ON THE AVERAGE NUMBER OF SUMMANDS IN PARTITION OF n

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1. Let $p_k(n)$ denote the number of partitions of n into exactly k positive integral parts, and $q_k(n)$ denote the number of partitions of n into exactly k positive unequal parts. Szekeres (1953) has determined the asymptotic behaviour of $p_k(n)$ for arbitrarily large value of k, and has proved that it possesses a unique maximum given by

$$k_0 = CN^{\frac{1}{2}}L - C^2\left(\tfrac{1}{4}L^2 - \tfrac{3}{2}L - \tfrac{3}{2}\right) - \tfrac{1}{2} + O(N^{-\frac{1}{2}} \log^4 N) \quad \ldots \quad \ldots \quad (1)$$

where

$$C = 6^{\frac{1}{2}}/\pi,\ L = \log\,(CN^{\frac{1}{2}}) \text{ and } N = \left(n - \frac{1}{24}\right).$$

He has also shown that $q_k(n)$ has a unique maximum for $k = k_1$, where k_1 is given by

$$k_1 = (2^{\frac{1}{2}} \log 2)CM^{\frac{1}{2}} + 2b(\log 2)^{-1} - \frac{\frac{1}{2}b}{1-2b} - 1 + O(M^{-\frac{1}{2}}) \quad \ldots \quad \ldots \quad (2)$$

with $b = c^2(\log 2)^2$ and $M = \left(n + \dfrac{1}{24}\right)$.

This proves a conjecture of Auluck, Chawla and Gupta (1942) on the uniqueness of the maximum of $p_k(n)$.

In this note we investigate $\bar{k}$, the average number of summands in a partition of n ; we prove that

$$\bar{k} = CN^{\frac{1}{2}}(L+\gamma) + \frac{c^2}{2}(L+\gamma) + \frac{1+c^2}{4} + O(N^{-\frac{1}{2}} \log N) \quad \ldots \quad \ldots \quad (3)$$

where γ is Euler's constant.*

If the summands are all unequal, the average number of summands in a partition of n becomes

$$\bar{k}_1 = (2^{\frac{1}{2}} \log 2)CM^{\frac{1}{2}} + \tfrac{1}{2}C^2 \log 2 - \tfrac{1}{4} + O(M^{-\frac{1}{2}}). \quad \ldots \quad \ldots \quad (4)$$

It may be noticed that the difference $k - k_0$ increases asymptotically as $n^{\frac{1}{2}}$ and that

$$\frac{\bar{k} - k_0}{k_0} \sim \gamma/\log\left(\frac{\sqrt{6N}}{\pi}\right).$$

However, in the case when all the summands are different, the difference between k_1 and $\bar{k}_1$ is less than unity for large values of n.

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* The value of k has also been calculated by Kodi Husani (1938). Our method is, however, different.

An application of Szekeres' result (and also the expression for $\bar{k}$ as given by the equation (3)) has been considered by Auluck and Kothari (1954) in connection with the problem of pion production during high energy collision of two nucleons.

In order to study the distribution functions $p_k(n)$ and $q_k(n)$ we have calculated in §4 the second, third and fourth moments when the summands are unrestricted and also when they are all different.

When the summands are unrestricted, we find that the second moment for $p_k(n)$ is given by

$$\mu_2 = N - \frac{C^3 N^{\frac{1}{2}}}{2}[(\gamma+L)^2+2(\gamma+L)+1] - \tfrac{1}{4}C^2 - \frac{C^4}{4}[(\gamma+L)^2+(\gamma+L)+\tfrac{1}{4}] - \frac{\gamma+L}{72} + O(N^{-\frac{1}{2}}) \quad \text{.. (5)}$$

and that the third moment is given by

$$\mu_3 \sim 2\zeta(3)C^3N^{\frac{3}{2}} - 3C^2N(\gamma+L+1) + O(N^{\frac{1}{2}}). \quad \text{.. .. (6)}$$

For the fourth moment we have

$$\mu_4 \sim C^4N^2\left[6\zeta(4)+\frac{3}{C^4}\right] + O(N^{\frac{3}{2}}). \quad \text{.. .. .. (7)}$$

Similarly, when the summands are all different, the second, third and the fourth moments for $q_k(n)$ are respectively given by

$$\bar{\mu}_2 = \frac{C^2}{4} + \frac{CM^{\frac{1}{2}}}{2^{\frac{1}{2}}} - \tfrac{1}{4}C^4(\log 2)^2 - 2^{\frac{1}{2}}C^3M^{\frac{1}{2}}(\log 2)^2 + O(M^{-\frac{1}{2}}), \quad \text{.. (8)}$$

and

$$\bar{\mu}_3 \sim CM^{\frac{1}{2}}\left[\frac{2^{\frac{1}{2}}}{4} + \frac{3\,.\,2^{\frac{1}{2}}}{2}C^4(\log 2)^3 - \frac{3\,.\,2^{\frac{1}{2}}C^2\log 2}{2}\right], \quad \text{.. .. (9)}$$

and

$$\bar{\mu}_4 \sim C^2M[\tfrac{3}{2} - 6C^2(\log 2)^2] + O(M^{\frac{1}{2}}). \quad \text{.. .. .. (10)}$$

We have also calculated $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ which gives the degree of departure from symmetry. In a symmetrical distribution, β_1 is zero, but for the distribution $p_k(n)$, β_1 for large values of n tends to the value $[2\zeta(3)]^2C^6 = 1{\cdot}2$ nearly; and for the distribution $q_k(n)$ we have

$$\bar{\beta}_1 \sim \frac{1}{2^{\frac{3}{2}}CM^{\frac{1}{2}}} \to 0$$

when $n \to \infty$.

Further, the kurtosis β_2, the ratio of the fourth moment about the mean of distribution to the square of the second moment, is also calculated. In a normal distribution the kurtosis has the value 3. It is interesting to notice that for the distribution $p_k(n)$ and for large values of n,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \sim C^4\left[6\zeta(4)+\frac{3}{C^4}\right] = 5{\cdot}4\,;$$

but when the partitions are unrestricted $\bar{\beta}_2 \sim 3$.

In the appendix, we have tabulated the values of k_0, $\bar{k}$; k_1, $\bar{k}_1$; μ_2, $\bar{\mu}_2$; μ_3, $\bar{\mu}_3$; μ_4, $\bar{\mu}_4$; β_1, $\bar{\beta}_1$; β_2, $\bar{\beta}_2$, for values of n up to 100. These values have been calculated from the tables of partitions of $p_k(n)$ by Y. R. Bhalotra.

