ON CHANDRASEKHAR'S THEORY OF TURBULENCE

by P. C. JAIN, *Hindu College, Delhi University, Delhi*

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INTRODUCTION

In the present paper, the theory of axisymmetric turbulence is discussed. Using the notations of S. Chandrasekhar's paper (1950, hereafter denoted as I), the defining scalars of the tensors

$$Q_{ij}; \qquad T_{ijk}; \qquad T_{ij}; \quad \text{and} \quad \nabla^2 T_{ijk}$$

have been obtained in the gauge-invariant form.

Further, on the basis of the statistical hypothesis that the fourth order moment is related to the second order moment as in a normal distribution, the defining scalars of curl $\left(\text{curl}\ \frac{\partial}{\partial \xi_k} Q_{ik;jl}\right)$ have been derived in the gauge-invariant form.

Then, starting with Stoke's Navier Equation of motion, two equations have been deduced. These two equations replace the equation of Von Karman and Howarth in the theory of isotropic turbulence. It is quite interesting to note that these two equations do not depend upon the pressure term which occurs in the basic equation of motion, while two similar equations derived by S. Chandrasekhar (1950; cf. equations 118 and 119) do contain the effects of this pressure term.

In the end, following the procedure of S. Chandrasekhar's paper (1955, hereafter denoted as II) eight implicit simultaneous differential equations in eight defining scalars Q_1, Q_2, T_1, T_2, T_3, T_4, T_5 and T_6 have been obtained. Although the equations seem to be far from being in the form of soluble equations, these equations bring the theory of axisymmetric turbulence to the same degree of completeness that S. Chandrasekhar brought the theory of isotropic turbulence. These are the basic equations for a deductive theory of axisymmetric turbulence.

1. DEFINING SCALARS OF VARIOUS TENSORS

As given in I, the defining scalars of

$$Q_{ij} = A\xi_i\xi_j + B\delta_{ij} + C\lambda_i\lambda_j + D\lambda_i\xi_j + E\xi_i\lambda_j \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

which is symmetrical and solenoidal in its indices, are Q_1 and Q_2, and the coefficients are given by

$$\left.\begin{aligned} A &= (D_r - D_{\mu\mu})Q_1 + D_r Q_2 \\ B &= [-(r^2D_r + r\mu D_\mu + 2) + r^2(1-\mu^2)D_{\mu\mu} - r\mu D_\mu]Q_1 \\ &\quad -[r^2(1-\mu^2)D_r + 1]Q_2 \\ C &= -r^2 D_{\mu\mu}Q_1 + (r^2D_r + 1)Q_2 \\ D &= E = (r\mu D_\mu + 1)D_\mu Q_1 - r\mu D_r Q_2 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2)$$

where

$$D_r \equiv \frac{1}{r}\frac{\partial}{\partial r} - \frac{\mu}{r^2}\frac{\partial}{\partial \mu}$$

$$D_\mu \equiv \frac{1}{r}\frac{\partial}{\partial \mu} \text{ and } D_{\mu\mu} \equiv D_\mu D_\mu .$$

The defining scalars of $\nabla^2 Q_{ij}$, in a gauge-invariant form, are (cf. equation (61), I)

$$\Delta Q_1 \text{ and } \Delta Q_2 + 2D_{\mu\mu}Q_1 \quad \ldots \quad (3)$$

where

$$\Delta \equiv r^2 D_{rr} + 2r\mu D_{r\mu} + D_{\mu\mu} + 5D_r$$

The third order axisymmetric tensor $T_{ij;k}$ which is solenoidal in k and symmetrical in i and j can be derived in a gauge-invariant form from the skew tensor (cf. equations (92) and (94), I)

$$\begin{aligned} t_{ijk} = {} & [T_1\xi_i\xi_j + T_2\lambda_i\lambda_j + T_3\delta_{ij}]\epsilon_{klm}\lambda_l\xi_m \\ & + T_4\xi_i\epsilon_{jkl}\xi_l + T_5\lambda_i\epsilon_{jkl}\xi_l \\ & + T_6\xi_i\lambda_j\epsilon_{klm}\lambda_l\xi_m + T_4\xi_j\epsilon_{ikl}\xi_l \\ & + T_5\lambda_j\epsilon_{ikl}\xi_l + T_6\lambda_i\xi_j\epsilon_{klm}\lambda_l\xi_m \end{aligned} \quad \ldots \quad (4)$$

where T_1 to T_6 are six arbitrary functions of r and $r\mu$.

