PRODUCTS OF SUMMABILITY METHODS AND MERCERIAN TRANSFORMATIONS

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1.1. Introduction

If A and B denote any two regular methods of summability, we denote by AB their iteration product which associates with any given sequence $\{S_n\}$ or function $S(x)$ the A-transform of its B-transform. Following the standard notations of the algebra of transformations, let us consider the transformation U_α, defined by

$$U_\alpha = \alpha E + (1-\alpha)B, \tag{1.1.1}$$

where α is any real number, E and B denote the identity transform and a regular transform respectively, and are compatible in the sense that they are both sequence-to-sequence or function-to-function. If A denotes a regular transform such that the iteration product AB can be defined, we have

$$AU_\alpha = \alpha A + (1-\alpha)AB. \tag{1.1.2}$$

Now, if $A \Longrightarrow AB$, that is, A-summability implies AB-summability, then evidently

$$A \Longrightarrow AU_\alpha. \tag{1.1.3}$$

If $\alpha = 0$, the statement becomes tautologous.

We observe that (except in the trivial case $\alpha = 1$), unless $A \Longrightarrow AB$, we do not necessarily have (1.1.3).

The object of the present paper is to investigate into the conditions under which, over and above (1.1.3), we shall have

$$AU_\alpha \Longrightarrow A, \tag{1.1.4}$$

so that finally we shall obtain the equivalence relation

$$AU_\alpha \approx A. \tag{1.1.5}$$

Such a relation indicates the translative character of the operation of iteration multiplication by A with respect to the equivalence relation

$$U_\alpha \approx E, \tag{1.1.6}$$

under the conditions specified. We notice that the classical theorem of Mercer gives a special case of (1.1.6).*

1.2. Definitions of the Methods A and B

According to the choice of A- and B-transforms, we shall treat in the next section the following two cases separately.

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* Mercer (1907).

CASE I. A is taken to be a regular sequence-to-function transform, defined by

$$A\{s_n, x\} = \frac{\sum_{0}^{\infty} p_n x^n s_n}{\sum_{0}^{\infty} p_n x^n}, \quad p_n \geqslant 0, 0 < x < \rho, \tag{1.2.1}$$

where ρ ($\rho \leqslant \infty$) is the radius of convergence of the power series $\sum_{0}^{\infty} p_n x^n$, which represents within its circle of convergence the function $f(x)$, say, and the A-summability of $\{S_n\}$ to the sum s is defined by

$$A\{s_n, x\} \longrightarrow s, \text{ as } x \longrightarrow \rho - 0. \tag{1.2.2}$$

In the theorem of this paper we shall be concerned with the following two important special cases of the A-transform.

Case (i): $A = A_1$, defined by

$$p_n = (-1)^n \binom{-k}{n}, \; k > 0;$$

$$f(x) = (1-x)^{-k}, \; \rho = 1.$$

The particular case of A_1, for $k = 1$, is the well-known Abel-transform.

Case (ii): $A = A_2$, defined by

$$p_n = 1/n!, \; f(x) = e^x, \; \rho = +\infty.$$

This is the familiar Borel-transform.

B is taken to be a regular sequence-to-sequence Hausdorff transform, defined by

$$\sigma_n = \sum_{\nu=0}^{\infty} \lambda_{n,\nu} s_\nu \quad (n = 0, 1, 2, \ldots.), \tag{1.2.3}$$

where

$$\lambda_{n,\nu} = \begin{cases} \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu & (\nu \leqslant n) \\ 0 & (\nu > n), \end{cases} \tag{1.2.4}$$