2. We define the average value $\bar{k}$ as

$$\bar{k} = \sum_{k=1}^{n} kp_k(n) \Big| \sum_{k=1}^{n} p_k(n) = \frac{1}{p(n)} \sum_{k=1}^{n} kp_k(n) \qquad \ldots \qquad \ldots \quad (11)$$

where $p(n)$ represents the total number of unrestricted partitions of n into positive integral summands. Consider the identity

$$\frac{1}{(1-Zx)(1-Zx^2)(1-Zx^3)\ldots\ldots} = \sum_{=0}^{\infty} \frac{Z^k x^k}{(1-x)(1-x^2)\ldots\ldots(1-x^k)}.$$

Differentiating with respect to Z, we obtain

$$\frac{1}{(1-Zx)(1-Zx^2)(1-Zx^3)\ldots\ldots} \left\{ \frac{x}{1-Zx} + \frac{x^2}{1-Zx^2} + \frac{x^3}{1-Zx^3} + \ldots\ldots \right\}$$

$$= \sum_{k=1}^{\infty} \frac{kZ^{k-1}x^k}{(1-x)(1-x^2)\ldots\ldots(1-x^k)} \quad \ldots \quad (12)$$

which gives, on putting $Z = 1$, the relation

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots} \left\{ \frac{x}{1-x} + \frac{x^2}{1-x^2} + \frac{x^3}{1-x^3} + \ldots\ldots \right\}$$

$$= \sum_{k=1}^{\infty} \frac{kx^k}{(1-x)(1-x^2)\ldots\ldots(1-x^k)} \qquad \ldots \quad (13)$$

since

$$\frac{x^k}{(1-x)(1-x^2)\ldots\ldots(1-x^k)} = \sum_{n=k}^{\infty} p_k(n)x^n$$

we have, by picking out the coefficient of x^n in (13),

$$\sum_{k=1}^{n} kp_k(n) = \text{Coefficient of } x^n \text{ in } \prod_{r=1}^{\infty} (1-x^r)^{-1} \sum_{s=1}^{\infty} \frac{x^s}{1-x^s}$$

$$= \text{Coefficient of } x^n \text{ in } \sum_{m=1}^{\infty} p(m)x^m \sum_{r=1}^{\infty} d(r)x^r$$

where $d(r)$ is the number of divisors of r. For example,

$$d(1) = 1, \qquad d(2) = 2, \qquad d(3) = 2, \qquad d(4) = 3.$$

Then we have

$$\sum_{k=1}^{n} kp_k(n) = \sum_{r=1}^{n} d(r)\, p(n-r) \qquad \ldots \qquad \ldots \qquad \ldots \quad (14)$$

and

$$\bar{k} = \sum_{r=1}^{n} d(r) \frac{p(n-r)}{p(n)}$$

$$= \left[\sum_{r=1}^{[n]} + \sum_{[n^{\frac{2}{3}}]+1}^{n-1} + \sum_{n=1}^{n} \right] \frac{d(r)p(n-r)}{p(n)}$$

$$= I_1 + I_2 + \frac{d(n)}{p(n)}$$

where $\frac{d(n)}{p(n)}$ is negligible. We have to consider I_1 and I_2. Before considering I_1 and I_2, let us find $\frac{p(n-r)}{p(n)}$.

We know

$$p(n) \approx \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{\frac{2}{3}N}}}{N^{\frac{1}{2}}} \right) \quad \text{where} \quad N = \left(n - \frac{1}{24}\right)$$

$$= \frac{e^{\pi\sqrt{\frac{2}{3}N}}}{4\pi\sqrt{2N}} \left[\pi\sqrt{\tfrac{2}{3}} - \frac{1}{N^{\frac{1}{2}}} \right].$$

For $r > n^{\frac{2}{3}}$, we have

$$\frac{p(n-r)}{p(n)} = \frac{N}{N-r} e^{\pi\sqrt{\frac{2}{3}}[(N-r)^{\frac{1}{2}} - N^{\frac{1}{2}}]} \cdot \frac{\pi\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{N-r}}}{\pi\sqrt{\frac{2}{3}} - \frac{1}{\sqrt{N}}}$$

$$= \left(1 + \frac{r}{N} + \frac{r^2}{N^2} + \cdots\right) e^{\pi\sqrt{\frac{2}{3}}N^{\frac{1}{2}}\left\{1 - \frac{r}{2N} - \frac{1}{8}\frac{r^2}{N^2} \cdots - 1\right\}}$$

$$= \left(1 + \frac{r}{N} + \frac{r^2}{N^2} + \cdots\right) \exp\left\{-\frac{rk}{2N^{\frac{1}{2}}}\right\} \exp\left\{-\frac{kr^2}{8N^{\frac{3}{2}}}\right\}$$

$$= e^{-\frac{kr}{2N^{\frac{1}{2}}}} \left(1 + \frac{r}{N}\right) \left(1 - \frac{kr^2}{8N^{\frac{3}{2}}}\right) + O\left(\frac{1}{N^2}\right)$$

$$= e^{-\frac{kr}{2N^{\frac{1}{2}}}} \left\{ 1 + \frac{r}{N} - \frac{k}{8} \frac{r^2}{N^{\frac{3}{2}}} + O\left(\frac{r^3}{N^{\frac{5}{2}}}\right) \right\}$$

where $k = \pi\sqrt{\frac{2}{3}}$.

$$\text{Now consider } I_2 = \sum_{[n^{\frac{2}{3}}]+1}^{n-1} d(r) \frac{N}{N-r} \exp\left\{ \pi\sqrt{\tfrac{2}{3}} \left(\sqrt{N-r} - \sqrt{N}\right) \right\}$$

$$< \sum_{[n^{\frac{2}{3}}]+1}^{n-1} N \,.\, d(r) \exp\left\{ -\pi\sqrt{\tfrac{2}{3}} \frac{r}{\sqrt{N-r} + \sqrt{N}} \right\}$$

$$< \sum_{[n^{\frac{2}{3}}]+1}^{n} N \,.\; d(r) \exp\left(-\frac{\pi}{\sqrt{6N}} \,.\, r\right)$$

$$= O\left(N^2 e^{-\frac{\pi N^{\frac{1}{6}}}{\sqrt{6}}}\right) = O(1)$$

We now consider the range $1 \leqslant r \leqslant [n^{\frac{2}{3}}]$

$$p(n) \sim \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \left(\frac{e^{\pi\sqrt{\frac{2}{3}N}}}{N^{\frac{1}{2}}}\right) \text{ where } N = n - \frac{1}{24}$$

$$= \frac{e^{\pi\sqrt{\frac{2N}{3}}}}{4\sqrt{3}N} \left[1 + O\left(N^{-\frac{1}{2}}\right)\right]$$

$$\therefore \; I_1 = \sum_{r=1}^{[n^{\frac{2}{3}}]} d(r) \exp\left(-\frac{\pi}{\sqrt{6N}} r\right)$$

$$= \sum_{r=1}^{\infty} d(r) \exp\left(-\frac{\pi r}{\sqrt{6N}}\right) - \sum_{[n^{\frac{2}{3}}]+1}^{\infty} d(r) \left(-\frac{\pi r}{\sqrt{6N}}\right)$$