Taking the curl of this skew tensor, we get

$$T_{ij;k} = \text{curl}_k\, t_{ijk} \quad \ldots \quad (5)$$

and the defining scalars of $T_{ij;k}$ are

$$T_1, T_2, T_3, T_4, T_5 \text{ and } T_6. \quad \ldots \quad (6)$$

We are interested in the tensor T_{ij} (solenoidal in j) and given as

$$T_{ij} = \frac{\partial}{\partial \xi_k} T_{ik;j} = \frac{\partial}{\partial \xi_k}\left[\epsilon_{jlm}\frac{\partial t_{ikm}}{\partial \xi_l}\right]$$

$$= \epsilon_{jlm}\frac{\partial}{\partial \xi_l}\frac{\partial}{\partial \xi_k} t_{ikm}$$

or

$$T_{ij} = \epsilon_{jlm}\frac{\partial}{\partial \xi_l} t_{im} \quad \ldots \quad (7)$$

where

$$t_{im} = \frac{\partial}{\partial \xi_k} t_{ikm} . \quad \ldots \quad (8)$$

On evaluating the R.H.S. of equation (8), we get

$$t_{ij} = \Gamma_1\epsilon_{ijk}\xi_k + \Gamma_2\lambda_j\epsilon_{ilm}\lambda_l\xi_m + \Gamma_3\xi_j\epsilon_{ilm}\lambda_l\xi_m \quad \ldots \quad (9)$$

where Γ_1, Γ_2 and Γ_3 are functions of T_1 to T_6 (cf. equations (100) and (101), I).

$$\begin{aligned} \therefore \quad T_{ij} &= \text{curl}_j\, t_{ij} \\ &= \xi_i\xi_j[D_r\Gamma_1 - D_\mu\Gamma_3 + D_r\Gamma_2] \\ &\quad + \lambda_i\lambda_j[\Gamma_2 - r^2 D_\mu\Gamma_3 + r^2 D_r\Gamma_2] \\ &\quad + \delta_{ij}[-(r^2 D_r + r\mu D_\mu + 2)\Gamma_1 \\ &\qquad - \{r^2(1-\mu^2)+1\}\Gamma_2 \\ &\qquad + \{r^2(1-\mu^2)D_\mu - r\mu\}\Gamma_3] \\ &\quad + \xi_j\lambda_i D_\mu\Gamma_1 + \xi_i\lambda_j\Gamma_3 \\ &\quad + r\mu(\lambda_i\xi_j + \lambda_j\xi_i)(D_\mu\Gamma_3 - D_r\Gamma_2) \end{aligned} \quad \ldots \quad (10)$$

If T_{ij} is symmetrical in i and j, we obtain $\Gamma_3 = D_\mu \Gamma_1$ from equation (10).

Hence the defining scalars of solenoidal and symmetrical tensor T_{ij}, in a gauge-invariant form, are

$$\Gamma_1 \text{ and } \Gamma_2 \quad \ldots \quad (11)$$

where

$$\Gamma_1 = -r\mu(r^2D_r + r\mu D_\mu + 5)T_1 - (r\mu D_r + D_\mu)T_2 - (r\mu D_r + D_\mu)T_3 + (r^2D_r + 2r\mu D_\mu + 5)T_4 + (r\mu D_r + 2D_\mu)T_5 - (r^2D_r + r^2\mu^2D_r + 2r\mu D_\mu + 5)T_6$$

and

$$\Gamma_2 = (r\mu D_r + D_\mu)T_2 + D_\mu T_3 - D_\mu T_5 + (r^2D_r + r\mu D_\mu + 5)T_6 + D_\mu[r^2(r^2D_r + r\mu D_\mu + 5)T_1 + r\mu(r\mu D_r + D_\mu)T_2 + (r^2D_r + r\mu D_\mu + 1)T_3 - r^2D_\mu T_4 - r\mu D_\mu T_5 + T_5 + r\mu(r^2D_r + r\mu D_\mu + 5)T_6 + r^2(r\mu D_r + D_\mu)T_6] \quad \ldots \quad (12)$$

Putting the value of Γ_3 in (10), we get the explicit representation of T_{ij} as

$$T_{ij} = \xi_i\xi_j[(D_r - D_{\mu\mu})\Gamma_1 + D_r\Gamma_2] + \lambda_i\lambda_j[-r^2D_{\mu\mu}\Gamma_1 + (r^2D_r + 1)\Gamma_2] + \delta_{ij}[-(r^2D_r + r\mu D_\mu + 2)\Gamma_1 + r^2(1-\mu^2)D_{\mu\mu}\Gamma_1 - r\mu D_\mu\Gamma_1 - \{r^2(1-\mu^2)D_r + 1\}\Gamma_2] + (\lambda_i\xi_j + \lambda_j\xi_i)[(r\mu D_\mu + 1)D_\mu\Gamma_1 - r\mu D_r\Gamma_2] \quad \ldots \quad (13)$$

which differs from the representation of Q_{ij} in having Γ_1 and Γ_2 in place of Q_1 and Q_2 respectively.