μ_n being a regular moment constant of rank n, defined by

$$\mu_n = \int_0^1 t^n d\chi(t) \; (n = 0, 1, 2, \ldots\ldots), * \tag{1.2.5}$$

where $\chi(t)$ is a real function of bounded variation in $0 \leqslant t \leqslant 1$, and $\chi(1) = 1$, $\chi(+0) = \chi(0) = 0$, so that $\chi(t)$ is continuous at $t = 0$. We adopt the notations:

$$\Delta^0 \mu_k = \mu_k; \; \Delta^p \mu_k = \Delta^{p-1} \mu_k - \Delta^{p-1} \mu_{k+1}, p \geqslant 1; \; \mu_{k,p} = \Delta^p \mu_k, \tag{1.2.6}$$

so that, for $0 \leqslant \nu \leqslant n$,

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* The function t^0 is defined at $t = 0$ so as to be continuous; thus

$$\mu_0 = \int_0^1 d\chi(t).$$

(1.2.7) $$\lambda_{n,\nu} = \binom{n}{\nu} \mu_{\nu,\, n-\nu} = \binom{n}{\nu} \int_0^1 t^\nu (1-t)^{n-\nu} d\chi(t).$$

Following the familiar terminology of Hausdorff summability, we call the transform, thus defined, the (H, μ)-transform of $\{S_n\}$, and say that the sequence $\{S_n\}$ is summable (H, μ) to the sum s, if $\sigma_n \longrightarrow s$, as $n \longrightarrow \infty$.

CASE II. In this case A is the function-to-function transform, defined by

(1.2.8) $$A\{s(x), t\} = \frac{1}{t} \int_0^\infty \psi\left(\frac{x}{t}\right) s(x) dx,$$

where $\psi(x)$ is a positive monotonic decreasing function of x for $x \geqslant 0$, and the A-summability of $s(x)$ to sum s is defined by

$$A\{s(x), t\} \longrightarrow s, \text{ as } t \longrightarrow \infty,$$

with the implication that, for every $t < \infty$, $A\ \{s(x), t\}$ is absolutely convergent.

B is taken to be a regular continuous function-to-function Hausdorff transform, defined by

(1.2.9) $$\sigma_s(x) = \int_0^1 s(xu) d\chi(u),$$

where $\chi(u)$ is the regular mass-function which generates the moment-constant μ_n of (1.2.6). We call the transform defined by (1.2.9) the (H, χ)-transform of $s(x)$, and say that $s(x)$ is (H, χ)-summable to the sum s, if

$$\sigma_s(x) \longrightarrow s, \text{ as } x \longrightarrow \infty.$$

The most important special case of the (H, χ)-transform is the (C, α)-transform, corresponding to the mass-function

$$\chi(u) = 1-(1-u)^\alpha,\ \alpha > 0.$$

In case I the transformation U_α is sequence-to-sequence and in case II it is function-to-function, while in these cases AU_α is sequence-to-function and function-to-function, respectively. We have in case I

(1.2.10) $$AU_\alpha\{s_n\} = A\{U_\alpha\{s_n\}, x\} = \alpha A\{s_n, x\} + (1-\alpha) A\{\sigma_n, x\},$$

and in case II

(1.2.11) $$AU_\alpha\{s(x)\} = A\{U_\alpha\{s(x)\}, t\} = \alpha A\{s(x), t\} + (1-\alpha) A\{\sigma(x), t\}.$$

1.3. THE THEOREM

We establish the following theorem.

Theorem. *If* $$U_\alpha = \alpha E + (1-\alpha) B,$$

and

$$\alpha E + (1-\alpha)(H, \chi) \approx E,$$

then

$$AU_\alpha \approx A.$$

We shall require the following lemma for the proof of our theorem.

Lemma.* *The necessary and sufficient conditions for the transformation*

$$G(y) = \int_0^1 g(ty)d\chi(t)$$

to be regular are

$$\chi(1) = 1, \chi(+0) = \chi(0) = 0.$$

Proof of the theorem.

