ON DECAY OF ENERGY SPECTRUM OF ISOTROPIC TURBULENCE

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1. Introduction

The concept of similarity applied to the decay process of homogeneous isotropic turbulence led Heisenberg (1948*a*, *b*) to propose a spectrum energy function $F(k, t)$ which is governed by the integro-differential equation:

$$-\frac{\partial}{\partial t}\int_0^k F(k', t)dk' = 2(\nu+\eta_K)\int_0^k k'^2F(k', t)dk' \qquad .. \qquad .. \quad (1)$$

where ν is the kinematic viscosity coefficient, and η_k the turbulent viscosity coefficient defined by Heisenberg as

$$\eta_k = K\int_0^\infty \{F(k', t)/k'^3\}^{\frac{1}{2}}dk'. \qquad .. \qquad .. \qquad .. \quad (2)$$

It has been shown (Sen, 1951) that during the earlier part of decay motion when the inertia terms of the equations of motion dominate the decay process, so that ν may be taken negligible compared to η_K (Reynolds number R large), a self-preserving type of solution of (1) and (2) is given by

$$F(k, t) = \frac{1}{K^2}\frac{1}{k_0^3t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left(k/k_0.(t/t_0)^c\right) \quad .. \qquad .. \qquad .. \quad (3)$$

where c is a constant, and $f(x)$ is given by the equation

$$2(1-c)\int_0^x f(x')dx' - cf(x) = 2\int_x^\infty \sqrt{f(x')/x'^3}dx'.\int_0^x x'^2f(x')dx'. \qquad .. \quad (4)$$

Solution (3) of equations (1) and (2) has the following properties

$$f(x) \sim x^{(2-3c)/c}(x\to 0), \text{ and } f(x) \sim x^{-5/3}(x\to\infty). \qquad .. \qquad .. \quad (5)$$

The solution of (1) and (2) given by Heisenberg (1948*b*), and later fully discussed by S. Chandrasekhar (1950) happens to correspond to the particular case $c = \frac{1}{2}$ (but with an equation different from (4)), of (3). Heisenberg's solution satisfies the complete equation (1), with $\nu \neq 0$. In fact it can be easily shown that a function of form (3) is the solution of the complete equation (1) only when $c = \frac{1}{2}$ ($f(x)$ then will be given by an equation different from (4)). Further one notes the interesting point that the linear Heisenberg spectrum for $x \to 0$, in the case $\nu \neq 0$, is also given by (5) when one puts in it $c = \frac{1}{2}$. The linear spectrum is not, however, consistent with the fourth power law, discovered by Lin (1947), and Batchelor (1949), which may be expressed as follows

$$\lim_{k\to 0}\frac{F(k, t)}{k^4} = J \qquad .. \qquad .. \qquad .. \qquad .. \quad (6)$$

J being the Loitsiansky's constant.

It is clear from above that during the early stage of decay of turbulence, when one may take $\nu \ll \eta_K$, a family of self-preserving spectrum functions represented by (3), with an open parameter c, is possible as solution of the Heisenberg equations (1) – (2). For the particular case $c = 2/7$, one notices from (5) that $f(x) \sim x^4$, as $x \to 0$, so that the fourth power law (6) appears as a limiting consequence of one of this family of self-preserving spectrum functions (3). The entire family of these solutions converges to the Kolmogoroff spectrum as $x \to \infty$, as is indicated by the second equation of (5).

The member $c = 2/7$ of this family has the further properties that the decay law of energy is given by

$$u'^2 \propto t^{-10/7} \qquad (7)$$

and in Taylor's formula for overall decay rate, viz.

$$\frac{1}{u'^2}\frac{du'^2}{dt} = -\frac{10\nu}{\lambda^2} \qquad (8)$$

one finds for this case

$$\lambda^2 = 7\nu t, \quad \text{and} \quad \epsilon = \frac{1}{u'^2}\frac{du'^2}{dt} \propto t^{-1}. \qquad (9)$$

Kolmogoroff in fact showed before that (7) and (9) follow provided one assumes the validity of the fourth power law which is here shown as a limiting form of the solution of the Heisenberg equation for $\nu = 0$.

2. Stability of Solution giving the Fourth Power Law

Thus if the motion be characterized by high Reynolds number and the effect of inertia terms remains dominant, the fourth power law is also the asymptotic form of a self-preserving spectrum function corresponding to $c = 2/7$ in (3). Hence if the similarity character of the spectrum function is to extend even up to the stage of validity of the fourth power law for large Reynolds number, the function may be represented by a form (3), with $c = 2/7$.