Further, we want to find the defining scalars of $\nabla^2 T_{ij;k}$ when the third order tensor is symmetrical in i and j, and solenoidal in k.

$$\nabla^2 T_{ij;k} = \nabla^2 \operatorname{curl}_k t_{ijk} = \operatorname{curl}_k(\nabla^2 t_{ijk}) = \operatorname{curl}_k\left(\frac{\partial}{\partial \xi_s}\frac{\partial}{\partial \xi_s} t_{ijk}\right) \quad \ldots \quad (14)$$

But

$$\frac{\partial}{\partial \xi_s} t_{ijk} = T_1[\xi_i\xi_j\epsilon_{kls}\lambda_l + \delta_{is}\xi_j\epsilon_{klm}\lambda_l\xi_m + \xi_i\delta_{js}\epsilon_{klm}\lambda_l\xi_m] + [\xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m\xi_sD_rT_1 + \xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m\lambda_sD_\mu T_1] + \lambda_i\lambda_j\epsilon_{klm}\lambda_l[\delta_{ms} + \xi_m\xi_sD_r + \xi_m\lambda_sD_\mu]T_2 + \delta_{ij}\epsilon_{klm}\lambda_l[\delta_{ms} + \xi_m\xi_sD_r + \xi_m\lambda_sD_\mu]T_3 + [\delta_{is}\epsilon_{jkl}\xi_l + \xi_i\epsilon_{jkl}\delta_{ls} + \delta_{js}\epsilon_{ikl}\xi_l + \xi_j\epsilon_{ikl}\delta_{ls}]T_4 + (\xi_i\epsilon_{jkl}\xi_l + \xi_j\epsilon_{ikl}\xi_l)(\xi_sD_r + \lambda_sD_\mu)T_4 + (\lambda_i\epsilon_{jkl} + \lambda_j\epsilon_{ikl})(\delta_{ls} + \xi_l\xi_sD_r + \xi_l\lambda_sD_\mu)T_5 + \lambda_j\epsilon_{klm}\lambda_l\xi_m[\delta_{is} + \xi_i\xi_sD_r + \xi_i\lambda_sD_\mu]T_6 + \lambda_i\epsilon_{klm}\lambda_l\xi_m[\delta_{js} + \xi_j\xi_sD_r + \xi_j\lambda_sD_\mu]T_6 + (\xi_i\lambda_j + \xi_j\lambda_i)\epsilon_{kls}\lambda_l T_6. \quad \ldots \quad (15)$$

and

$$
\begin{aligned}
\frac{\partial}{\partial \xi_s}\frac{\partial}{\partial \xi_s} t_{ijk} = {} & \xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m[\Delta T_1+4D_r T_1] \\
& +\lambda_i\lambda_j\epsilon_{klm}\lambda_l\xi_m[\Delta T_2+4D_\mu T_6] \\
& +\delta_{ij}\epsilon_{klm}\lambda_l\xi_m[\Delta T_3+2T_1] \\
& +(\xi_i\epsilon_{jkl}\xi_l+\xi_j\epsilon_{ikl}\xi_l)(\Delta T_4+2D_r T_4) \\
& +(\lambda_i\epsilon_{jkl}\xi_l+\lambda_j\epsilon_{ikl}\xi_l)(\Delta T_5+2D_\mu T_4) \\
& +(\xi_i\lambda_j+\lambda_i\xi_j)\epsilon_{klm}\lambda_l\xi_m[\Delta T_6+2D_r T_6+2D_\mu T_1] \\
& +(\lambda_i\epsilon_{jkl}\lambda_l+\lambda_j\epsilon_{ikl}\lambda_l)(2D_\mu T_5+2T_6) \\
& +(\xi_j\epsilon_{ikl}\lambda_l+\xi_i\epsilon_{jkl}\lambda_l)(2D_\mu T_4+2T_1). \qquad (16)
\end{aligned}
$$

Since we have to find the defining scalars of $\nabla^2 T_{ij;k}$ in a gauge-invariant form, we replace the values of the last two terms on R.H.S. of equation (16) in terms of the remaining tensors with the help of the following divergence free combinations of third order tensors: (cf. equation (90), I)

$$
\left.
\begin{aligned}
& Q\lambda_i\epsilon_{jkl}\lambda_l+(D_rQ\lambda_i\xi_j+D_\mu Q\lambda_i\lambda_j)\epsilon_{klm}\lambda_l\xi_m \\
& Q\lambda_j\epsilon_{kil}\lambda_l-(D_rQ\xi_i\lambda_j+D_\mu Q\lambda_i\lambda_j)\epsilon_{klm}\lambda_l\xi_m \\
& Q(\xi_i\epsilon_{ikl}\lambda_l+\delta_{ij}\epsilon_{klm}\lambda_l\xi_m)+(D_rQ\xi_j\xi_i+D_\mu Q\lambda_j\xi_i)\epsilon_{klm}\lambda_l\xi_m \\
& Q(\xi_j\epsilon_{kli}\lambda_l+\delta_{ji}\epsilon_{klm}\lambda_l\xi_m)+(D_rQ\xi_j\xi_i+D_\mu Q\xi_j\lambda_i)\epsilon_{klm}\lambda_l\xi_m
\end{aligned}
\right\} \qquad (17)
$$

where Q is any function of r and $r\mu$.

