THE EQUIVALENT CHARGE METHOD IN THE GENERAL THEORY OF COMPOSITE CHARGES—II

by J. N. Kapur, *Hindu College, Delhi University, Delhi*

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received March 12; read May 3, 1957*)

1. Introduction

The equivalent charge method in the general theory of composite charges was first studied by Clemmow (1920, 1951) and Corner (1950). Their method has been recently generalized in a paper by the present author (Kapur, 1956*a*). This paper will be referred to in the present paper as Part I.

The definition of the equivalent charge we adopted was that the ballistic equations should be the same for the equivalent charge and the composite charge both during the period of burning of the charge as well as in the period after 'burnt', and we tried to derive the shape, size and composition of the equivalent charge from this definition. However, since the shape, size and composition are defined by means of a number of parameters, viz. F, C, D, β, α, θ, b, δ, γ, it is obvious that all these may not be determined from the four fundamental equations of Internal Ballistics. Actually we find that while all these quantities can be determined, some of these do not come out to be constant during the burning period, though we are able to show that after 'burnt' all these parameters have constant values. This is what we expect on general considerations, since during burning, the composition of the product gases goes on changing due to the different proportions in which the component charges burn till at any time t, while after burnt this composition remains fixed.

The equations of Internal Ballistics can be integrated even in the most general case when some of the parameters vary with time. We discuss this integration in the present paper and are able to show that all the cases discussed so far by Aggarwal (1955, 1956), Venkatesan (1956), Venkatesan and Patni (1953), Kapur (1956*b*, 1956*c*), Tawakley (1956) follow as particular cases of our general system. It is, however, obvious that the integration would be considerably simplified when the parameters are constant throughout the burning period, for in this case, all the classical methods of Internal Ballistics can, with the help of the theory developed in Part I, be easily extended to apply to composite charges. It is, therefore, of the utmost importance that we investigate the cases when all the parameters for the equivalent charge have constant values. We call this the existence of the equivalent charge 'in the strict sense'. We find that the conditions for this are satisfied in, among others, the following important cases: (i) when the component charges have the same composition but have possibly different shapes and sizes, (ii) when they possibly have different compositions, but have same 'effective' sizes and shapes, (iii) when covolume correction terms and variations in γ_i's can be neglected.

In the more general case, when an equivalent charge does not exist in the strict sense, we find the best possible average values of the varying quantities. Since the effect of some of the quantities like the covolume correction terms is in the nature of a correction and the effect of taking the variation in these terms into account would be in the nature of a correction to a correction, such average values should, in practice, be quite satisfactory.

In section 7, we have extended the theory developed in Part I for the pressure-index law with same pressure-index to the case when the component charges burn according to the experimentally and theoretically more plausible linear law $r = a+bp$ and investigated the conditions for simultaneous and non-simultaneous burning out when a and b for component charges are same or different.

2. The basic equations

The four basic equations of the composite charges are:

$$\sum_{i=1}^{n} \frac{F_i C_i z_i}{\gamma_i - 1} = p \left\{ \left[K_0 - \sum_{i=1}^{n} \frac{C_i}{\delta_i} + Ax - \sum_{i=1}^{n} C_i z_i \left(b_i - \frac{1}{\delta_i} \right) \right] \times \left[\frac{\sum_{i=1}^{n} \frac{m_i C_i z_i}{\gamma_i - 1}}{\sum_{i=1}^{n} m_i C_i z_i} \right] \right\} + \tfrac{1}{2} W_1 v^2, \quad \ldots \quad (1)$$

where

$$W_1 = 1{\cdot}05W + \tfrac{1}{3}[C_1 + C_2 + \ldots + C_n], \quad \ldots \quad (1a)$$

and m_i denotes the number of gram molecules per gram of the gases produced by the burning of the ith component charge.

$$D_i \frac{df_i}{dt} = -\beta_i p^{\alpha} [i = 1, 2, \ldots, n] \quad \ldots \quad (2)$$

$$z_i = (1-f_i)(1+\theta_i f_i + \psi_i f_i^2)[i = 1, 2, \ldots, n] \quad \ldots \quad (3)$$

or

$$z_i = \phi_i(f_i)[i = 1, 2, \ldots n] \quad \ldots \quad (3a)$$

$$W_1 \frac{dv}{dt} = Ap \quad \ldots \quad (4)$$

The basic equations for the equivalent charge are:

$$\frac{FCz}{\gamma - 1} = p \left[K_0 - \frac{C}{\delta} + Ax - Cz\left(b - \frac{1}{\delta} \right) \right] + \tfrac{1}{2}[1{\cdot}05W + \tfrac{1}{3}C]v^2 \quad \ldots \quad (5)$$