This one may also understand as follows. C. C. Lin (1953) has shown that the Heisenberg spectrum function corresponding to $c = \frac{1}{2}$ in (3) but with corresponding $f(x)$ has the property that if a disturbance be produced in a narrow band of the spectrum, then the ratio of initial rate of disappearance of this perturbation energy to the overall rate of decay of energy in the quasi-equilibrium spectrum is less than unity for a considerable part of the spectrum (represented by the fraction of the total energy up to the perturbed band of the spectrum) but the total dissipation in this part is negligibly small. Outside of this part for the rest of the spectrum the dissipation of the perturbation energy is very much quicker than the overall rate of dissipation of the entire spectrum. This suggests that the Heisenberg spectrum is possibly quite stable (quasi-stable).

We may apply the same test to the energy spectrum (3) with $c = \frac{2}{7}$. K. M. Ghosh (1955) has worked out two numerical solutions for $c = \frac{2}{7}$. The function $f(x)$ has been calculated numerically for two different values of $\{x^3 f(x)\}_{\max}$. Using these solutions we calculate, as with C. C. Lin (1947), the functions

$$\left.\begin{aligned} P(x^*) &= \frac{\displaystyle\int_0^{x^*} f(x)dx}{\displaystyle\int_0^{\infty} f(x)dx} \\ \text{and} \qquad & \\ \gamma(x^*) &= \frac{\dfrac{d}{dt}(\log e)}{\dfrac{d}{dt}(\log u'^2)} \end{aligned}\right\} \qquad (10)$$

e being the perturbation energy of turbulence assumed to have been introduced in a narrow band at x^*. The (P, γ) curves (Fig. 1) suggest that at high Reynolds number ($\nu \sim o$), for solution corresponding to $c = \frac{2}{7}$ the decay of energy of perturbation is everywhere very much slower than the overall decay of the Heisenberg energy spectrum. This seems to indicate that the spectrum function (3) for $c = \frac{2}{7}$, which gives the fourth power law for $k \to 0$, represents an unstable situation.

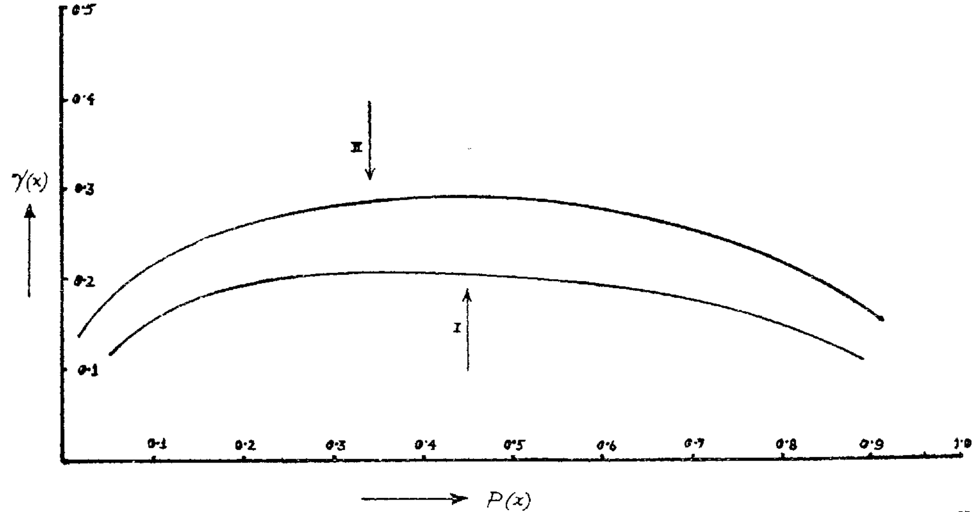

FIG. 1. γ, the ratio of the rate of decay of energy of localized perturbation to the overall rate of decay of energy of the turbulence spectrum (reaction time), plotted against P, the fraction of turbulence energy, for two motions with $R = \infty$.

One may suggest that for high Reynolds number the ultimate more or less stable Heisenberg spectrum (corresponding to $c = \frac{1}{2}$) is developed through a series of labile stages represented by varying values of c in the spectrum function (3).

It may be noted that the energy corresponding to (3) is given by

$$u'^2 = \text{const. } t^{-2(1-c)} \int_0^\infty f(x)dx. \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

If one supposes that c starts from smaller values at the labile stages conditioned by the turbulence generating apparatus and increases to the value $\frac{1}{2}$, corresponding to quasi-stable Heisenberg spectrum, then the rate of decay with time of the energy of the spectra of the labile stages also reaches its lowest value when the quasi-stable state is attained.

The above suggestion of stability was made in previous papers (1951, 1955).

The present discussion seems to support this idea.

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ABSTRACT

The family of self-preserving solutions of Heisenberg's equation for decay of isotropic turbulence has been examined. It appears from an examination of the stability of the spectrum associated with the fourth power law for small k that the solution of Heisenberg and Chandrasekhar represents the most stable member of the family to which possibly all previous labile motions converge.

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