$$
\begin{aligned}
\therefore\quad (2D_\mu T_5+2T_6)(\lambda_i\epsilon_{jkl}\lambda_l+\lambda_j\epsilon_{ikl}\lambda_l) = {} & -2D_r(D_\mu T_5+T_6)(\lambda_i\xi_j+\xi_i\lambda_j)\epsilon_{klm}\lambda_l\xi_m \\
& -2D_\mu(D_\mu T_5+T_6)(2\lambda_i\lambda_j)\epsilon_{klm}\lambda_l\xi_m \\
2(D_\mu T_4+T_1)(\xi_j\epsilon_{ikl}\lambda_l+\xi_i\epsilon_{jkl}\lambda_l) = {} & -4(D_\mu T_4+T_1)\delta_{ij}\epsilon_{klm}\lambda_l\xi_m \\
& -2D_r(D_\mu T_4+T_1)2\xi_i\xi_j\epsilon_{klm}\lambda_l\xi_m \\
& -2D_\mu(D_\mu T_4+T_1)(\lambda_j\xi_i+\xi_j\lambda_i)\epsilon_{klm}\lambda_l\xi_m \qquad (18)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\nabla^2 t_{ijk} = {} & [(\Delta T_1-4D_{r\mu}T_4)\xi_i\xi_j+(\Delta T_4-4D_{\mu\mu}T_5)\lambda_i\lambda_j \\
& \qquad +(\Delta T_3-4D_\mu T_4-2T_1)\delta_{ij}]\epsilon_{klm}\lambda_l\xi_m \\
& +(\Delta+2D_r)T_4(\xi_i\epsilon_{jkl}\xi_l+\xi_j\epsilon_{ikl}\xi_l) \\
& +(\Delta T_5+2D_\mu T_4)(\lambda_i\epsilon_{jkl}\xi_l+\xi_j\epsilon_{ikl}\xi_l) \\
& +(\Delta T_6-2D_{r\mu}T_5-2D_{\mu\mu}T_4)(\xi_i\lambda_j+\lambda_i\xi_j)\epsilon_{klm}\lambda_l\xi_m. \qquad (19)
\end{aligned}
$$

Thus the defining scalars of $\nabla^2 T_{ij;k}$ (symmetrical in i and j; solenoidal in k) in a gauge-invariant form are

$$
\begin{aligned}
& \Delta T_1-4D_{r\mu}T_4\,; && \Delta T_4+2D_r T_4 \\
& \Delta T_2-4D_{\mu\mu}T_5\,; && \Delta T_5+2D_\mu T_4 \\
& \Delta T_3-4D_\mu T_4-2T_1\,; && \Delta T_6-2D_{r\mu}T_5-2D_{\mu\mu}T_4 \qquad (20)
\end{aligned}
$$

In order to find the defining scalars of $\text{curl}_i\ \text{curl}_i\ X_{jl;i}$ in a gauge-invariant form where $X_{jl;i}$ is a third order axisymmetric tensor which is solenoidal in i and symmetrical in j and l, we have

$$\text{curl}_i\ \text{curl}_i\ X_{jl;i} = -\nabla^2 X_{jl;i}$$

Since $X_{jl;i}$ is of the same form as $T_{jl;i}$ the defining scalars of $\text{curl}_i \text{ curl}_i X_{jl;i}$ are of the same type as of $\nabla^2 T_{jl;i}$ but with opposite signs.

Thus with the help of (20), the defining scalars of $\text{curl}_i \text{ curl}_i X_{jl;i}$ in a gauge-invariant form are

$$\left.\begin{array}{ll} -\Delta X_1+4D_{r\mu}X_4; & -\Delta X_4-2D_r X_4 \\ -\Delta X_2+4D_{\mu\mu}X_5; & -\Delta X_5-2D_\mu X_4 \\ -\Delta X_3+4D_\mu X_4+2X_1; & -\Delta X_6+2D_{r\mu}X_5+2D_{\mu\mu}X_4. \end{array}\right\} \quad \ldots \quad (21)$$

where X_1, X_2, X_3, X_4, X_5, X_6 are the defining scalars of the tensor $X_{jl;i}$

Now, we shall obtain the defining scalars of $\text{curl}_i \left(\text{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl}\right)$ on the basis of the hypothesis that

$$Q_{ij;kl} = Q_{ik}Q_{jl}+Q_{il}Q_{jk}+Q_{ij}(0,0)Q_{kl}(0,0) \quad \ldots \quad (22)$$

$$\therefore \quad \frac{\partial}{\partial \xi_k} Q_{ik;jl} = Q_{kl}\frac{\partial}{\partial \xi_k} Q_{ij}+Q_{kj}\frac{\partial}{\partial \xi_k} Q_{il} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

as Q_{ij} is symmetrical and solenoidal in its indices.