$$D \frac{df}{dt} = -\beta p^{\alpha} \quad \ldots \quad (6)$$

$$z = \phi(f) \quad \ldots \quad (7)$$

$$[1{\cdot}05W + \tfrac{1}{3}C] \frac{dv}{dt} = Ap \quad \ldots \quad (8)$$

From (2) and (6), we get

$$\frac{1-f_1}{\beta_1'} = \frac{1-f_2}{\beta_2'} = \ldots = \frac{1-f_n}{\beta_n'} = \frac{1-f}{\beta'}, \quad \ldots \quad (9)$$

where

$$\beta_i' = \frac{\beta_i}{D_i}, \beta' = \frac{\beta}{D} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Without loss of generality, we assume

$$\beta_1' \geqslant \beta_2' \geqslant \ldots\ldots \geqslant \beta_n', \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

so that

$$\beta' = \beta_n' \text{ or } \frac{D}{\beta} = \frac{D_n}{\beta_n} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Now (1) and (5) will be identical, if

$$\sum_{i=1}^{n} \frac{F_i C_i z_i}{\gamma_i - 1} = \frac{FCz}{\gamma - 1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$\sum_{i=1}^{n} C_i z_i \left(b_i - \frac{1}{\delta_i}\right) = Cz\left(b - \frac{1}{\delta}\right) \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$\sum_{i=1}^{n} \frac{m_i C_i}{\gamma_i - 1} z_i = \frac{1}{\gamma - 1} \sum_{i=1}^{n} m_i C_i z_i \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$\sum_{i=1}^{n} \frac{C_i}{\delta_i} = \frac{C}{\delta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$\sum_{i=1}^{n} C_i = C \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In virtue of (17), we easily verify that the dynamical equations (4) and (8) also become identical.

Now for the equivalent charge, we determine C from (17), δ from (16), $\frac{D}{\beta}$ from (12), γ from (15), b from (14), F from (13) and finally form-function $\phi(f)$ from (3), (7), (9) and (13).

In Part I, we determined $\phi(f)$, but we did not determine b and γ. We stated that these may be determined from Thermodynamical Considerations. We also implicitly assumed these to be constant. We find, however, that we can determine these from purely ballistic considerations also and that as (14) and (15) show, these are not, in general, constant. We find, however, in the next section some important cases where these are constant and where, accordingly, the integration of the equations of Internal Ballistics is considerably simplified.

3. Some cases where the equivalent charge with constant values for its parameters exists

Case I: *When all the component charges have same composition (Clemmow's case)*

In this case, making use of the above equations, we get, for all values of i,

$$F = F_i, \gamma = \gamma_i, \beta = \beta_i, \delta = \delta_i, b = b_i, m = m_i \quad \ldots \quad \ldots \quad (18)$$

so that equations (13) to (17) are satisfied if

$$C_1 + C_2 + \ldots\ldots + C_n = C \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

and

$$C_1 z_1 + C_2 z_2 + \dots\dots + C_n z_n = Cz \qquad (19)$$

Equation (17) shows that the composite charge and the equivalent charge burn out simultaneously, since at all-burnt

$$z_1 = z_2 = \dots = z_n = z = 1.$$

The case when all the component charges have the same composition is, in practice, the most important. This was the case discussed by Clemmow (1920, 1951) and in this case an equivalent charge in the strict sense exists. The component charges, while having the same composition, can have different shapes and sizes. The effective ballistic sizes will determine the order in which the different component charges burn out and if these are also the same, i.e. if the component charges differ in their shapes only, all the component charges burn out simultaneously and we get the interesting result that the form-function of the equivalent charge is

$$z = (1-f)(1+\theta f+\psi f^2), \qquad (20)$$

where

$$\theta = \frac{\sum_{i=1}^{n} C_i \theta_i}{\sum_{i=1}^{n} C_i}, \quad \psi = \frac{\sum_{i=1}^{n} C_i \psi_i}{\sum_{i=1}^{n} C_i} \qquad (21)$$

Thus if we have two charges of the same composition and ballistic size, one in the form of tubes and the other in the form of cords, a suitable choice of $\frac{C_1}{C_2}$ would enable us to get a composite charge which will behave as a single charge with any desired form-factor $\theta (0 \leqslant \theta \leqslant 1)$.

CASE II: *Charges with same effective ballistic sizes and same shapes but possibly with different compositions*

In this case (9) gives

$$f_1 = f_2 = \dots\dots = f_n = f,$$

and since the component charges have same shape

$$z_1 = z_2 = \dots\dots = z_n = z \qquad (22)$$

In this case (13), (14) and (15) give

$$\sum_{i=1}^{n} \frac{F_i C_i}{\gamma_i - 1} = \frac{FC}{\gamma - 1} \qquad (23)$$

$$\sum_{i=1}^{n} C_i \left(b_i - \frac{1}{\delta_i}\right) = C\left(b - \frac{1}{\delta}\right) \qquad (24)$$

$$\sum_{i=1}^{n} \frac{m_i C_i}{\gamma_i - 1} = \frac{1}{\gamma - 1} \sum_{i=1}^{n} m_i C_i \qquad (25)$$

Also from (16) and (24)

$$\sum_{i=1}^{n} b_i C_i = bC \qquad (26)$$

Thus in this case, the equivalent charge exists in the strict sense. For it we determine C from (17), γ from (25), F from (23), δ from (16), b from (26). Its shape is the same as that of each component charge and from (22) all the component charges and the equivalent charge burn out simultaneously.