CASE I. Let

$$A\{s_n, x\} \longrightarrow l, \text{ as } x \longrightarrow \rho - 0.$$

For $0 < x < \rho$, we have

$$f(x)A\{\sigma_n, x\} = \sum_0^\infty p_n x^n \sigma_n$$

(1.3.1)
$$= \sum_0^\infty p_n x^n \sum_0^n \binom{n}{k} s_k \int_0^1 t^k(1-t)^{n-k}\, d\chi(t)$$

$$= \int_0^1 \left\{ \sum_0^\infty p_n x^n \sum_0^n \binom{n}{k} s_k t^k(1-t)^{n-k} \right\} d\chi(t).$$

In order to justify the inversion of the order of integration and summation, we need only prove that

(1.3.2)
$$\int_0^1 \left(\sum_0^\infty p_n x^n \sum_0^n \binom{n}{k} |s_k| t^k(1-t)^{n-k} \right) d\chi(t) < \infty,$$

assuming (as we may, on account of the bounded variation of $\chi(t)$ in $(0, 1)$) that $\chi(t)$ is monotonic increasing in $(0, 1)$. Now, we treat the cases (i) $A = A_1$ and (ii) $A = A_2$, separately.

Case (i): $A = A_1$.

By direct calculation, we find that, for $0 < t < 1$, the integrand in the Stieltjes integral (1.3.2) equals

$$f(x)A\{\,|s_n|, \xi\},$$

where

$$\xi = xt/\{1 - x(1-t)\} \ (< x),$$

which is Borel-measurable, majorized by $\{f(x)\}^2 A_1\{\,|s_n|, x\}$, since

$$A_1\{\,|s_n|, \xi\} < f(\xi)A_1\{\,|s_n|, \xi\} < f(x)A_1\{\,|s_n|, x\}.$$

Therefore (1.3.2) holds, and we finally have the result of (1.3.1), that is

(1.3.3)
$$A_1\{\sigma_n, x\} = \int_0^1 A_1\{s_n, \xi\}\, d\chi(t),$$

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* Hardy (1949), Theorem 217, p. 276.

where
$$\xi = xt/\{1-x(1-t)\}.$$
Putting
$$x = 1-1/(y+1) = y/(y+1),$$
so that
$$y \longrightarrow +\infty, \text{ as } x \longrightarrow 1-0, \text{ and}$$
$$\xi = yt/(yt+1),$$
and setting
$$A_1\left\{s_n, \frac{y}{y+1}\right\} = \bar{A}_1\{s_n, y\},$$
we finally obtain

(1.3.4)
$$A_1 U_\alpha\{s_n\} = \alpha \bar{A}_1\{s_n, y\} + (1-\alpha)\int_0^1 \bar{A}_1\{s_n, yt\} d\chi(t).$$

Now, using the lemma, we conclude that if
$$A_1\{s_n, x\} \longrightarrow l, \text{ as } x \longrightarrow 1-0,$$
that is, if
$$\bar{A}_1\{s_n, y\} \longrightarrow l, \text{ as } y \longrightarrow +\infty,$$
then
$$\int_0^1 \bar{A}_1\{s_n, yt\} d\chi(t) \longrightarrow l, \text{ as } y \longrightarrow +\infty,\text{*}$$
and, therefore, from (1.3.4),
$$A_1 U_\alpha\{s_n\} \longrightarrow l, \text{ as } x \longrightarrow 1-0,$$
that is

(1.3.5)
$$A_1 \Longrightarrow A_1 U_\alpha.$$

Let us now assume that
$$A U_\alpha\{s_n\} \longrightarrow l, \text{ as } x \longrightarrow 1-0,$$
that is, by (1.2.10),
$$\alpha A\{s_n, x\} + (1-\alpha) A\{\sigma_n, x\} \longrightarrow l, \text{ as } x \longrightarrow 1-0.$$
We have
$$\alpha A\{s_n, x\} + (1-\alpha) A\{\sigma_n, x\}$$
$$= \frac{1}{f(x)}\left[\alpha \sum_0^\infty p_n x^n s_n \int_0^1 d\chi(t) + (1-\alpha)\sum_0^\infty p_n x^n \sum_0^n \binom{n}{k}\right.$$

