·8110	6·0503	4·5253
1·0	2·7180	1·0359	7·3875	6·0714

Similar tables can easily be prepared for $\bar{M} = 3$ and $\bar{M} = 4$. We note that both $P(z)$ and $V_1(z)$ are functions of $\bar{M}z$. This will considerably reduce the number of calculations.

The table shows that the power series solution will have greater accuracy for small values of z; and for moderate values of z, it should give satisfactory results even up to all-burnt, provided b is small. If $\bar{M}$ and b are larger, we may have to take the contribution by the coefficient of b^2 into account. This would mean more calculations, but even then the labour would be much less than for numerical integration.

We can, therefore, regard $V_1(z)$ as a known function of z and we can express the equations (134), (135), (136) in the following alternative form, without the use of an infinite series

$$\xi = P(z)+bV_1(z) = e^{\bar{M}z}+bV_1(z) \quad \ldots \quad (142)$$

$$\zeta = \frac{z}{e^{\bar{M}z}}\left[1-be^{-\bar{M}z}V_1(z)\right] \quad \ldots \quad (143)$$

$$\eta = z\left\{\bar{M}+b\left[2\bar{M}e^{-\bar{M}z}-\bar{M}-\frac{1-e^{-\bar{M}z}}{z}-\bar{M}e^{-\bar{M}z}V_1(z)\right]\right\} \quad \ldots \quad (144)$$

Now $\zeta = \dfrac{z}{\xi}$, and, therefore, ζ is maximum when

$$\zeta - z\frac{d\xi}{dz} = 0$$

or

$$e^{\bar{M}z}+bV_1(z)-\bar{M}ze^{-\bar{M}z}-b\left[2\bar{M}z-\bar{M}ze^{\bar{M}z}-(e^{\bar{M}z}-1)\right] = 0 \quad \ldots \quad (145)$$

Let the solution of (145) be

$$z = \frac{1}{\bar{M}}+bK \quad \ldots \quad (146)$$

Substituting in (146) and neglecting squares and higher powers of b, we get

$$e(1+bK\bar{M})+bV_1\left(\frac{1}{\bar{M}}\right)-(1+2bK\bar{M})e-b(3-2e) = 0$$

or

$$K = \frac{1}{e\overline{M}}\left[2e-3+V_1\left(\frac{1}{\overline{M}}\right)\right] \quad \ldots \quad (147)$$

From (141), we find that $V_1\left(\frac{1}{\overline{M}}\right)$ is independent of $\overline{M}$ and from Table VII we find its value to be $-1{\cdot}0359$, so that

$$K = \frac{0{\cdot}5152}{\overline{M}} \quad \text{and} \quad z = \frac{1+0{\cdot}5152b}{\overline{M}} \quad \ldots \quad (148)$$

Thus up to a first approximation, the effect of tapering is that maximum pressure occurs when relatively more charge has been and burnt and the ratio of this charge to that for the corresponding untapered gun is independent of $\overline{M}$.

Also

$$\zeta_{\text{max.}} = \frac{\frac{1}{\overline{M}}+bk}{e^{\overline{M}\left(\frac{1}{\overline{M}}+bk\right)}+bV_1\left(\frac{1}{\overline{M}}+bk\right)}$$

$$= \frac{1}{e\overline{M}}\left[1-\frac{b}{e}V_1\left(\frac{1}{\overline{M}}\right)\right]$$

$$= \frac{1}{e\overline{M}}\left[1+\frac{1{\cdot}0359b}{e}\right] \quad \ldots \quad (149)$$

Thus the maximum pressure is greater for the tapered-bore gun and the ratio of this maximum pressure to that for the untapered gun is independent of $\overline{M}$.

If $\overline{M}'$ is the adjusted ballistic parameter for the orthodox gun to give the same maximum pressure

$$\frac{1}{e\overline{M}'} = \frac{1}{e\overline{M}}\left[1+\frac{1{\cdot}0359b}{e}\right]$$

or

$$\overline{M} = \overline{M}'\left[1+\frac{1{\cdot}0359b}{e}\right] = \overline{M}'[1+0{\cdot}3811b] \quad \ldots \quad (150)$$

Thus the ballistic parameter for the tapered-bore gun is greater.