CASE III: *When covolume correction terms and differences in ratios of specific heats for the component charges can be neglected*

In this case, we assume

$$\text{(i)}\ \gamma_1 = \gamma_2 = \ldots\ldots = \gamma_n \qquad (27a)$$

$$\text{(ii)}\ b_1 - \frac{1}{\delta_1} = b_2 - \frac{1}{\delta_2} = \ldots\ldots = b_n - \frac{1}{\delta_n} = 0 \qquad (27b)$$

(iii) pressure-index same for each component charge.

These are the usual assumptions for the solution of the equations of Internal Ballistics for composite charges [Venkatesan and Patni (1953), Kapur (1956*b*, *c*, 1957), Patni (1955), Aggarwal and Mehta (1955), Tawakley (1956)]. Subject to these assumptions equations (13) to (17) are satisfied if we take γ for the equivalent charge to be equal to the common value in (27*a*) and covolume for the equivalent charge equal to its specific volume.

Thus subject to the assumptions stated, an equivalent charge exists in the strict sense in this case also.

CASE IV: *When covolume correction terms are proportional to the force-constants and the differences in ratios of specific heats for the component charges can be neglected (Corner's case)*

Here we assume

$$\text{(i)}\ \gamma_1 = \gamma_2 = \ldots\ldots = \gamma_n \qquad (28a)$$

$$\text{(ii)}\ \frac{F_1}{b_1 - \frac{1}{\delta_1}} = \frac{F_2}{b_2 - \frac{1}{\delta_2}} = \ldots\ldots = \frac{F_n}{b_n - \frac{1}{\delta_n}} \qquad (28b)$$

In this case C is determined from (17), δ from (16), F from

$$C_1F_1 + C_2F_2 + \ldots\ldots + C_nF_n = C \qquad (29)$$

and γ is the common value in (28*a*).

$$\frac{F}{b - \frac{1}{\delta}}$$

is common value from (28*b*) and knowing F and δ, this determines b. The form-function is determined as in Part I. Thus, in this case also, an equivalent charge in the strict sense exists.

4. AVERAGE VALUES OF b AND γ

In the general case, it is not always possible to choose the form-functions so as to satisfy (13), (14) and (15) simultaneously. Since the variations in γ and $b - \frac{1}{\delta}$ are much less important than variations in F, we chose our form-function in Part I so as to satisfy (13). Our object in the present section is to choose such average values of b and γ so as to satisfy (14) and (15) as nearly as possible.

From (9) and (13), during the rth stage of burning,

$$z = A_r + B_r (1-f) - C_r (1-f)^2 + D_r (1-f)^3, \quad \ldots \quad (30)$$

where

$$\left.\begin{aligned} A_r &= 1 - \sum_{i=r}^{n} \lambda_i, \\ B_r &= \sum_{i=r}^{n} \lambda_i k_i (1+\theta_i+\psi_i) \\ C_r &= \sum_{i=r}^{n} \lambda_i k_i^2 (\theta_i+2\psi_i) \\ D_r &= \sum_{i=r}^{n} \lambda_i k_i^3 \psi_i \end{aligned}\right\} \quad \ldots \quad (31)$$

and

$$\left.\begin{aligned} \lambda_i &= \frac{F_i C_i/(\gamma_i-1)}{FC/(\gamma-1)} \\ k_i &= \frac{\beta_i'}{\beta'} \end{aligned}\right\} [i = 1, 2, \ldots . n] \quad \ldots \quad (32)$$

Similarly from (9) and (14)

$$z = A_r' + B_r' (1-f) - C_r' (1-f)^2 + D_r' (1-f)^3, \quad \ldots \quad (33)$$

where A_r', B_r', C_r', D_r' are obtained from A_r, B_r, C_r, D_r respectively by replacing λ_i by μ_i where

$$\mu_i = \frac{\left(b_i - \frac{1}{\delta_i}\right) C_i}{\left(b - \frac{1}{\delta}\right) C} [i = 1, 2, \ldots, n] \quad \ldots \quad (34)$$