$$= \sum_{r=1}^{\infty} d(r)\, x^r - \sum_{[n^{\frac{2}{3}}]+1}^{\infty} d(r)\, x^r \text{ where } x = e^{-\frac{\pi}{\sqrt{6N}}}$$

$$= I_4 - I_5 \,.$$

Consider

$$I_5 = \sum_{[n^{\frac{2}{3}}]+1}^{\infty} d(r) e^{-\frac{\pi r}{\sqrt{6N}}} = \sum_{m+1}^{\infty} d(r) e^{-\frac{\pi r}{\sqrt{6N}}}$$

$$= e^{-\frac{\pi N^{\frac{1}{6}}}{2\sqrt{6}}} \sum_{m+1}^{\infty} d(r) e^{-\frac{\pi}{\sqrt{6N}}\left(r - \frac{N^{\frac{2}{3}}}{2}\right)}$$

$$< e^{-\frac{\pi}{2\sqrt{6}} N^{\frac{1}{6}}} \sum_{1}^{\infty} d(r) e^{-\frac{\pi r}{2\sqrt{6N}}} = O(1)$$

where $m = [n^{\frac{2}{3}}]$. The value of the series is of the order of $N^{\frac{1}{2}}$ as is shown below.
Consider

$$I_4 = \sum_{r=1}^{\infty} d(r)\, x^r = \sum_{s=1}^{\infty} \frac{x^s}{1-x^s} \,.$$

Let

$$x = e^{-\lambda} = e^{-\frac{k}{2N^{\frac{1}{2}}}} .$$

$$\sum_{r=1}^{\infty} d(r) e^{-r\lambda} = \sum_{r=1}^{\infty} \frac{x^r}{1-x^r} = \sum_{r=1}^{\infty} \frac{1}{e^{r\lambda}-1}$$

$$= \frac{1}{\lambda}(\gamma - \log \lambda) + \frac{1}{4} + O(\lambda)$$

$$= \frac{2N^{\frac{1}{2}}}{k}\left(\gamma + \frac{\log 2N^{\frac{1}{2}}}{k}\right) + \frac{1}{4} + O\left(\frac{k}{2N^{\frac{1}{2}}}\right). \quad .. \quad (15)$$

Differentiating (15) with respect to λ,

$$\sum_{r=1}^{\infty} d(r) e^{-\lambda r}(-r) = -\frac{\gamma}{\lambda^2} + \frac{\log \lambda}{\lambda^2} - \frac{1}{\lambda^2} + O(1)$$

or

$$\sum_{r=1}^{\infty} d(r) e^{-\lambda r}\left(\frac{r}{N}\right) = \left(\frac{\gamma+1}{\lambda^2} - \frac{\log \lambda}{\lambda^2}\right)\frac{1}{N} + o(1) \quad .. \quad .. \quad .. \quad (16)$$

Differentiating again, with respect to λ,

$$\sum_{r=1}^{\infty} d(r) e^{-\lambda r} r^2 = [(2\gamma+3) - 2\log \lambda]\frac{1}{\lambda^3} + o(1). \quad .. \quad .. \quad .. \quad (17)$$

Adding (15), (16) and (17)

$$\bar{k} = 2N^{\frac{1}{2}}\left(\gamma + \log \frac{2N^{\frac{1}{2}}}{k}\right) + \frac{1}{4} + o\left(\frac{1}{N^{\frac{1}{2}}}\right)$$

$$+ \frac{(\gamma+1)4}{k^2} + \frac{4}{k^2}\log \frac{2N^{\frac{1}{2}}}{k} + o(1)$$

$$+ \left\{\frac{2\log k}{2N^{\frac{1}{2}}} - (2\gamma+3)\right\}\frac{8N^{\frac{3}{2}}k}{8N^{\frac{3}{2}}k^3} + \ldots .$$

Substituting $\frac{2}{k} = C$ and $CN^{\frac{1}{2}} = L$ and $\left(n - \frac{1}{24}\right) = N$

$$\bar{k} = CN^{\frac{1}{2}}(\gamma + L) + \frac{1}{4} + o\left(\frac{1}{N^{\frac{1}{2}}}\right) + (\gamma+1)C^2 + C^2 L + o(1)$$

$$- \frac{C^2 L}{2} - \frac{1}{2}C^2\gamma - \frac{3}{4}C^2$$

$$= CN^{\frac{1}{2}}(L+\gamma) + \frac{1+C^2}{4} + \frac{C^2}{2}(\gamma + L) + o\left(\frac{1}{N^{\frac{1}{2}}}\right). \quad .. \quad .. \quad .. \quad (3)$$

3. We now consider the partitions into summands which are all different. We start with the identity

$$(1+Zx)(1+Zx^2)(1+Zx^3)\ldots . = \sum_{k=0}^{\infty} \frac{Z^k x^{\frac{k(k+1)}{2}}}{(1-x)(1-x^2)\ldots .(1-x^k)} \quad .. \quad (18)$$

and differentiate it with respect to Z and put $Z = 1$, we obtain

$$\prod_{r=1}^{\infty}(1+x^r)\left(\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \ldots .\right)$$

$$= \sum_{k=1}^{\infty} \frac{k x^{\frac{k(k+1)}{2}}}{(1-x)(1-x^2)\ldots .(1-x^k)}, \quad .. \quad .. \quad .. \quad (19)$$

comparing coefficients of x^n, we get

$$\sum_{k=1}^{n} kq_k(n) = \prod_{r=1}^{\infty} (1+x^r) \sum_{m=1}^{\infty} \frac{x^m}{1+x^m}$$

$$= \sum_{s=1}^{\infty} q(s)x^s \sum_{r=1}^{n} D(r)x^r \quad \ldots \quad (20)$$

where $M = n+\frac{1}{24}$ and $D(r)$ is the coefficient of x^r in $\sum_{m=1}^{\infty} \frac{x^m}{1+x^m}$

For example $D(1) = 1$, $D(2) = 0$, $D(3) = 1$, $D(4) = -2$.
Also

$$\bar{k}_1 = \sum_{k=1}^{n} \frac{kq_k(n)}{q(n)} = \sum_{r=1}^{n} D(r)q(n-r)$$

$$= \sum_{r=1}^{n} D(r) \exp\left(-\frac{\pi r}{\sqrt{12M}}\right)\left(1+\frac{3r}{4M}-\frac{kr^2}{8\sqrt{2}M^{\frac{3}{2}}}+\ldots\right)$$

In (19) for $\sum_{r=0}^{\infty} D(r)x^r = \sum_{m=1}^{\infty} \frac{x^m}{1+x^m}$, making the transformation $x = e^{-\lambda} = e^{-\pi/\sqrt{12M}}$

as before,

$$\sum_{r=0}^{\infty} D(r)e^{-\lambda r} = \sum_{m=1}^{\infty} \frac{e^{-\lambda m}}{1+e^{-\lambda m}} = \sum_{m=1}^{\infty} \frac{1}{1+e^{\lambda m}} = \log\frac{2}{\lambda}-\frac{1}{4}+o(\lambda). \quad \ldots \quad (21)$$