But

$$\begin{aligned} \frac{\partial}{\partial \xi_k} Q_{il} &= \frac{\partial}{\partial \xi_k}[A\xi_i\xi_j+B\delta_{ij}+C\lambda_i\lambda_j+D(\lambda_i\xi_j+\xi_i\lambda_j)] \\ &= \xi_i\xi_j(\xi_k D_r A+\lambda_k D_\mu A)+(\lambda_i\xi_j+\xi_i\lambda_j)\xi_k D_r D \\ &\quad +\lambda_i\lambda_j(\xi_k D_r C+\lambda_k D_\mu C)+(\lambda_i\xi_j+\xi_i\lambda_j)\lambda_k D_\mu D \\ &\quad +\delta_{jk}\xi_i A+\delta_{ki}\xi_j A+\delta_{ij}(\xi_k D_r B+\lambda_k D_\mu B) \\ &\quad +\delta_{jk}\lambda_i D+\delta_{ki}\lambda_j D \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24) \end{aligned}$$

$$\left.\begin{aligned} \therefore \quad \frac{\partial}{\partial \xi_k} Q_{ik;jl} &= 2A_1\xi_i\xi_j\xi_l+2A_2\lambda_i\lambda_j\lambda_l+2A_3\lambda_i\xi_j\xi_l \\ &\quad +A_4(\xi_i\lambda_j\xi_l+\xi_i\xi_j\lambda_l)+2A_5\xi_i\lambda_j\lambda_l \\ &\quad +A_6(\lambda_i\xi_j\lambda_l+\lambda_i\lambda_j\xi_l)+A_7(\delta_{ij}\xi_l+\delta_{il}\xi_j) \\ &\quad +2A_8\delta_{lj}\xi_i+2A_9\delta_{lj}\lambda_i \\ &\quad +A_{10}[\delta_{il}\lambda_j+\delta_{ij}\lambda_l] \end{aligned}\right\} \quad \ldots \quad \ldots \quad (25)$$

where

$$\left.\begin{array}{ll} A_1 = \Delta_1 A; & A_6 = \Delta_1 C+\Delta_2 D-(AC-D^2) \\ A_2 = \Delta_2 C; & A_7 = \Delta_1 B-AB \\ A_3 = \Delta_1 D; & A_8 = AB \\ A_4 = \Delta_1 D+\Delta_2 A; & A_9 = DB \\ A_5 = \Delta_2 D+AC-D^2; & A_{10} = \Delta_2 B-DB \end{array}\right\} \quad \ldots \quad (26)$$

and

$$\left.\begin{aligned} \Delta_1 &\equiv A(r^2D_r+r\mu D_\mu+2)+BD_r+D(r\mu D_r+D_\mu) \\ \Delta_2 &\equiv D(r^2D_r+r\mu D_\mu+2)+BD_\mu+C(r\mu D_r+D_\mu) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (27)$$

Thus

$$\left.\begin{aligned}\operatorname{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl} = {} & \xi_j\xi_l\epsilon_{ips}\lambda_p\xi_s(2D_\mu A_1-2D_r A_3) \\ & +\lambda_j\lambda_l\epsilon_{ips}\lambda_p\xi_s(-2D_r A_2+2D_\mu A_5) \\ & +\delta_{lj}\epsilon_{ips}\lambda_p\xi_s(2D_\mu A_8-2D_r A_9) \\ & +(\lambda_j\xi_l+\xi_j\lambda_l)\epsilon_{ips}\lambda_p\xi_s(D_\mu A_4-D_r A_6) \\ & +(\xi_l\epsilon_{ijs}\xi_s+\xi_j\epsilon_{ils}\xi_s)(2A_1-D_r A_7) \\ & +(\lambda_j\epsilon_{ils}\xi_s+\lambda_l\epsilon_{ijs}\xi_s)(A_4-D_r A_{10}) \\ & +(\xi_l\epsilon_{ijs}\lambda_s+\xi_j\epsilon_{ils}\lambda_s)(2A_3-D_\mu A_7) \\ & +(\lambda_l\epsilon_{ijs}\lambda_s+\lambda_j\epsilon_{ils}\lambda_s)(A_6-D_\mu A_{10})\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Expressing the last two tensors in terms of others with the help of (17), we get