Equations (30) and (33) are not always the same except when (28b) is satisfied. We choose $b - \frac{1}{\delta}$ so that the area under the curve (30) from $f = 1$ to $f = 0$ is the same as the area under the curve (31) between the same limits. This gives

$$\sum_{r=1}^{n} \int_{1-\frac{1}{k_{r-1}}}^{1-\frac{1}{k_r}} [A_r + B_r(1-f) - C_r(1-f)^2 + D_r(1-f)^3] \, df$$

$$= \sum_{r=1}^{n} \int_{1-\frac{1}{k_{r-1}}}^{1-\frac{1}{k_r}} [A_r' + B_r'(1-f) - C_r'(1-f)^2 + D_r'(1-f)^3] \, df. \quad (35)$$

Integrating, using the two lemmas proved in Part I, simplifying and remembering that

$$\frac{1}{k_0} = 0, \ \frac{1}{k_n} = 1,$$

we get

$$\left(b-\frac{1}{\delta}\right)\left[-\sum_{i=1}^{n}\frac{F_iC_i}{\gamma_i-1}+\frac{1}{2}\sum_{i=1}^{n}\frac{C_iF_i}{\gamma_i-1}\frac{1}{k_i}-\frac{1}{6}\sum_{i=1}^{n}\frac{C_iF_i}{\gamma_i-1}\frac{\theta_i}{k_i}+\frac{5}{12}\sum_{i=1}^{n}\frac{C_iF_i}{\gamma_i-1}\frac{\psi_i}{k_i}\right]$$

$$=\frac{F}{\gamma-1}\left[-\sum_{i=1}^{n}C_i\left(b_i-\frac{1}{\delta_r}\right)+\frac{1}{2}\sum_{i=1}^{n}\frac{C_i\left(b_i-\frac{1}{\delta_r}\right)}{k_i}-\frac{1}{6}\sum_{i=1}^{n}\frac{C_i\left(b_i-\frac{1}{\delta_r}\right)\theta_i}{k_i}\right.$$

$$\left.+\frac{5}{12}\sum_{i=1}^{n}\frac{C_i\left(b_i-\frac{1}{\delta_r}\right)\psi_i}{k_i}\right] \quad .. \quad .. \quad (36)$$

From (3), (9) and (15), we choose γ so that

$$\sum_{r=1}^{n}\int_{1-\frac{1}{k_{r-1}}}^{1-\frac{1}{k_r}}\left[A_r''+B_r''(1-f)-C_r''(1-f)^2+D_r''(1-f)^3\right]df$$

$$=\frac{1}{\gamma-1}\sum_{r=1}^{n}\int_{1-\frac{1}{k_{r-1}}}^{1-\frac{1}{k_r}}\left[A_r'''+B_r'''(1-f)-C_r'''(1-f)^2+D_r'''(1-f)^3\right]df \quad .. \quad (37)$$

where A_r'', B_r'', C_r'', D_r'' and A_r''', B_r''', C_r''', D_r''' are obtained from A_r, B_r, C_r, D_r by replacing λ_i by ν_i and ρ_i respectively, where

$$\nu_i = \frac{m_iC_i}{\gamma_i-1}, \ \rho_i = m_iC_i \ [i = 1, 2, \ldots; n] \quad .. \quad .. \quad (38)$$

Integrating, using the lemmas and simplifying, we get

$$\frac{1}{\gamma-1}\left[-\sum_{i=1}^{n}\rho_i+\frac{1}{2}\sum_{i=1}^{n}\frac{\rho_i}{k_i}-\frac{1}{6}\sum_{i=1}^{n}\frac{\rho_i\theta_i}{k_i}+\frac{5}{12}\sum_{i=1}^{n}\frac{\rho_i\psi_i}{k_i}\right]$$

$$=\left[-\sum_{i=1}^{n}\nu_i+\frac{1}{2}\sum_{i=1}^{n}\frac{\nu_i}{k_i}-\frac{1}{6}\sum_{i=1}^{n}\frac{\nu_i\theta_i}{k_i}+\frac{5}{12}\sum_{i=1}^{n}\frac{\nu_i\psi_i}{k_i}\right] \quad .. \quad .. \quad (39)$$

It is easily verified that in the four cases discussed in section 3, (36) and (39) confirm the results obtained there.

In the above analysis, we used Corner's method of fitting. If we want to use Clemmow's principle (which is applicable in the case of two component charges only) we choose $\left(b-\frac{1}{\delta}\right)$ so that

$$z = A_1'+B_1'(1-f)-C_1'(1-f)^2+D_1'(1-f)^3$$

passes through

$$f = 1-\frac{1}{k_1},\ z = A_1+\frac{B_1}{k_1}-\frac{C_1}{k_1^2}+\frac{D_1}{k_1^3},$$

giving

$$\left(b-\frac{1}{\delta}\right)\left[\frac{F_1C_1}{\gamma_1-1}+\frac{F_2C_2}{\gamma_2-1}\frac{1}{k_1}(1+\theta_2+\psi_2)-\frac{F_2C_2}{\gamma_2-1}\frac{1}{k_1^2}(\theta_2+2\psi_2)+\frac{F_2C_2}{\gamma_2-1}\frac{\psi_2}{k_1^3}\right]$$

$$= \frac{F}{\gamma-1}\left[\left(b_1-\frac{1}{\delta_1}\right)C_1+\left(b_2-\frac{1}{\delta_2}\right)C_2\frac{1}{k_1}(1+\theta_2+\psi_2)\right.$$

$$\left.-\left(b_2-\frac{1}{\delta_2}\right)C_2\frac{1}{k_1^2}(\theta_2+2\psi_2)+\left(b_2-\frac{1}{\delta_2}\right)C_2\frac{\psi_2}{k_1^3}\right] \quad \ldots \quad (40)$$