Differentiating, $\sum_{r=0}^{\infty} D(r)e^{-\lambda r}(-r) = -\frac{\log 2}{\lambda^2}+o(\lambda)$

or

$$\sum_{r=0}^{\infty} D(-r)e^{-\lambda r}\cdot\frac{3r}{4M} = \frac{3}{4}\frac{\log 2}{\lambda^2 M} - o(\lambda)$$

Differentiating again,

$$\sum_{r=0}^{\infty} D(r)e^{-\lambda r}r^2 = \frac{2\log 2}{\lambda^3}+o(\lambda)$$

or

$$\sum_{r=0}^{\infty} D(r)e^{-\lambda r}r^2\left(-\frac{k}{8\sqrt{2}M^{\frac{3}{2}}}\right) = -\frac{\log 2}{\lambda^3}\cdot\frac{k}{8\sqrt{2}M^{\frac{3}{2}}}.$$

Adding and using $M = n+\frac{1}{24}$,

$$\bar{k}_1 = \frac{\log 2}{\lambda}-\frac{1}{4}+\frac{3}{4}\frac{\log 2}{\lambda^2 M}-\frac{2\log 2}{\lambda^3}\cdot\frac{k}{8\sqrt{2}M^{\frac{3}{2}}}+o(\lambda)$$

$$= \sqrt{2}CM^{\frac{1}{2}}\log 2-\tfrac{1}{4}+\tfrac{1}{k}C^2\log 2+o(M^{-\frac{1}{2}}). \quad \ldots \quad (4)$$

7. We define $\Sigma k^2 p_k(n)$ = coefficient of x^n in

$$\sum_{m=1}^{\infty} p(m)x^m \sum_{r=1}^{\infty} d(r)x^r.$$

In order to calculate k^{-2}, we differentiate (12) once again, with respect to Z,

$$\frac{1}{(1-Zx)(1-Zx^2)(1-Zx^3)\ldots.}\left[\left\{\frac{x}{1-Zx}+\frac{x^2}{1-Zx^2}+\ldots.\right\}^2+\left\{\frac{x^2}{(1-xz)^2}+\frac{x^4}{(1-Zx^2)^2}+\right.\right.$$

$$\left.\left.+\frac{x^6}{(1-Zx^3)^2}+\ldots.\right\}\right]=\sum_{k=1}^{\infty}\frac{k(k-1)Z^{k-2}x^k}{(1-x)(1-x^2)\ldots.(1-x^k)}. \qquad \ldots \quad \ldots \quad (22)$$

Put $Z = 1$, as before

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots.}\left[\left\{\frac{x}{1-x}+\frac{x^2}{1-x^2}+\ldots.\right\}^2+\left\{\frac{x^2}{(1-x)^2}+\frac{x^4}{(1-x^2)^2}+\right.\right.$$

$$\left.\left.+\frac{x^6}{(1-x^3)^2}+\ldots.\right\}\right]=\sum_{k=1}^{\infty}\frac{k^2x^k}{(1-x)(1-x^2)\ldots.(1-x^k)}-$$

$$-\sum_{k=1}^{\infty}\frac{kx^k}{(1-x)(1-x^2)\ldots.(1-k^k)}$$

or coefficient of x^n in

$$\prod_{r=1}^{\infty}(1-x^r)^{-1}\left[\left\{\sum_{s=1}^{\infty}\frac{x^s}{1-x^s}\right\}^2+\sum_{s=1}^{\infty}\frac{x^{2s}}{(1-x^s)^2}\right]=\sum_{k=1}^{\infty}k^2p_k(n)-\sum_{k=1}^{\infty}kp_k(n)$$

or $\sum_{k=1}^{n} k^2 p_k(n)$ = coefficient of x^n in

$$\prod_{r=1}^{\infty}(1-x^r)^{-1}\left[\left\{\sum_{s=1}^{\infty}\frac{x^s}{1-x^s}\right\}^2+\sum_{s=1}^{\infty}\frac{x^{2s}}{(1-x^s)^2}+\sum_{s=1}^{\infty}\frac{x^s}{1-x^s}\right]. \qquad \ldots \quad (23)$$

Applying the same argument as in §6, we consider

$$\sum_{r=1}^{\infty} d(r)x^r=\left\{\sum_{s=1}^{\infty}\frac{x^s}{1-x^s}\right\}^2+\sum_{s=1}\frac{x^{2s}}{(1-x^s)^2}+\sum_{s=1}^{\infty}\frac{x^s}{1-x^s}.$$

Let $x = e^{-\lambda} = e^{-\frac{k}{2N^{\frac{1}{2}}}}$ as before

$$\sum_{r=1}^{\infty} d(r)e^{-r\lambda}=\left(\sum_{r=1}^{\infty}\frac{1}{e^{r\lambda-1}}\right)^2+\sum_{r=1}^{\infty}\frac{r}{e^{r\lambda-1}}$$

$$=\left\{\frac{1}{\lambda}(\gamma-\log\lambda)+\frac{1}{4}+o(\lambda)\right\}^2+\frac{\Pi^2}{6\lambda^2}-\frac{1}{2\lambda}$$

from (15) and proof of $\sum_{r=1}^{\infty}\frac{r}{e^{r\lambda}-1}=\frac{\Pi^2}{6\lambda^2}-\frac{1}{2\lambda}$ is given in the appendix or

$$\sum_{r=1}^{\infty} d(r)e^{-r\lambda} = \frac{1}{\lambda^2}\left[(\gamma^2-2\gamma\log\lambda)+(\log\lambda)^2+\frac{\pi^2}{6}\right]+\frac{1}{2\lambda}(\gamma-\log\lambda-1)+\frac{1}{16}-\frac{\gamma}{72}+$$
$$+\frac{\log\lambda}{72}-\frac{\lambda}{288}+o(\lambda^2). \quad .. \quad (24)$$

Differentiating (24) as before and adding the values of

$$\sum_{r=1}^{\infty} d(r)e^{-r\lambda}, \quad \sum_{r=1}^{\infty} d(r)e^{-r\lambda}\left(\frac{r}{N}\right), \quad \sum_{r=1}^{\infty} d(r)e^{-r\lambda}\left(-\frac{r^2k}{8N^{\frac{3}{2}}}\right)$$

and using previous notation

$$\bar{k}^2 = C^2N[(\gamma+L)^2]+N+\frac{C^3N^{\frac{1}{2}}}{2}[(\gamma+L)^2-(\gamma+L)-1]+\frac{CN^{\frac{1}{2}}(\gamma+L)}{2}$$
$$+\frac{C^2(\gamma+L)}{4}-\frac{C^2}{8}+\frac{1}{16}-\frac{\gamma+L}{72}+o(N^{-\frac{1}{2}}). \quad .. \quad (25)$$