If

$$1+\frac{0{\cdot}5152b}{\overline{M}} \geqslant 1,$$

the maximum pressure occurs at all-burnt. In this case

$$\zeta_{\text{max.}} = \frac{1}{\xi_B} \quad \ldots \quad (151)$$

In this case also, since ξ_B is smaller for the tapered-bore gun, the maximum pressure would be larger for it.

For $b = 0{\cdot}05$, (150) gives

$$\overline{M} = \overline{M}'[1+0{\cdot}0191]$$

TABLE VIII

$\bar{M}$	0·49	1	1·96	2·89	4
$\bar{M}'$	0·48	0·98	1·92	2·84	3·92

For the volume behind the projectile at all-burnt, we get, for the same maximum pressure:

TABLE IX

$\bar{M}$	0·49	1	1·96	2·89
ξ_B [Tapered-bore]	1·62	2·66	6·79	17·05
ξ_B [Orthodox]	1·62	2·66	6·82	17·11

Table IX confirms the result of Table VI.

Thus we find that all the results for the moderately tapered-bore gun of this section are consistent with and present the same trend as the results for the more steeply tapered gun of the previous section.

It is also interesting to observe that the effect of tapering with same A_0 is of the same type as of using the law

$$D\frac{df}{dt} = -\beta(p+p_1),$$

instead of the law

$$D\frac{df}{dt} = -\beta p$$

with same β (Kapur (1956*d*)).

8. CONICAL-BORED GUN

This is the most important shape from the practical point of view. For this from (55)

$$\frac{A}{A_0} = \left(1+\frac{1}{\mu}-\frac{1}{\mu}\xi\right)^{\frac{2}{3}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots (152)$$

For most practical tapered-bore guns $\frac{1}{\mu} < 0{\cdot}1$. If we neglect squares and higher powers of $\frac{1}{\mu}$, (152) gives

$$\frac{A}{A_0} = 1+\frac{2}{3\mu}-\frac{2}{3\mu}\xi \quad \ldots \quad \ldots \quad \ldots \quad \ldots (153)$$

(153) is the same as (118), provided $b = \frac{2}{3\mu}$.

Thus up to a first approximation, all the results of the last sub-section are applicable for a conical bore gun. If, however, we want to keep up to rth power of b, we expand $\frac{A}{A_0}$ in powers of $\frac{1}{\mu}$ up to $\left(\frac{1}{\mu}\right)^r$ and substitute this in (106) and the using a solution of the form (123), we can proceed as in the last sub-section. In practice, keeping terms up to b^2 should give quite satisfactory results.

For more accurate results, the equation

$$\frac{d}{dz}\left[z\frac{\xi'}{\xi}\left(1+\frac{1}{\mu}-\frac{1}{\mu}\xi\right)^{-\frac{2}{3}}\right] = \bar{M}\left(1+\frac{1}{\mu}-\frac{1}{\mu}\xi\right)^{\frac{2}{3}} \quad \ldots \quad \ldots (154)$$

can be integrated numerically subject to initial conditions (108), and since $\xi > 1$ and $1+\frac{1}{\mu}-\frac{1}{\mu}\xi > 0$, no singularity should arise. When ξ is small, the steps of integration can be taken as large, but near the position of burnt, the steps will have to be smaller.

9. Extension to Composite Charges

Let F_i, C_i D_i, β_i, θ_i, z_i, f_i refer to the ith component charge and let F, C, D, β, θ, z, f refer to the equivalent charge which is defined as the charge which would give the same ballistic equations as the composite charge both during and after burning.

We shall assume that the pressure-index α in the pressure-index law and the constant p_1 in the linear law have the same value for each component charge. In particular this condition will be satisfied in the practically important case when all the component charges have the same composition. Then if

$$\beta_1' > \beta_2' > \ldots . > \beta_n', \qquad \ldots \quad (155)$$

there are n distinct stages of burning and the form-function for the rth stage of burning is (Kapur (1956a)):

$$z = A_r + B_r(1-f) - E_r(1-f)^2, \qquad \ldots \quad (156)$$

when

$$1-\frac{1}{k_{r-1}} \geqslant f \geqslant 1-\frac{1}{k_r}, \qquad \ldots \quad (157)$$