It is easily verified that it gives results consistent with those obtained in section 3.

We may point out here that just like the form-function (30), the $z-f$ relation (32) is also continuous throughout and in general (32) is discontinuous at the end of each stage of burning and would be continuous at the end of the rth stage if and only if $\theta_r-\psi_r = 1$; in particular this would be satisfied if the rth charge is in cord form.

However, (32) cannot be regarded as the form-function. When $f = 0$, (32) gives

$$z = \sum_{i=1}^{n}\mu_i,$$

which, in general, does not give $z = 1$. However, (32) will become identical with the form-function when either all the component charges have same composition or same ballistic sizes and shapes or when (28a) and (28b) are satisfied.

5. Solution of the equations in the general case

From (1), we get during the rth stage of burning

$$\frac{FC}{\gamma-1}[A_r+B_r(1-f)-C_r(1-f)^2+D_r(1-f)^3]$$

$$= p\left\{\left[A(x+l)-C\left(b-\frac{1}{\delta}\right)[A_r'+B_r'(1-f)-C_r'(1-f)^2+D_r'(1-f)^3]\right]\right.$$

$$\left.\times\left[\frac{A_r''+B_r''(1-f)-C_r''(1-f)^2+D_r''(1-f)^3}{A_r'''+B_r'''(1-f)-C_r'''(1-f)^2+D_r'''(1-f)^3}\right]\right\}$$

$$+\tfrac{1}{2}W_1v^2, \quad \ldots \quad (41)$$

where

$$Al = K_0-\frac{C_1}{\delta_1}-\frac{C_2}{\delta_2}-\ldots\ldots-\frac{C_n}{\delta_n}.$$

Assuming the rate of burning proportional to pressure, we get, on using the dynamical equation (8) and integrating,

$$v = \frac{AD}{W_1\beta}(1-f) \quad \ldots \quad (42)$$

and

$$p = -\frac{AD^2}{W_1\beta^2}(1-f)\frac{df}{dx} \qquad (43)$$

Substituting from (42), (43) in (41), we get an ordinary linear differential equation, the integration of which would give the relation between f (or v) and x. The solution is easily obtained and is not being written here for reasons of space. However, if we want to use a shot-start pressure assumption, i.e. if we assume that the shot does not start to move unless a certain pressure p_0 is built, (42) is replaced by

$$v = \frac{AD}{W_1\beta}(f_0-f) \qquad (44)$$

where f_0 is obtained from (41) by putting there $f = f_0$, $x = 0$, $p = p_0$, $v = 0$, if the shot starts moving in the rth stage and this stage is the first for which the value of f_0 obtained from this equation lies between 0 and 1.

For the particular case of the quadratic form-function, the differential equation reduces to the form

$$\frac{d\xi}{dv} - \frac{W_1\xi K_r''(a_r''-v)(b_r''+v)v}{K_rK_r'''(a_r-v)(b_r+v)(a_r'''-v)(b_r'''+v)}$$
$$= -W_1\frac{vK_r'(a_r'-v)(b_r'+v)K_r''(a_r''-v)(b_r''+v)}{K_r(a_r-v)(b_r+v)K_r'''(a_r'''-v)(b_r'''+v)}, \qquad (45)$$

where

$$\xi = 1+\frac{x}{l} \qquad (46)$$

(45) is an ordinary linear differential equation of the first order and can be easily integrated to give ξ as a function v.

The integration of (45) or the corresponding equation for the cubic form-function can be simplified in the following cases:

(i) When covolume correction terms are neglected, $K_r' = 0$ and (45) reduces to the form discussed by Aggarwal (1955).

(ii) When covolume correction terms are considered, but we assume $\gamma_1 = \gamma_2 = \ldots = \gamma_n$, (45) reduces to the form

$$\frac{d\xi}{dv} - \frac{W_1v\xi}{K_r(a_r-v)(b_r+v)} = \frac{-W_1vK_r'(a_r'-v)(b_r'+v)}{K_r(a_r-v)(b_r+v)}, \qquad (47)$$

an equation discussed by Aggarwal (1956).

The corresponding equation for the cubic form-function has been discussed by Venkatesan (1956).