$$\begin{aligned}\operatorname{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl} = {} & \xi_j\xi_l\,\epsilon_{ips}\lambda_p\xi_s\,[2D_\mu A_1+2D_r A_3-2D_{r\mu}A_7] \\ & +\lambda_j\lambda_l\,\epsilon_{ips}\lambda_p\xi_s\,[-2D_r A_2+2D_\mu A_5+2D_\mu A_6-2D_{\mu\mu}A_{10}] \\ & +\delta_{lj}\,\epsilon_{ips}\lambda_p\xi_s\,[2D_\mu A_8-2D_r A_9+4A_3-2D_\mu A_7] \\ & +(\xi_j\,\epsilon_{lis}\xi_s+\xi_l\,\epsilon_{jis}\xi_s)(-2A_1+D_r A_7) \\ & +(\lambda_j\,\epsilon_{lis}\xi_s+\lambda_l\,\epsilon_{jis}\xi_s)(-A_4+D_r A_{10}) \\ & +(\xi_j\lambda_l+\lambda_j\xi_l)\,\epsilon_{ips}\lambda_p\xi_s\,[D_\mu A_4-2D_\mu A_3-D_{\mu\mu}A_7-D_{r\mu}A_{10}] \quad (29)\end{aligned}$$

Hence the defining scalars of $\operatorname{curl}_i\left(\operatorname{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl}\right)$ in a gauge-invariant form are

$$\left.\begin{aligned} & 2D_\mu A_1+2D_r A_3-2D_{r\mu}A_7\,; \\ & -2D_r A_2+2D_\mu A_5+2D_\mu A_6-2D_{\mu\mu}A_{10}\,; \\ & 4A_3-2D_\mu A_7+2D_\mu A_8-2D_r A_9\,; \\ & -2A_1+D_r A_7\,; \\ & -A_4+D_r A_{10}\,; \\ & -2D_\mu A_3+D_\mu A_4-D_{\mu\mu}A_7-D_{r\mu}A_{10}\,.\end{aligned}\right\} \quad \ldots \quad \ldots \quad (30)$$

At the end of this section, we proceed to obtain the defining scalars of $\nabla^4 Q_{ij}$ in a gauge-invariant form. As obtained in paper I (cf. eq. 55),

if
$$Q_{ij} = \operatorname{curl} q_{ij}$$

where
$$q_{ij} = Q_1\epsilon_{ijk}\xi_k+Q_2\lambda_j\epsilon_{ilm}\lambda_l\xi_m+Q_3\xi_j\epsilon_{ilm}\lambda_l\xi_m \quad \ldots \quad \ldots \quad (31)$$

then

$$\begin{aligned}\nabla^2 q_{ij} = {} & \Delta Q_1\epsilon_{ijk}\xi_k+(\Delta Q_2+2D_{\mu\mu}Q_1)\lambda_j\epsilon_{ilm}\lambda_l\xi_m \\ & +(\Delta Q_3+2D_{r\mu}Q_1)\xi_j\epsilon_{ilm}\lambda_l\xi_m. \quad \ldots \quad \ldots \quad \ldots \quad (32)\end{aligned}$$

Due to the symmetry of Q_{ij} in its indices,

$$Q_3 = D_\mu Q_1$$

Let
$$\nabla^2 q_{ij} = Q_1'\epsilon_{ijk}\xi_k+Q_2'\lambda_j\epsilon_{ilm}\lambda_l\xi_m+Q_3'\xi_j\epsilon_{ilm}\lambda_l\xi_m \quad \ldots \quad \ldots \quad (32a)$$

where we have denoted the defining scalars by Q_1', Q_2' and Q_3'. In order to obtain the expression for $\nabla^4 q_{ij}$, we proceed in the same manner as in the case of equations (31) and (32). Thus, we get

$$\nabla^4 q_{ij} = \Delta Q_1' \epsilon_{ijk}\xi_k + (\Delta Q_2' + 2D_{\mu\mu}Q_1')\lambda_j \epsilon_{ilm}\lambda_l \xi_m + (\Delta Q_3' + 2D_{r\mu}Q_1')\xi_j \epsilon_{ilm}\lambda_l \xi_m \quad \ldots \quad (33)$$

As

$$Q_3' = \Delta Q_3 + 2D_{r\mu}Q_1 = \Delta D_\mu Q_1 + 2D_{r\mu}Q_1 = D_\mu(\Delta Q_1) = D_\mu Q_1'$$

Therefore,

$$\Delta Q_3' + 2D_{r\mu}Q_1' = D_\mu \Delta Q_1' = D_\mu \Delta^2 Q_1$$

where

$$\Delta^2 Q_1 = \Delta(\Delta Q_1)$$

But

$$\nabla^4 Q_{ij} = \text{curl } \nabla^4 q_{ij} = \text{curl } \big[(\Delta^2 Q_1)\epsilon_{ijk}\xi_k + \{\Delta^2 Q_2 + 2(\Delta D_{\mu\mu} + D_{\mu\mu}\Delta)Q_1\}\lambda_j \epsilon_{ilm}\lambda_l \xi_m + D_\mu(\Delta^2 Q_1)\xi_j \epsilon_{ilm}\lambda_l \xi_m\big]. \quad \ldots \quad (34)$$