where

$$A_r = \sum_{i=1}^{r-1} \lambda_i \qquad \ldots \quad (158)$$

$$B_r = \sum_{i=r}^{n} \lambda_i k_i(\theta_i+1) \qquad \ldots \quad (159)$$

$$E_r = \sum_{i=r}^{n} \lambda_i k_i^2 \theta_i \qquad \ldots \quad (160)$$

$$\lambda_i = \frac{F_i C_i}{\sum_{i=1}^{n} F_i C_i} \qquad \ldots \quad (161)$$

$$k_i = \frac{\beta_i'}{\beta_n'} \qquad \ldots \quad (162)$$

Also for the equivalent charge

$$C = C_1 + C_2 + \ldots . + C_n \qquad \ldots \quad (163)$$

$$\frac{D}{\beta} = \frac{D_n}{\beta_n} \qquad \ldots \quad (164)$$

$$CF = C_1F_1 + C_2F_2 + \ldots . + C_nF_n \qquad \ldots \quad (165)$$

From (156), during the rth stage

$$\frac{dz}{df} = -\nu_r\sqrt{1-q_r z}, \qquad \ldots \quad (166)$$

where

$$\nu_r = \sqrt{B_r^2+4A_rE_r} \qquad \ldots \quad (167)$$

$$q_r = \frac{4E_r}{B_r^2+4A_rE_r} \qquad \ldots \quad (168)$$

From (50) and (156), we see that the fundamental equation for either of the two methods for the equivalent charge for the rth stage is obtained from the corresponding differential equation (54) or (75) or (94) or (96) or (98) or (99) or (100) or (101) for a single charge by replacing ν and q by ν_r and q_r respectively.

Assuming that the shot starts moving in the first stage, the initial conditions for this stage would be obtained in the same way as the initial conditions for a single charge. Next we determine the initial conditions for the rth stage for each of the methods presented here.

Initial Conditions for the rth Stage

(i) *First Method. Pressure-index law:*

Since $Y = \zeta\xi^\gamma$

$$\frac{1}{Y} = \frac{\zeta'}{\zeta}+\gamma\frac{\xi'}{\xi}$$

$$\therefore \quad \frac{\zeta'}{\zeta} = \frac{1}{\xi^\gamma}\left[\frac{1}{\zeta}+\frac{\gamma\eta}{\zeta^\alpha}\frac{A}{A_0}\frac{1}{\frac{dz}{df}}\right]$$

or

$$\frac{d\zeta}{dY} = \frac{1}{\xi^\gamma}\left[1+\frac{\gamma\eta\zeta}{\zeta^\alpha}\frac{A}{A_0}\frac{1}{\frac{dz}{df}}\right] \qquad \ldots \quad (169)$$

In crossing from $(r-1)$th stage to rth, ξ, η, ζ, A are continuous, but $\frac{dz}{df}$ is in general discontinuous unless $\theta_{r-1} = 1$. Accordingly in writing the initial conditions for the rth stage, we should write the value of $\frac{dz}{df}$ at the beginning of the rth stage and not at the end of the previous stage. Thus the initial conditions for the rth stage are:

$$Y = (Y)_{r-1}, \qquad \zeta = (\zeta)_{r-1}$$

$$\frac{d\zeta}{dY} = \frac{1}{[(\xi)_{r-1}]^{\gamma-1}}\left[1+\frac{\gamma(\eta)_{r-1}(\zeta)_{r-1}}{(\zeta)^\alpha_{r-1}}\frac{(A)_{r-1}}{A_0}\frac{1}{-B_r+\frac{2E_r}{k_{r-1}}}\right] \qquad \ldots \quad (170)$$

(ii) *Second Method. Pressure-index law:*

$$\frac{d\xi}{dY} = \frac{d\xi}{d(\zeta\xi^\gamma)} = \frac{d\xi}{df}\frac{df}{dz}\frac{dz}{d(\zeta\xi^\gamma)} = -\frac{\eta}{\zeta^\alpha}\frac{A}{A_0}\frac{1}{\frac{dz}{df}}\xi^{1-\gamma} \qquad \ldots \quad (171)$$

Remembering the possible discontinuity in $\frac{dz}{df}$, the initial conditions for the rth stage are:

$$\xi = (\xi)_{r-1}, \qquad Y = (Y)_{r-1}$$

$$\frac{d\xi}{dY} = -\frac{(\eta)_{r-1}}{[(\zeta)_{r-1}]^{\alpha}} [(\xi)_{r-1}]^{1-\gamma} \frac{(A)_{r-1}}{A_0} \frac{1}{-B_r + \frac{2E_r}{k_{r-1}}}$$

$$= \frac{-B_{r-1} + \frac{2E_{r-1}}{k_{r-1}}}{-B_r + \frac{2E_r}{k_{r-1}}} \left(\frac{d\xi}{dY}\right)_{r-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (172)$$

(iii) *Linear law:*

The initial conditions for the two methods of this law are obtained from (170) and (172) by replacing $[(\zeta)_{r-1}]^{\alpha}$ by $(\zeta)_{r-1} + \zeta_1$.