(iii) When covolume terms are neglected and we also assume $\gamma_1 = \gamma_2 = \ldots = \gamma_n$, (45) reduces to the form

$$\frac{d\xi}{\xi} = \frac{W_1vdv}{K_r(a_r-v)(b_r+v)} \qquad (48)$$

the equation discussed by Venkatesan and Patni (1953).

For the cubic form-function the corresponding equation is

$$\frac{dx}{x+l} = -\frac{M(1-f)df}{[A_r+B_r(1-f)-[C_r+\frac{1}{2}(\gamma-1)M](1-f)^2+D_r(1-f)^3]}, \qquad (49)$$

where

$$M = \frac{A^2D^2}{W_1\beta^2FC}.$$

It has been discussed by Tawakley (1956) for the particular case of an Isothermal Model ($\gamma = 1$) when both the charges are spheres.

(iv) During the first stage $b_1 = 0$, $b_1' = 0$, $b_1'' = 0$, $b_1''' = 0$ and (45) gives

$$\frac{d\xi}{dv} - \frac{W_1\xi K_1''(a_1''-v)}{K_1K_1'''(a_1-v)(a_1''-v)} = \frac{-W_1vK_1'K_1''(a_1'-v)(a_1''-v)}{K_1K_1'''(a_1-v)(a_1'''-v)} \quad .. \quad (50)$$

(v) When all the effective ballistic sizes are equal, there is only one stage of burning and (50) holds for the entire period of burning.

(vi) For tubular charges, $C_r = 0$, $C_r' = 0$, $C_r'' = 0$, $C_r''' = 0$, and the equation is again considerably satisfied.

Maximum pressure and all-burnt position

$$p = \frac{K_rK_r'''(a_r-v)(b_r+v)(a_r'''-v)(b_r'''+v)}{[l\xi-K_r'(a_r'-v)(b_r'+v)][K_r''(a_r''-v)(b_r''+v)]} \quad .. \quad .. \quad (51)$$

Since ξ is a known function of v, p is determined as a function of v and thus maximum pressure can be easily determined. Even without finding the maximum pressure, we can discuss the conditions for the maximum pressure to occur in or at the end of any stage. Thus if we neglect the covolume correction terms, we get

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a_r-v}+\frac{1}{b_r+v}-\frac{1}{a_r'''-v}+\frac{1}{b_r'''+v}+\frac{1}{a_r''-v}-\frac{1}{b_r''+v}$$
$$-\frac{w_1vK_r''(a_r''-v)\,(b_r''+v)}{K_rK_r'''(a_r-v)\,(b_r+v)\,(a_r'''-v)\,(b_r'''+v)} \equiv P_r(v) \quad .. \quad .. \quad (52)$$

Thus the maximum pressure will occur in the rth stage if

$$\text{(i) } [P_r(v)]_{v=\frac{AD}{\beta W_1}\frac{1}{k_{r-1}}} > 0 \quad \text{and} \quad \text{(ii) } [P_r(v)]_{v=\frac{AD}{\beta W_1}\frac{1}{k_r}} < 0 \quad .. \quad .. \quad (53)$$

and it will occur at the end of the rth stage if

$$\text{(i) } [P_r(v)]_{v=\frac{AD}{\beta W_1}\frac{1}{k_r}} \geqslant 0 \quad \text{and} \quad \text{(ii) } [P_r(v)]_{v=\frac{AD}{\beta W_1}\frac{1}{k_r}} \leqslant 0 \quad .. \quad .. \quad (54)$$

where $\gamma_1 = \gamma_2 = \ldots = \gamma_n$, (52) reduces to

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a_r-v}+\frac{1}{b_r+v}-\frac{W_1v}{K_r(a_r-v)(b_r+v)} \quad .. \quad .. \quad (55)$$

This equation has been discussed by Kapur (1956*b*) and the uniqueness of maximum pressure has been established on its basis. At all-burnt position (42) gives

$$v_B = \frac{AD}{W_1\beta}.$$

Integration of (45) determines ξ at this instant and then (51) determines pressure at all-burnt.

In the earlier literature conditions corresponding to (54) have not, in general, been discussed.

6. Motion after all-burnt

After all-burnt equations (2) and (3) for the composite charge are no longer operative and equation (1) becomes

$$\sum_{i=1}^{n} \frac{F_i C_i}{\gamma_i - 1} = p \left\{ \left[K_0 - \sum_{i=1}^{n} \frac{C_i}{\delta_i} + Ax - \sum_{i=1}^{n} C_i \left(b_i - \frac{1}{\delta_i} \right) \right] \times \left[\frac{\sum_{i=1}^{n} \frac{m_i C_i}{\gamma_i - 1}}{\sum_{i=1}^{n} m_i C_i} \right] \right\} + \tfrac{1}{2} W_1 v^2 \quad \ldots \quad (52)$$