(1.3.6)
$$\left. \times\, s_k \int_0^1 t^k(1-t)^{n-k} d\chi(t)\right]$$
$$= \frac{1}{f(x)}\left[\sum_0^\infty p_n x_n \int_0^1 \left\{\alpha s_n + (1-\alpha)\sum_0^n \binom{n}{k} s_k t^k (1-t)^{n-k}\right\} d\chi(t)\right]$$
$$= \frac{1}{f(x)}\int_0^1 \left\{\sum_0^\infty p_n x^n \left\{\alpha s_n + (1-\alpha)\sum_0^n \binom{n}{k} s_k t^k (1-t)^{n-k}\right\} d\chi(t)\right.$$

* Cf. Pati (1954).

$$= \alpha \int_0^1 \frac{\sum_0^\infty p_n x^n s_n}{f(x)} d\chi(t) + (1-\alpha) \int_0^1 \frac{\sum_0^\infty p_n x^n \sum_0^n \binom{n}{k} s_k t^k (1-t)^{n-k}}{f(x)} d\chi(t)$$

$$= \alpha A\{s_n, x\} + (1-\alpha) \int_0^1 \frac{\sum_0^\infty p_n x^n \sum_0^n \binom{n}{k} s_k t^k (1-t)^{n-k}}{f(x)} d\chi(t).$$

Therefore, we have the identity (1.3.4), and, in view of (1.3.5), we conclude that

$$A_1 U_\alpha \approx A_1,$$

if $\quad$ '$\alpha \bar{A}_1 \{s_n, y\} + (1-\alpha) \int_0^1 \bar{A}_1 \{s_n, yt\} d\chi(t) \longrightarrow l$, as $y \longrightarrow \infty$'

implies that

$$\bar{A}_1 \{s_n, y\} \longrightarrow l, \text{ as } y \longrightarrow \infty,$$

and this is so, by virtue of the hypothesis:

$$\alpha E + (1-\alpha)(H, \chi) \approx E.$$

Case (ii): $A = A_2$. Let us assume that

$$A_2\{s_n, x\} \longrightarrow l, \text{ as } x \longrightarrow \infty.$$

In this case, by direct calculation, we find that, for $0 < t < 1$, the integrand in the Stieltjes integral (1.3.2) equals $f(x)A_2\{ |s_n|, xt\}$, which is Borel-measurable, majorized by $\{f(x)\}^2 A_2\{ |s_n|, x\}$.

Thus (1.3.2) holds, and we have the result of (1.3.1), that is

$$A_2\{\sigma_n, x\} = \int_0^1 A_2\{s_n, xt\} d\chi(t), \tag{1.3.7}$$

and, therefore, finally

$$A_2 U_\alpha \{s_n\} = \alpha A_2\{s_n, x\} + (1-\alpha) \int_0^1 A_2\{s_n, xt\} d\chi(t). \tag{1.3.8}$$

From (1.3.8) we infer that if

$$A_2\{s_n, x\} \longrightarrow l, \text{ as } x \longrightarrow \infty,$$

then

$$A_2 U_\alpha \{s_n\} \longrightarrow l, \text{ as } x \longrightarrow \infty,$$

by virtue of the lemma, that is

$$A_2 \Longrightarrow A_2 U_\alpha. \tag{1.3.9}$$

Let us now assume that

$$A_2 U_\alpha \{s_n\} \longrightarrow l, \text{ as } x \longrightarrow \infty.$$

Then, as in case (i), we shall have the identity (1.3.8) on an inversion of the order of integration and summation by virtue of the convergence of $A_2 U_\alpha \{ s_n \}$, and in view of (1.3.9) we conclude that

$$A_2 U_\alpha \approx A_2,$$

if only

$$`\alpha A_2\{s_n, x\} + (1-\alpha)\int_0^1 A_2\{s_n, xt\}d\chi(t) \longrightarrow l, \text{ as } x \longrightarrow \infty'$$

implies that

$$A_2\{s_n, x\} \longrightarrow l, \text{ as } x \longrightarrow \infty,$$

and this is so by virtue of the hypothesis:

$$\alpha E + (1-\alpha)(H, \chi) \approx E.$$

CASE I. In this case

$$AU_\alpha\{s(x)\} = \alpha A\{s(x), t\} + (1-\alpha)A\{\sigma_s(x), t\}. \tag{1.3.10}$$