Therefore, second moment $\mu_2 = (\bar{k})^2-\bar{k}^2$

or

$$\mu_2 = N-\frac{C^3N^{\frac{1}{2}}}{2}[(\gamma+L)^2+2(\gamma+L)+1]-\frac{C^2}{4}-\frac{\gamma+L}{72}$$
$$-\frac{C^4}{4}\left[(\gamma+L)^2+(\gamma+L)+\frac{1}{4}\right]+o(N^{-\frac{1}{2}}) \cdot \quad .. \quad (5)$$

In the case, when the partitions into summands are different, we differentiate (19), with respect to Z (before putting it equal to one), once again, then

$$(1+Zx)(1+Zx^2)\ldots\left[\left(\frac{x}{1+Zx}+\frac{x^2}{1+Zx^2}+\ldots\right)^2-\frac{x^2}{(1+Zx)^2}-\frac{x^4}{(1+Zx^2)^2}\right.$$
$$\left.-\frac{x^6}{(1+Zx^2)^3}\ldots\right] = \sum_{k=1}^{\infty}\frac{k(k-1)Z^{k-2}x^{\frac{k(k+1)}{2}}}{(1-x)(1-x^2)\ldots(1-x^k)}$$

Put $Z = 1$,

$$(1+x)(1+x^2)\ldots\left[\left(\frac{x}{1+x}+\frac{x^2}{1+x^2}+\ldots\right)^2-\frac{x^2}{(1+x)^2}-\right.$$
$$\left.-\frac{x^4}{(1+x^2)^2}-\frac{x^6}{(1+x^3)^2}-\ldots\right]$$
$$=\sum_{k=1}^{\infty}\frac{k^2x^{\frac{k(k+1)}{2}}}{(1-x)(1-x^2)\ldots(1-x^k)}-\sum_{k=1}^{\infty}\frac{kx^{\frac{k(k+1)}{2}}}{(1-x)(1-x^2)\ldots(1-x^k)}.$$

or

$$\sum_{k=1}^{n} k_1^2 q_k(n) = \text{coefficient of } x^n \text{ in } \prod_{r=1}^{\infty}(1+x^r)\left[\left(\sum_{s=1}^{\infty}\frac{x^s}{1+x^s}\right)^2-\sum_{s=1}^{\infty}\frac{x^{2s}}{(1+x^s)^2}+\sum\frac{x^s}{1+x^s}\right]$$

Consider $\sum_{r=1}^{\infty} D(r)x^r = \left(\sum_{s=1}^{\infty} \frac{x^s}{1+x^s}\right)^2 - \sum_{s=1}^{\infty} \frac{x^{2s}}{(1+x^s)^2} + \sum_{s=1}^{\infty} \frac{x^s}{1+x^s}$

Let $x = e^{-\lambda}$, $\sum_{r=1}^{\infty} D(r)e^{-\lambda r} = \left(\sum_{r=1}^{\infty} \frac{1}{e^{\lambda r}+1}\right)^2 - \sum_{r=1}^{\infty} \frac{r}{e^{r\lambda}-1} + \sum_{r=1}^{\infty} \frac{2r}{e^{r\lambda}+1}$

$$= \left\{\frac{\log 2}{\lambda} - \frac{1}{4} + o(\lambda)\right\}^2 - \left\{\frac{\pi^2}{6\lambda^2} - \frac{1}{2\lambda}\right\} + 2\left\{\frac{\pi^2}{12\lambda^2}\right\}$$

from (21) and the values of the other two summations is given in the appendix. Therefore

$$\sum_{r=1}^{\infty} D(r)e^{-\lambda r} = \frac{(\log 2)^2}{\lambda^2} - \frac{\log 2}{2\lambda} + \frac{1}{16} + \frac{1}{2\lambda} + o(\lambda).$$

Differentiating twice, and adding

$$\sum_{r=1}^{\infty} D(r)e^{-\lambda r}, \quad \sum_{r=1}^{\infty} D(r)e^{-\lambda r}\left(\frac{3r}{4M}\right) \text{ and } \sum_{r=1}^{\infty} D(r)e^{-\lambda r}\left(-\frac{k}{8\sqrt{2}M^{\frac{3}{2}}}\right)$$

and putting

$$\lambda = \frac{k}{2\sqrt{2}M^{\frac{1}{2}}} \text{and } k = \pi\sqrt{\frac{2}{3}},$$

$$\bar{k}_1^2 = 2C^2M(\log 2)^2 + \frac{CM^{\frac{1}{2}}}{\sqrt{2}}(1-\log 2) + \frac{1}{16} + \frac{C^2}{4}(1-\log 2) + o\left(\frac{1}{M^{\frac{1}{2}}}\right).$$

Therefore, the second moment

$$\bar{\mu}_2 = \frac{CM^{\frac{1}{2}}}{\sqrt{2}} - \sqrt{2}(\log 2)^2C^3M^{\frac{1}{2}} + \frac{C^2}{4} - \frac{1}{4}C^4(\log 2)^2 + o\left(\frac{1}{M^{\frac{1}{2}}}\right) \quad \ldots \quad (8)$$

4. Repeating these procedures given above, we find the values of $\bar{k}^3$, $\bar{k}^4$; $\bar{k}_1^3$ and $\bar{k}_4^1$. Also we know that the third moment μ_3 is equal to $\bar{k}^3 - 3\bar{k}\bar{k}^2 + 2(\bar{k})^3$ and fourth moment μ_4 is equal to $\bar{k}^4 - 4\bar{k}\bar{k}^3 + 6(\bar{k})^2\bar{k}^2 - 3\bar{k}(\bar{k})^4$

$$\bar{k}^3 = C^3N^{\frac{3}{2}}\left[(\gamma+L)^3 + 2\zeta(3) + \frac{3}{C^2}(\gamma+L)\right] - \frac{9}{4}C^4N\left[(\gamma+L)^2 + \frac{2(\gamma+L)}{3}\right]$$
$$+\frac{3}{4}C^2N[(\gamma+L)^2 - 2(\gamma+L) - 3] + \frac{3}{4}N + \frac{3}{8}C^3N^{\frac{1}{2}}[(\gamma+L)^2 - 3(\gamma+L)]$$
$$-\frac{CN^{\frac{1}{2}}}{48}\left[(\gamma+L)^2 - 9(\gamma+L) + 4 + \frac{1}{C^2}\right] - \frac{C^2}{96}\left[(\gamma+L)^2 - 8(\gamma+L) + \frac{33}{2}\right] -$$
$$-\frac{\gamma+L}{96} + \frac{1}{64} + o(N^{-\frac{1}{2}})$$

and

$$\bar{k}^4 \sim C^4N^2[(\gamma+L)^4 + 6\zeta(4) + 8\zeta(3)(\gamma+L)] + 3N^2 + 6C^2N^2(\gamma+L)^2 + o(N^{\frac{3}{2}}).$$

Also

$$\bar{k}_1^3 = 2^{\frac{3}{2}}C^3M^{\frac{3}{2}}(\log 2)^3 - 3C^4M(\log 2)^3 - \frac{3C^2}{2}M[(\log 2)^2 - 2\log 2]$$
$$+\frac{2^{\frac{1}{2}}}{16}CM^{\frac{1}{2}}[3\log 2 - 2] + \frac{C^2}{32}(3\log 2 - 2) - \frac{1}{64} + o(M^{-\frac{1}{2}})$$

and

$$\bar{k}_1^4 = 4C^4M^2(\log 2)^4 + 2^{\frac{3}{2}}C^3M^{\frac{3}{2}}[3(\log 2)^2 - (\log 2)^3] - 8\sqrt{2}C^5M^{\frac{3}{2}}(\log 2)^4 + o(M).$$

Therefore, we calculate the third and fourth moments in both the cases and get the results (6), (7), (9) and (10).