Similarly equation (5) for the equivalent charge becomes

$$\frac{FC}{\gamma - 1} = \frac{p}{\gamma - 1} \left[K_0 - \frac{C}{\delta} + Ax - C\left(b - \frac{1}{\delta} \right) \right] + \tfrac{1}{2}[1{\cdot}05W + \tfrac{1}{3}C]v^2 \quad \ldots \quad (53)$$

These two become identical if

$$\sum_{i=1}^{n} \frac{F_i C_i}{\gamma_i - 1} = \frac{FC}{\gamma - 1} \quad \ldots \quad (54)$$

$$\sum_{i=1}^{n} \frac{C_i}{\delta_i} = \frac{C}{\delta} \quad \ldots \quad (55)$$

$$\sum_{i=1}^{n} C_i \left(b_i - \frac{1}{\delta_i} \right) = C\left(b - \frac{1}{\delta} \right) \quad \ldots \quad (56)$$

$$\sum_{i=1}^{n} \frac{m_i C_i}{\gamma_i - 1} = \frac{1}{\gamma - 1} \sum_{i=1}^{n} m_i C_i \quad \ldots \quad (57)$$

$$C_1 + C_2 + \ldots + C_n = C \quad \ldots \quad (58)$$

From (55) and (56)

$$\sum_{i=1}^{n} C_i b_i = Cb \quad \ldots \quad (59)$$

Equations (4) and (8) also become identical in virtue of (58).

Thus after all-burnt an equivalent charge in the strict sense always exists. For it we determine C from (58), b from (59), F from (59), F from (54) and δ from (55).

7. Composite charges for linear law

The theory developed in Part I is applicable when the law of burning for the ith component charge is

$$D_i \frac{df_i}{dt} = -\beta_i p^{\alpha},$$

where α is the same for all the component charges. This last assumption holds when (i) all the component charges have same composition, (ii) when the rate of burning is assumed proportional to pressure for each component charge, (iii) we use the following pairs of component charges:

$$MD \text{ and } NFQ \text{ with } \alpha = 0{\cdot}91$$

$$W \text{ and } HSC \text{ with } \alpha = 0{\cdot}97$$

$$WM \text{ and } ASN \text{ with } \alpha = 1{\cdot}05$$

Another law of burning which has been found nearer the truth than the pressure-index law both on experimental and theoretical grounds is the linear law [(Mansell (1907), Dederick (1947, page 1001), Corner (1950, pages 44, 72, 206), Kapur (1956c)]

$$D_i \frac{df_i}{dt} = -\beta_i(p+p_0), \qquad \ldots \qquad (59)$$

for the ith component charge, where we assume p_0 to be the same for all the component charges. For this law also we get

$$\frac{D_1}{\beta_1}\frac{df_1}{dt} = \frac{D_2}{\beta_2}\frac{df_2}{dt} = \ldots\ldots = \frac{D_n}{\beta_n}\frac{df_n}{dt} = -(p+p_0),$$

so that on integration

$$\frac{1-f_1}{\beta_1'} = \frac{1-f_2}{\beta_2'} = \ldots\ldots = \frac{1-f_n}{\beta_n'} = \frac{1-f}{\beta'}, \qquad \ldots \qquad (60)$$

which is precisely the same equation as (9) for the pressure-index law of burning. Since the form-function for the equivalent charge was deduced from this equation, the form-function would be the same whether we use the pressure-index law (with same index) or the linear law (with same p_0). This theory will be of particular interest when all the component charges have the same composition.

Conditions for simultaneous and non-simultaneous burning out of composite charges :

If, however, the rates of burning equations are:

$$D_i \frac{df_i}{dt} = -\beta_i(p+p_{i0}), \qquad \ldots \qquad (61)$$

we get

$$\frac{D_1}{\beta_1}\frac{df_1}{dt}+p_{10} = \frac{D_2}{\beta_2}\frac{df_2}{dt}+p_{20} = \ldots\ldots = \frac{D_n}{\beta_n}\frac{df_n}{dt}+p_{n0}$$

$$= \frac{D}{\beta}\frac{df}{dt}+p_0 = -p, \qquad \ldots \qquad (62)$$

so that integrating and remembering that initially

$$t = 0,\ f_1 = 1,\ f_2 = 1, \ldots\ldots, f_n = 1,\ f = 1, \text{ we get}$$

$$\frac{1-f_1}{\beta_1'}+p_{10}t = \frac{1-f_2}{\beta_2'}+p_{20}t+\ldots\ldots = \frac{1-f_n}{\beta_n'}+p_{n0}t = \frac{1-f}{\beta'}+p_0t \qquad \ldots \qquad (63)$$

This gives when $f_1 = 0$,

$$\frac{f_2}{\beta_2'} = \frac{1}{\beta_2'} - \frac{1}{\beta_1'} + (p_{20} - p_{10}t_1) \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

(i) If $\beta_1' > \beta_2'$ and $p_{20} \geqslant p_{10}$ or if $\beta_1' > \beta_2'$ and $p_{20} > p_{10}$, f_2 is positive when f_1 is zero and so charge number 1 burns out before charge number 2.