Maximum Pressure

If $A = A_0$, $\zeta_1 = 0$ or $\alpha = 1$, we can easily show that maximum pressure is unique (Kapur (1956*b*, *c*)).

From (169), we have during the rth stage:

$$\frac{d\zeta}{dY} = \frac{1}{\xi^{\gamma}} \left(1 + \frac{\gamma\eta\zeta}{\zeta_{\alpha}} \frac{A}{A_0} \frac{1}{-B_r + 2E_r(1-f)}\right) \qquad \ldots \quad \ldots \quad \ldots (173)$$

A pressure maximum would arise during the rth stage if

$$1 + \frac{\gamma(\eta)_{r-1} (\zeta)_{r-1}}{[(\zeta)_{r-1}]^{\alpha}} \frac{(A)_{r-1}}{A_0} \frac{1}{-B_r + \frac{2E_r}{k_{r-1}}} > 0 \qquad \ldots \quad \ldots (174)$$

and

$$1 + \frac{\gamma(\eta)_r (\zeta)_r}{[(\zeta)_r]^{\alpha}} \frac{(A)_r}{A_0} \frac{1}{-B_r + \frac{2E_r}{k_r}} < 0 \qquad \ldots \quad \ldots \quad \ldots (175)$$

Again at the end of the rth stage, $\frac{dz}{df}$ receives, in general, a positive increment and therefore $\frac{d\zeta}{dY}$ receives a negative increment. It is possible, therefore, for $\frac{d\zeta}{dY}$ to change from positive to negative in crossing from one stage to another and thus a pressure maximum can arise at the end of a stage.

A pressure maximum would arise at the end of the rth stage if either

(i) $$1 + \frac{\gamma(\eta)_r (\zeta)_r}{[(\zeta)_r]^{\alpha}} \frac{(A)_r}{A_0} \frac{1}{-B_r + \frac{2E_r}{k_{r+1}}} = 0 \qquad \ldots \quad \ldots \quad \ldots (176)$$

or

(ii) $$1+\frac{\gamma(\eta)_r(\zeta)_r}{[(\zeta)_r]^{\alpha}}\frac{(A)_r}{A_0}\frac{1}{-B_r+\dfrac{2E_r}{k_{r+1}}}>0 \qquad (177)$$

and

$$1+\frac{\gamma(\eta)_r(\zeta)_r}{[(\zeta)_r]^{\alpha}}\frac{(A)_r}{A_0}\frac{1}{-B_{r+1}+\dfrac{2E_{r+1}}{k_{r+1}}}<0 \qquad (178)$$

or

(iii) $$1+\frac{\gamma(\eta)_r(\zeta)_r}{[(\zeta)_r]^{\alpha}}\frac{(A)_r}{A_0}\frac{1}{-B_{r+1}+\dfrac{2E_{r+1}}{k_{r+1}}}=0 \qquad (179)$$

For the linear law, the corresponding conditions are obtained from (173) to (179) by replacing ζ^{α} by $\zeta+\zeta_1$.

The *all-burnt position* and conditions at this instant are determined by the position and the conditions at the end of the nth stage of burning. The *muzzle velocity* is determined as for a single charge.

The case when some of the component charges burn out simultaneously needs no separate discussion as in this case these charges behave in all respects as a single charge (Kapur, 1957*a*).

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Summary

After showing that the variation of cross-section area of the bore of a gun introduces new problems for the solution of the equations of Internal Ballistics, two methods for solving them for the pressure-index law and for the linear law for a tapered-bore have been given. When the area of cross-section is constant these give two new methods for the orthodox gun for each of the laws. The Isothermal Model has been studied in details for three different types of tapered-bore guns and in each case the internal ballistics for a tapered-bore gun have been compared with those of the orthodox gun. In the final section the theory has been extended to composite charges.

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