Hence the defining scalars of the tensor $\nabla^4 Q_{ij}$ in a gauge-invariant form are

$$\Delta^2 Q_1 \quad \text{and} \quad \Delta^2 Q_2 + 2(\Delta D_{\mu\mu} + D_{\mu\mu}\Delta)Q_1 \quad \ldots \quad (34a)$$

It may be remarked also that the defining scalars of $\nabla^2 T_{ij}$ in a gauge-invariant form are [cf. eqs. (1), (3) and (13)]

$$\Delta \Gamma_1 \quad \text{and} \quad \Delta \Gamma_2 + 2D_{\mu\mu}\Gamma_1 \quad \ldots \quad (35)$$

2. The Dynamical Equations

Using the notations of II, equation of motion is

$$\frac{\partial u_i'}{\partial t'} + \frac{\partial}{\partial x_k'} u_i' u_k' = -\frac{\partial \omega'}{\partial x_i'} + \nu \nabla^2 u_i' \quad \ldots \quad (36)$$

where $\omega = p/\rho$.

Multiplying it by u_j'' and taking the averages, we obtain

$$\frac{\partial Q_{ij}}{\partial t'} + \frac{\partial}{\partial x_k'} T_{ik;j} = \nu \nabla^2 Q_{ij} + \frac{\partial \overline{\omega' u_j''}}{\partial x_i'} \quad \ldots \quad (37)$$

which transforms to

$$\pm \frac{\partial Q_{ij}}{\partial t} = \frac{\partial}{\partial \xi_k} T_{ik;j} + \nu \nabla_\xi^2 Q_{ij} + \frac{\partial \overline{\omega' u_j''}}{\partial \xi_i} \quad \ldots \quad (37a)$$

Taking curl curl of equation (37*a*), we have

$$\pm \frac{\partial}{\partial t} \text{curl}_i \text{ curl}_i\ Q_{ij} = \text{curl}_i \text{ curl}_i\ T_{ij} + \nu \nabla^2 \text{ curl}_i \text{ curl}_i\ Q_{ij} \quad \ldots \quad (38)$$

as

$$\text{curl}_i \frac{\partial \overline{\omega' u_j''}}{\partial \xi_i} = 0$$

Since Q_{ij} and T_{ij} are solenoidal tensors which are symmetrical in their indices, equation (38) can be written as

$$\pm \frac{\partial}{\partial t}(\nabla^2 Q_{ij}) = \nabla^2 T_{ij} + \nu \nabla^4 Q_{ij} \qquad \text{.. .. ..} \quad (39)$$

Putting this equation in terms of the defining scalars of the tensors, we finally obtain

$$\left.\begin{aligned} &\left(\pm \frac{\partial}{\partial t} - \nu \Delta\right) \Delta Q_1 = \Delta \Gamma_1 \\ \text{and} \quad &\left(\pm \frac{\partial}{\partial t} - \nu \Delta\right)(\Delta Q_2 + 2D_{\mu\mu} Q_1) - 2\nu D_{\mu\mu} \Delta Q_1 \\ &\qquad = \Delta \Gamma_2 + 2D_{\mu\mu} \Gamma_1 \end{aligned}\right\} \qquad \text{.. ..} \quad (40)$$

where Γ_1 and Γ_2 are functions of the defining scalars

$$T_1,\ T_2,\ T_3,\ T_4,\ T_5 \text{ and } T_6 .$$

These two equations are the fundamental equations in the theory of axisymmetric turbulence and replace the equation of Von Karman and Howarth (1938) in the theory of isotropic turbulence.

Actually, these are the modified forms of the equations (cf. equations (118) and (119), I)

$$\left.\begin{aligned} \frac{\partial Q_1}{\partial t} &= 2\nu \Delta Q_1 + S_1 \\ \frac{\partial Q_2}{\partial t} &= 2\nu(\Delta Q_2 + 2D_{\mu\mu} Q_1) + S_2 \end{aligned}\right\} \qquad \text{.. .. ..} \quad (41)$$

where S_1 and S_2 are the defining scalars of the second order tensor which is expressed in terms of T_{ij} and $\frac{1}{\rho}\frac{\partial \overline{p u_j'}}{\partial \xi_i}$. Thus equations (41) also contain the effects of the pressure term, while our equations (40) are quite free from such effects.

Equations (40) give us two equations in eight defining scalars Q_1, Q_2, T_1, T_2, T_3, T_4, T_5, T_6. In the next section we shall deduce six more equations in these eight defining scalars and the result will provide us with a set of eight equations in eight unknown scalars.