(ii) If $\beta_1' = \beta_2'$ and $p_{20} = p_{10}$, the charges burn out simultaneously. In general,

if $\qquad \beta_1' > \beta_2' > \ldots . > \beta_n'$ and $p_{10} \leqslant p_{20} \leqslant \ldots . \leqslant p_{n0}$,

or if $\qquad \beta_1' \geqslant \beta_2' \geqslant \ldots . \geqslant \beta_n'$ and $p_{10} < p_{20} < \ldots . < p_{n0}$

charge number 1 burns out first, charge number 2 burns out next and the last to burn out is charge number nth.

(iii) If $\beta_1' = \beta_2' = \ldots . = \beta_n'$ and $p_{10} = p_{20} = \ldots = p_{n0}$, all the charges burn out simultaneously.

Actually we can always arrange the charges so that

$$p_{10} \leqslant p_{20} \leqslant \ldots . \leqslant p_{n0},$$

and then if we want the charges to burn out in this order, we should so arrange the ballistic sizes of the component charges that the component charge with the smaller p_{i0} has also the smaller effective ballistic size.

8. Acknowledgements

The author is grateful to Professor P. L. Bhatnagar, F.N.I., for his useful advice and to Dr. R. S. Varma, F.N.I., for his encouragement.

Summary

In the present paper

(i) The existence of the equivalent charge in the strict sense has been defined and conditions have been obtained for such existence. In particular it has been shown that for motion before burnt, an equivalent charge in the strict sense always exists, if (*a*) all the component charges have same composition, (*b*) they have the same shapes and effective ballistic sizes, (*c*) if covolume correction terms and variations in γ_i's can be neglected.

(ii) For motion after all-burnt it has been shown that whatever the shapes, sizes and compositions of the component charges, an equivalent charge in the strict sense always exists.

(iii) In the most general case, satisfactory average values of b and γ have been obtained.

(iv) A method has been given for integrating the equations of Internal Ballistics in the most general case and it has been shown that most of the cases of integration discussed so far for a composite charge consisting of two component charges are particular cases of the system considered here, for here we consider (*a*) general values of n, (*b*) covolume correction terms are taken into account, (*c*) take variation of γ_i's into account.

(v) The Theory of Composite Charges has been extended to the theoretically and experimentally more plausible linear law of burning.

References

Aggarwal, S. P. (1955). Internal ballistics of composite charges. *PNISIPS*, **21** A, 321–330.

Aggarwal, S. P. (1956). Internal ballistics of composite charges taking covolume terms into account.

Aggarwal, S. P., and Mehta, A. K. (1955). On ballistics of composite charges for power law of burning. *PNISIPS*, **21** A, 55–62.

Clemmow, C. A. (1920). General Theory of Composite Charges. R.D. 4. H.M.S.O.
——— (1951). Internal Ballistics. H.M.S.O. Chap. XI.
Corner, J. (1950). Theory of Internal Ballistics of Guns. John Wiley & Sons, New York.
Dederick, L. S. (1947). 'Ballistics', Encyclopaedia Britannica, p. 1001.
Kapur, J. N. (1956*a*). Equivalent charge method in the general theory of composite charges. *PNISIPS*, **22** A, 63–81.
——— (1956*b*). Uniqueness of maximum pressure in the general theory of composite charges. *PNISIPS*, **22** A, 236–255.
——— (1956*c*). Uniqueness of maximum pressure for orthodox, H/L and leaking guns. *Proc. First Cong. Th. App. Mech.*, 227–238.
——— (1956*d*). Internal ballistics of composite charges for power law of burning. *PNISIPS*, **23** A, 16–39.
——— (1956*e*). Solution of the equations of internal ballistics when the rate of burning is a linear function of the pressure. *Trans. Nat. Inst. Sci. India*, **3**, No. **6**, 257–290.
——— (1957). General theory of composite charges. Mem. artill. franc (Paris). I. (*In press.*)
Patni, G. C. (1955). Internal ballistics of composite charges for non-linear law of burning. *PNISIPS*, **21** A, 104–119.
Patni, G. C., and Venkatesan, N. S. (1953). Influence of composite charges on maximum pressure and muzzle velocity. *Def. Sci. Jour.*, 51–60.
Tawakley, V. B. (1956). Internal ballistics of composite charges when they burn according to the form-function for spheres. *PNISIPS*, **22** A, 54–62.
Venkatesan, N. S. (1956). Internal ballistics of composite charges in the cubic form-function. *Proc. 2nd Cong. Th. App. Mech.* (*In press.*)

Issued December 14, 1957.