Let us assume that

$$A\{s(x), t\} \longrightarrow l, \text{ as } t \longrightarrow \infty.$$

Now

$$\begin{aligned} A\{\sigma_s(x), t\} &= \frac{1}{t}\int_0^\alpha \psi\left(\frac{x}{t}\right)\sigma_s(x)dx \\ &= \frac{1}{t}\int_0^\infty \psi\left(\frac{x}{t}\right)\left(\int_0^1 s(xu)d\chi(u)\right)dx \\ &= \frac{1}{t}\int_0^\infty \psi\left(\frac{x}{t}\right)dx \int_0^x s(v)d\left[\chi(v/x)\right] \qquad (1.3.11) \\ &= \frac{1}{t}\int_0^\infty s(v)dv \int_v^\infty \psi\left(\frac{x}{t}\right)d\left[\chi(v/x)\right]. \end{aligned}$$

To justify the equality of the last two iterated integrals in (1.3.11) we need only observe that, if we assume (as we may) that $\chi(t)$ is monotonic increasing, then

$$\begin{aligned} &\frac{1}{t}\int_0^\infty |s(v)|dv \int_v^\infty \psi\left(\frac{x}{t}\right)d\left[\chi(v/x)\right] \\ &< \frac{1}{t}\int_0^\infty |s(v)|\psi\left(\frac{v}{t}\right)dv, \end{aligned} \tag{1.3.12}$$

which is absolutely convergent by virtue of the proviso that the A-transform: $A\{s(x), t\}$ is absolutely convergent for every finite t. Thus $A\{\sigma_s(x), t\}$ exists, and by a change of order of integration in (1.3.11), we have

$$A\{\sigma_s(x), t\} = \frac{1}{t}\int_0^\infty \psi\left(\frac{x}{t}\right)\left(\int_0^1 s(xu)d\chi(u)\right)dx$$

$$(1.3.13) \qquad = \int_0^1 d\chi(u)\,\frac{1}{t}\int_0^\infty \psi\left(\frac{x}{t}\right) s(xu)\,dx$$

$$= \int_0^1 d\chi(u)\,\frac{1}{tu}\int_0^\infty \psi\left(\frac{x}{tu}\right) s(x)\,dx$$

$$= \int_0^1 A\,\{s(x),\, tu\}\,d\chi(u),$$

and, therefore, finally

$$(1.3.14) \qquad AU_\alpha\,\{s(x)\} = \alpha A\,\{s(x),\, t\} + (1-\alpha)\int_0^1 A\{s(x),\, tu\}\,d\chi(u).$$

From (1.3.14) we infer by appealing to the lemma that if

$$A\,\{s(x),\, t\} \longrightarrow l, \text{ as } t \longrightarrow \infty,$$

then

$$AU_\alpha\,\{s(x)\} \longrightarrow l, \text{ as } t \longrightarrow \infty,$$

that is

$$(1.3.15) \qquad A \Longrightarrow AU_\alpha.$$

Let us now assume that

$$AU_\alpha\,\{s(x)\} \longrightarrow l, \text{ as } t \longrightarrow \infty.$$

This is evidently equivalent to the assertion that

$$(1.3.16) \qquad \alpha A\,\{s(x),\, t\} + (1-\alpha)\int_0^1 A\,\{s(x),\, tu\}\,d\chi(u) \longrightarrow l,$$

as $t \longrightarrow \infty$. In view of (1.3.15),

$$AU_\alpha \approx A,$$

if (1.3.16) implies that

$$A\{s(x),\, t\} \longrightarrow l, \text{ as } t \longrightarrow \infty,$$

and this is so by virtue of the hypothesis:

$$\alpha E + (1-\alpha)\,(H,\, \chi) \approx E.$$

This completes the proof of the theorem.

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