3. Basic Equations

Multiplying equation (36) by $u_j'' u_l''$ and taking the averages we get (cf. equation (22), II)

$$\pm \frac{\partial T_{jl;i}}{\partial t} - \frac{\partial}{\partial \xi_k} Q_{ik;jl} = \frac{\partial}{\partial \xi_i} P_{jl} - \nu \nabla^2_{\xi} T_{jl;i}$$

or

$$\pm \frac{\partial T_{jl;i}}{\partial t} + \nu \nabla^2_{\xi} T_{jl;i} = X_{jl;i} \qquad \text{.. .. .. ..} \quad (42)$$

where

$$X_{jl;i} = \frac{\partial}{\partial \xi_k} Q_{ik;jl} + \frac{\partial}{\partial \xi_i} P_{jl} \qquad \text{.. .. .. ..} \quad (43)$$

Since the tensors on the left hand side of equation (42) are both third order axisymmetric tensors which are symmetrical in j and l, and solenoidal in i, therefore

$X_{jl;i}$ is also a third order axisymmetric tensor which is solenoidal in i and symmetrical in j and l. Hence $X_{jl;i}$ is of the same form as $T_{jl;i}$ and the defining scalars of $X_{jl;i}$ as defined in the first section are X_1, X_2, X_3, X_4, X_5 and X_6. Writing equation (42) in terms of the defining scalars we obtain

$$\left.\begin{aligned}
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_1-4\nu D_{r\mu}T_4 &= X_1\\
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_2-4\nu D_{\mu\mu}T_5 &= X_2\\
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_3-4\nu D_{\mu}T_4-2\nu T_1 &= X_3\\
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_4+2\nu D_r T_4 &= X_4\\
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_5+2\nu D_{\mu}T_4 &= X_5\\
\left(\pm\frac{\partial}{\partial t}+\nu\Delta\right)T_6-2\nu D_{r\mu}T_5-2\nu D_{\mu\mu}T_4 &= X_6
\end{aligned}\right\} \quad .. \quad .. \quad (44)$$

Next, taking the curl of equation (43) with respect to index i, we have

$$\text{curl}_i\, X_{jl;i} = \text{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl} \quad .. \quad .. \quad .. \quad (45)$$

In deriving equation (45), the use has been made of the fact that

$$\text{curl}_i \frac{\partial}{\partial \xi_i} P_{jl} = 0 \quad .. \quad .. \quad .. \quad .. \quad (46)$$

Taking the curl_i of equation (45) again, we have

$$\text{curl}_i\, \text{curl}_i\, X_{jl;i} = \text{curl}_i \left[\text{curl}_i \frac{\partial}{\partial \xi_k} Q_{ik;jl}\right] \quad .. \quad .. \quad (47)$$

Writing this equation in terms of the defining scalars, we get (cf. (30) and (21))

$$\left.\begin{aligned}
-\Delta X_1+4D_{r\mu}X_4 &= 2D_{\mu}A_1+2D_rA_3-2D_{r\mu}A_7\\
-\Delta X_2+4D_{\mu\mu}X_5 &= -2D_rA_2+2D_{\mu}A_5+2D_{\mu}A_6-2D_{\mu\mu}A_{10}\\
-\Delta X_3+4D_{\mu}X_4+2X_1 &= 4A_3-2D_{\mu}A_7+2D_{\mu}A_8-2D_rA_9\\
-\Delta X_4-2D_rX_4 &= -2A_1+D_rA_7\\
-\Delta X_5-2D_{\mu}X_4 &= -A_4+D_rA_{10}\\
-\Delta X_6+2D_{r\mu}X_5+2D_{\mu\mu}X_4 &= -2D_{\mu}A_3+D_{\mu}A_4-D_{\mu\mu}A_7-D_{r\mu}A_{10}
\end{aligned}\right\} \quad .. \quad (48)$$

where A_1 to A_{10} are functions of Q_1 and Q_2 and are given explicitly by equations (2) and (26). Also X_1 to X_6 are given explicitly in terms of T_1 to T_6 by equations (44). Hence equations (48) give six relations between Q_1, Q_2, T_1, T_2, T_3, T_4, T_5 and T_6. Combining equations (40) with equations (48), we obtain a set of eight equations in eight defining scalars of Q_{ij} and $T_{jk;i}$. Equations (40) and (48) are the basic equations for the development of a deductive theory of axisymmetric turbulence.

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Summary

In the present paper, the theory of S. Chandrasekhar (1955) has been generalised to the case of axisymmetric turbulence, and eight differential equations (40) and (48) in eight defining scalars of double and triple correlation tensors have been deduced. Two of these equations (40) are the modified forms of the equations obtained by S. Chandrasekhar (1950) on the basis of old theory and, therefore, these equations replace the equation of Von Karman and Howarth in the theory of isotropic turbulence.

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