Appendix

1. Because

$$\prod_{r=1}^{\infty} (1-e^{-r\lambda})^{-1} \sim e^{\frac{\pi^2}{6\lambda}} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}$$

differentiating,

$$-\sum_{r=1}^{\infty} \frac{re^{-r\lambda}}{1-e^{-r\lambda}} = -\frac{\pi^2}{6\lambda^2} + \frac{1}{2\lambda}$$

or

$$\sum_{r=1}^{\infty} \frac{r}{e^{r\lambda}-1} = \frac{\pi^2}{6\lambda^2} - \frac{1}{2\lambda}.$$

2. Because

$$\sum_{r=1}^{\infty} (1+e^{-r\mu}) \sim \frac{1}{\sqrt{2}} e^{\frac{\pi^2}{12\mu}} \text{ for } \mu \to 0$$

we have differentiating, with respect to μ,

$$\sum_{r=1}^{\infty} \frac{re^{-r\mu}}{1+e^{-r\mu}} \sim \frac{\pi^2}{12\mu^2}$$

or

$$\sum \frac{r}{e^{r\mu}+1} = \frac{\pi^2}{12\mu^2} + o(\mu).$$

3. Using, result of Wright (1931)

$$\sum_{r=1}^{\infty} r \log (1-e^{-r\mu}) = \frac{du}{d\mu} = -\frac{\zeta(3)}{\mu^2} - \frac{1}{12}\log \mu - \zeta^1(-1) + o(\mu^2)$$

$$\frac{d^2u}{d\mu^2} = \sum_{r=1}^{\infty} \frac{r^2}{e^{r\mu}-1} = \frac{2\zeta(3)}{\mu^3} - \frac{1}{12\mu} + o(\mu).$$

4. Because

$$-\sum_{r=1}^{\infty} r^2 \log (1-e^{-r\mu}) \sim \frac{2\zeta(4)}{\mu^3}.$$

differentiating with respect to μ,

$$\sum_{r=1}^{\infty} \frac{r^3e^{-r\mu}}{1-e^{-r\mu}} = \sum_{r=1}^{\infty} \frac{r^3}{e^{r\mu}-1} \sim \frac{6\zeta(4)}{\mu^4}.$$

Appendix

Table I $p_k(n)$

n	Max. k_0	Av. $\bar{k}$	μ_2	μ_3	μ_4	β_1	β_2
1	1	1·0000	0	0	0	0	0
2	1	1·5000	·2500	0	·0625	0	1·0000
3	1	2·0000	·6667	0	·6665	·0054	1·4997
4	2	2·4000	1·0400	·2880	2·1152	·0737	1·9556
5	2, 3	2·8571	1·5510	·5247	5·1100	·0738	2·1241
6	2, 3	3·1818	1·9670	1·2846	9·4025	·2168	2·4302
7	3, 4	3·6000	2·5067	1·8721	16·0293	·2225	2·5511
8	3	3·9091	2·9917	2·9076	24·6360	·3157	2·7525
9	3	4·2667	3·5290	4·0243	35·7989	·3685	2·8746
10	4	4·5714	4·0545	5·2911	49·6680	·4200	3·0214
11	4	4·9107	4·6171	6·7193	66·3775	·4587	3·1138
12	4	5·1818	5·1618	8·4248	86·2547	·5161	3·2373
13	4, 5	5·5050	5·7351	10·1636	109·0983	·5476	3·3169
14	4, 5	5·7778	6·3063	12·0812	135·5851	·5820	3·4093
15	5	6·0682	6·8931	14·2374	165·4375	·6189	3·4818
16	5	6·3333	7·4777	16·4547	198·9415	·6476	3·5579
17	5	6·6162	8·0749	18·8699	236·0159	·6763	3·6196
18	6	6·8675	8·6760	21·3953	277·3244	·7009	3·6842
19	6	7·1388	9·2829	24·1035	322·1139	·7263	3·7381
20	6	7·3844	9·8953	26·9172	371·3286	·7478	3·7923
21	6	7·6414	10·5129	29·9128	424·5058	·7701	3·8409
22	6	7·8832	11·1331	32·9818	481·8881	·7883	3·8879
23	7	8·1323	11·7594	36·2493	543·5078	·8081	3·9304
24	7	8·3644	12·3917	39·5976	609·8769	·8240	3·9717
25	7	8·6062	13·0242	43·1285	680·1969	·8419	4·0099
26	7	8·8350	13·6625	46·7279	755·3388	·8562	4·0465
27	7	9·0678	14·3045	50·5104	835·0231	·8717	4·0809
28	7	9·2913	14·9500	54·3698	919·3698	·8847	4·1135
29	8	9·5187	15·5975	58·3975	1008·3034	·8987	4·1446
30	8	9·7364	16·2507	62·5033	1102·2859	·9103	4·1740
31	8	9·9583	16·9038	66·7743	1200·7466	·9231	4·2023
32	8	10·1721	17·5624	71·1217	1304·3642	·9338	4·2289
33	8	10·3881	18·2223	75·6259	1412·7804	·9452	4·2547
34	8	10·5981	18·8857	80·2099	1526·2131	·9551	4·2791
35	9	10·8094	19·5513	84·9424	1644·7103	·9654	4·3027
36	9	11·0149	20·2206	89·7571	1768·3682	·9744	4·3250
37	9	11·2221	20·8903	94·7171	1896·9499	·9841	4·3468
38	9	11·4242	21·5642	99·7547	2030·9209	·9924	4·3674
39	9	11·6270	22·2394	104·9335	2169·9816	1·0011	4·3874
40	10	11·8256	22·9177	110·1959	2314·3564	1·0088	4·4064
41	10	12·0249	23·5968	115·5931	2463·9102	1·0170	4·4251
42	10	12·2201	24·2801	121·0541	2619·3679	1·0238	4·4432
43	10	12·4158	24·9631	126·6770	2779·3041	1·0316	4·4601
44	10	12·6081	25·6497	132·3686	2945·1126	1·0383	4·4765
45	10	12·8003	26·3373	138·1843	3116·3288	1·0452	4·4926
46	10	12·9898	27·0276	144·0825	3293·0891	1·0515	4·5080
47	11	13·1790	27·7188	150·1061	3475·2323	1·0580	4·5231
48	11	13·3655	28·4128	156·2113	3663·0676	1·0639	4·5375
49	11	13·5519	29·1075	162·4356	3856·3539	1·0699	4·5516
50	11	13·7359	29·8049	168·7425	4055·3677	1·0754	4·5652
51	11	13·9194	30·5034	175·1607	4260·0445	1·0810	4·5785
52	11	14·1009	31·2038	181·6683	4470·3278	1·0863	4·5912
53	11	14·2819	31·9053	188·2845	4686·3218	1·0915	4·6037
54	12	14·4609	32·6092	194·9826	4908·1728	1·0964	4·6157
55	12	14·6395	33·3136	201·7960	5135·5608	1·1014	4·6275
56	12	14·8162	34·0203	208·6880	5368·9254	1·1061	4·6389
57	12	14·9924	34·7278	215·6894	5607·9702	1·1108	4·6500
58	12	15·1669	35·4374	222·7700	5853·0167	1·1151	4·6608
59	12	15·3409	36·1478	229·9565	6103·8234	1·1196	4·6713
60	13	15·5134	36·8599	237·2296	6360·5081	1·1238	4·6915

Appendix

Table I $p_k(n)$—*contd.*

n	Max. k_0	Av. $\bar{k}$	μ_2	μ_3	μ_4	β_1	β_2
61	13	15·6853	37·5730	244·6005	6623·1358	1·1279	4·6915
62	13	15·8557	38·2878	252·0551	6891·7462	1·1319	4·7012
63	13	16·0255	39·0036	259·6084	7166·2925	1·1359	4·7107
64	13	16·1940	39·7211	267·2409	7446·9741	1·1396	4·7199
65	13	16·3619	40·4394	274·9733	7733·5523	1·1433	4·7290
66	13	16·5286	41·1590	282·7931	8026·0701	1·1469	4·7378
67	13	16·6946	41·8798	290·7017	8324·7838	1·1505	4·7464
68	14	16·8595	42·6019	298·6940	8629·5808	1·1539	4·7548
69	14	17·0237	43·3246	306·7878	8940·2016	1·1574	4·7630
70	14	17·1868	44·0492	314·9517	9257·3170	1·1606	4·7710
71	14	17·3494	44·7744	323·2114	9580·4272	1·1638	4·7789
72	14	17·5109	45·5009	331·5539	9909·6876	1·1669	4·7865
73	14	17·6717	46·2283	339·9859	10245·1216	1·1700	4·7940
74	14	17·8316	46·9569	348·4991	10586·7721	1·1730	4·8014
75	15	17·9909	47·6863	357·1033	10934·5549	1·1760	4·8085
76	15	18·1493	48·4170	365·7859	11288·6625	1·1789	4·8156
77	15	18·3070	49·1486	374·5518	11649·1220	1·1817	4·8225
78	15	18·4638	49·8814	383·3979	12015·8781	1·1844	4·8292
79	15	18·6201	50·6146	392·3402	12388·5902	1·1871	4·8358
80	15	18·7755	51·3493	401·3516	12767·9503	1·1897	4·8423
81	15	18·9303	52·0849	410·4473	13153·6491	1·1923	4·8487
82	15	19·0844	52·8213	419·6246	13545·6211	1·1948	4·8549
83	16	19·2378	53·5585	428·8867	13943·9096	1·1973	4·8610
84	16	19·3905	54·2966	438·2317	14348·4403	1·1997	4·8670
85	16	19·5426	55·0355	447·6569	14759·4295	1·2022	4·8729
86	16	19·6939	55·7758	457·1498	15177·1704	1·2044	4·8786
87	16	19·8447	56·5164	466·7373	15600·9076	1·2068	4·8843
88	16	19·9948	57·2584	476·3891	16031·5226	1·2089	4·8899
89	16	20·1443	58·0006	486·1335	16468·1929	1·2112	4·8953
90	17	20·2932	58·7441	495·9479	16911·5624	1·2133	4·9007
91	17	20·4415	59·4883	505·8415	17361·4327	1·2154	4·9059
92	17	20·5892	60·2331	515·8178	17817·5999	1·2175	4·9111
93	17	20·7363	60·9789	525·8651	18280·5347	1·2196	4·9162
94	17	20·8828	61·7252	536·0006	18749·5732	1·2216	4·9212
95	17	21·0288	62·4724	546·2044	19225·4918	1·2236	4·9261
96	17	21·1741	63·2205	556·4846	19707·9242	1·2256	4·9309
97	17	21·3190	63·9697	566·8299	20197·3531	1·2274	4·9357
98	17	21·4632	64·7192	577·2621	20692·9333	1·2293	4·9403
99	18	21·6070	65·4693	587·7728	21195·0365	1·2311	4·9449
100	18	21·7502	66·2203	598·3564	21703·7951	1·2330	4·9494

Appendix

Table II $q_h(n)$

n	Max. k_1	Av. $\bar{k}_1$	$\bar{\mu}_2$	$\bar{\mu}_3$	$\bar{\mu}_4$	$\bar{\beta}_1$	$\bar{\beta}_2$
1	1	1	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1, 2	1·5000	·2500	0	·0625	0	1·0000
4	1, 2	1·5000	·2500	0	·0625	0	1·0000
5	2	1·6667	·2223	—·0742	·0743	·5005	1·5044
6	2	2·0000	·5000	0	·5000	0	2·0000
7	2	2·0000	·4000	0	·4	0	2·5000
8	2	2·1667	·4723	—·0741	·4701	·0521	2·1078
9	2	2·2500	·4375	—·0938	·4258	·1049	2·2245
10	2, 3	2·5000	·6500	0	1·0625	0	2·5148
11	2	2·5000	·5833	0	·8957	0	2·6323
12	3	2·6667	·6223	—·0742	1·0080	·0228	2·6032
13	3	2·7222	·6451	—·0521	1·0264	·0101	2·4666
14	3	2·8636	·6633	—·1354	1·0796	·0628	2·4542
15	3	2·9630	·7764	·0493	1·6677	·0052	2·7664
16	3	3·0625	·7461	—·0777	1·5021	·0015	2·0133
17	3	3·1316	·7985	—·0282	1·6955	·0016	2·6590
18	3	3·2391	·7906	—·0808	1·6937	·0013	2·7095
19	3	3·3148	·8454	—·0705	1·8546	·0082	2·5952
20	3	3·4063	·8350	—·1160	1·8346	·0023	2·6316
21	3, 4	3·5000	·9342	0	2·4045	0	2·7551
22	4	3·5730	·8964	—·0775	2·2257	·0083	2·7700
23	4	3·6538	·9572	—·0610	2·4841	·0042	2·7115
24	4	3·7295	·9515	—·0691	2·4956	·0055	2·7567
25	4	3·8099	·9991	—·0785	2·7034	·0062	2·7081
26	4	3·8788	·9914	—·0865	2·6966	·0077	2·7435
27	4	3·9583	1·0400	—·0993	2·8903	·0088	2·6724
28	4	4·0315	1·0576	—·0416	3·1663	·0015	2·8309
29	4	4·1016	1·0835	—·0891	3·2201	·0062	2·7431
30	4	4·1689	1·0932	—·0659	3·3316	·0033	2·7880
31	4	4·2412	1·1242	—·0924	3·4888	·0060	2·7603
32	4	4·3051	1·1352	—·0705	3·5971	·0034	2·7914
33	4	4·3750	1·1630	—·1022	3·7239	·0066	2·7535
34	4	4·4375	1·1758	—·0761	3·8396	·0036	2·7774
35	5	4·5043	1·1970	—·1120	3·9518	·0073	2·7579
36	5	4·5689	1·2243	—·0610	4·2173	·0020	2·8135
37	5	4·6316	1·2380	—·1033	4·2724	·0056	2·7874
38	5	4·6921	1·2571	—·0765	4·4262	·0029	2·8007
39	5	4·7546	1·2748	—·1048	4·5443	·0053	2·7962
40	5	4·8149	1·2955	—·0797	4·7038	·0029	2·8026
41	5	4·8746	1·3097	—·1071	4·8059	·0051	2·8018
42	5	4·9334	1·3315	—·0863	4·9632	·0031	2·7996
43	5	4·9919	1·3447	—·1100	5·0698	·0050	2·8037
44	5	5·0496	1·3655	—·0947	5·2167	·0035	2·7979
45	5	5·1074	1·3830	—·0995	5·4098	·0037	2·8283
46	5	5·1632	1·4005	—·0947	5·5053	·0033	2·8068
47	5	5·2189	1·4143	—·1072	5·6503	·0041	2·8247
48	5	5·2746	1·4349	—·0972	5·7918	·0032	2·8129
49	5	5·3290	1·4482	—·1070	5·9309	·0038	2·8281
50	5	5·3830	1·4666	—·1021	6·0559	·0033	2·8157
51	5	5·4367	1·4811	—·1077	6·2101	·0036	2·8312
52	6	5·4900	1·4987	—·1059	6·3264	·0033	2·8166
53	6	5·5422	1·5124	—·1113	6·4790	·0036	2·8326
54	6	5·5944	1·5304	—·1104	6·6085	·0034	2·8215
55	6	5·6461	1·5451	—·1074	6·7863	·0031	2·8425
56	6	5·6972	1·5609	—·1119	6·8848	·0033	2·8259
57	6	5·7479	1·5757	—·1109	7·0530	·0031	2·8406
58	6	5·7981	1·5915	—·1141	7·1721	·0032	2·8317
59	6	5·8478	1·6056	—·1129	7·3293	·0031	2·8431
60	6	5·8974	1·6215	—·1166	7·4551	·0032	2·8353

APPENDIX

TABLE II $q_h(n)$—*contd.*

n	Max. k_1	Av. $\bar{k}_1$	$\bar{\mu}_2$	$\bar{\mu}_3$	$\bar{\mu}_4$	$\bar{\beta}_1$	$\bar{\beta}_2$
61	6	5·9463	1·6357	−·1146	7·6160	·0030	2·8466
62	6	5·9948	1·6504	−·1193	7·7325	·0032	2·8387
63	6	6·0431	1·6649	−·1231	8·0525	·0033	2·9049
64	6	6·0910	1·6793	−·1213	8·0126	·0031	2·8412
65	6	6·1384	1·6934	−·1185	8·1615	·0029	2·8461
66	6	6·1856	1·7084	−·1212	8·3073	·0029	2·9187
67	6	6·2322	1·7220	−·1200	8·4536	·0028	2·8509
68	6	6·2786	1·7359	−·1243	8·6222	·0029	2·8613
69	6	6·3247	1·7501	−·1222	8·7381	·0028	2·8531
70	6	6·3704	1·7637	−·1254	8·8684	·0029	2·8509
71	6	6·4158	1·7774	−·1242	9·0193	·0027	2·8550
72	6	6·4609	1·7911	−·1278	9·1579	·0028	2·8548
73	7	6·5057	1·8045	−·1260	9·3020	·0027	2·8566
74	7	6·5501	1·8178	−·1296	9·4393	·0028	2·8567
75	7	6·5943	1·8313	−·1273	9·5811	·0026	2·8569
76	7	6·6382	1·8443	−·1309	9·7215	·0027	2·8581
77	7	6·6818	1·8576	−·1305	9·8693	·0027	2·8603
78	7	6·7251	1·8708	−·1317	10·0160	·0026	2·8620
79	7	6·7681	1·8837	−·1320	10·1570	·0026	2·8626
80	7	6·8109	1·8963	−·1332	10·2893	·0026	2·8614
81	7	6·8534	1·9094	−·1336	10·4419	·0026	2·8640
82	7	6·8957	1·9220	−·1350	10·5813	·0026	2·8645
83	7	6·9376	1·9347	−·1355	10·7265	·0025	2·8657
84	7	6·9794	1·9472	−·1360	10·8636	·0025	2·8653
85	7	7·0209	1·9598	−·1371	11·0111	·0025	2·8669
86	7	7·0621	1·9721	−·1380	11·1536	·0025	2·8680
87	7	7·1031	1·9845	−·1378	11·2896	·0024	2·8667
88	7	7·1439	1·9967	−·1395	11·4390	·0024	2·8693
89	7	7·1844	2·0089	−·1405	11·5824	·0024	2·8699
90	7	7·2248	2·0212	−·1417	11·7353	·0024	2·8727
91	7	7·2649	2·0332	−·1406	11·8606	·0024	2·8692
92	7	7·3047	2·0450	−·1419	12·0073	·0024	2·8711
93	7	7·3444	2·0571	−·1429	12·1538	·0023	2·8720
94	7	7·3839	2·0690	−·1445	12·3075	·0024	2·8750
95	7	7·4231	2·0808	−·1452	12·4476	·0023	2·8748
96	7	7·4621	2·0926	−·1460	12·5968	·0023	2·8766
97	7	7·5010	2·1042	−·1457	12·7253	·0023	2·8741
98	8	7·5396	2·1158	−·1463	12·8722	·0023	2·8755
99	8	7·5780	2·1275	−·1478	13·0210	·0023	2·8768
100	8	7·6163	2·1390	−·1489	13·1719	·0023	2·8789

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Abstract

Asymptotic expressions are derived for the average number of summands in a partition of n in the two cases when the summands are all different and when the summands are repeated. Also their second, third and fourth moments are calculated, along with their skewness and kurtosis. The values of all these up to 100 have been tabulated in the appendix.

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