A RADIATING MASS PARTICLE IN AN EXPANDING UNIVERSE

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1. Introduction

McVittie (1933) derived a solution of the field equations of general relativity which was a generalization of Schwarzschild's solution in isotropic co-ordinates and which represented the field of a mass particle in the expanding universe. He and later Järnefelt (1941) worked out the geodesics of this field and concluded that the expanding space had no appreciable effect on the orbits of test particles round mass singularities. McVittie derived the metric

$$ds^2 = \left(\frac{1-\mu(t)/2r}{1+\mu(t)/2r}\right)^2 . dt^2-(1+\mu(t)/2r)^4 e^{\beta(t)} . [dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)] \quad .. \quad (1.1)$$

with $\dot{\beta}/2 = -\dot{\mu}/\mu$

a dot indicating differentiation with regard to t. The space round this mass particle is not empty but is occupied by a static distribution of matter with non-zero density and isotropic pressure, the distribution being spherically symmetric round the mass particle.

Later, Einstein and Straus (1945) and recently Gilbert (1956), working on the same problem of the effect of the expanding space on the gravitational fields of individual stars, derived solutions which were distinct from McVittie's solution in the sense that the space immediately surrounding the star was empty (not occupied by matter) and that the field in this space passed over continuously to the expanding cosmological field over a definite boundary $r = P$. Since Einstein and Straus actually found a solution in which for $r \leq P$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = 0$$

and for $r > P$

$$-\frac{1}{8\pi} G_{\mu\nu} = \rho v_{\mu} v_{\nu}, \quad .. \quad .. \quad .. \quad .. \quad (1.2)$$

they concluded that the effect of the expansion of space was cut off at the boundary $r = P$ so that the field surrounding an individual star was unaffected by the expanding space. Qvist (1947) later studied this solution in greater detail and reached the conclusion that such a cut-off radius $r = P$ for the effect of the expansion of space could be found by Einstein and Straus only because they had taken the

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pressure p of the smoothed out universe to be zero, i.e. because they had assumed (1.2) in place of the more general form

$$-\frac{1}{8\pi}G_{\mu\nu} = (p+\rho)v_\mu v_\nu - pg_{\mu\nu}. \qquad (1.3)$$

If we try to study the Einstein-Straus problem in the case when the mass singularity at the origin of co-ordinates is a source of radiation, we immediately come across a fundamental difficulty over the cut-off boundary $r = P$. For now in the space surrounding the singularity, i.e. in the region $r \leqslant P$, $G_{\mu\nu} \neq 0$, this space is being traversed by radially flowing radiation. For radial flow of radiation we have, instead,

$$-\frac{1}{8\pi}G_{\mu\nu} = \sigma w_\mu w_\nu, \quad w_\mu w^\mu = 0, \quad (w^\mu)_\nu w^\nu = 0,$$

σ being the density of flowing radiation. The boundary sphere $r = P$ will now not be static but be the wavefront of the emitted radiation. The equations of fit for such boundaries have been studied by Sen (1924), O'Brien and Synge (1952) and one of the conditions to be satisfied at such boundaries is (in the notations of O'Brien and Synge) the continuity of $T^\nu_\mu w^\mu$. It can now be verified that if for $r < P$,

$$T^\nu_\mu = \sigma w_\mu w^\nu$$

and for $r > P$,

$$T^\nu_\mu = \rho v_\mu v^\nu$$

then, for $r < P$,

$$T^4_\mu w^\mu = 0$$

while for $r > P$,

$$T^4_\mu w^\mu = \rho w^4 \neq 0\,.$$

Hence this particular O'Brien-Synge condition cannot be satisfied at the boundary $r = P$. It is thus not possible to locate a definite cut-off radius $r = P(t)$ such that on one side of it we have a region traversed by pure flowing radiation and on the other side the expanding cosmic space (Vaidya and Shah, 1956).

A source of radiation embedded in an expanding universe presents a problem which is distinct from the Einstein-Straus problem of a cold static singularity. In other words the field outside a singularity, which is a source of radiation, is rather of the McVittie type than of the Einstein-Straus type. Such a field is presented in the following pages.

2. The Field Equations

The Einstein-de Sitter universe with zero curvature can be described by a line-element of the form

$$ds^2 = e^{g(T)}[-dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + dT^2]. \qquad (2.1)$$

The null-spheres (Synge and McConnell, 1928) in the field (2.1) are given by the equation

$$r - T = \text{constant}.$$

Choosing a new time co-ordinate t such that these null-surfaces become t = constant, we can transform the line-element (2.1) to the form

$$ds^2 = e^{g(r-t)}[-r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2 - 2drdt].$$

Now for the field of an isolated particle we have the line-element

$$ds^2 = -r^2(d\theta^2+\sin^2\theta d\phi^2)+\left(1-\frac{2m}{r}\right)dt^2-2drdt \quad \ldots \quad (2.2)$$

Here $m =$ a constant gives the Schwarzschild solution of a static singularity while $m =$ an undetermined function of t gives the field of a radiating mass particle at the origin (Vaidya, 1953).

In order to consider the field of a mass particle (static or radiating) embedded in an expanding universe we choose a line-element of the form

$$ds^2 = e^g[-r^2(d\theta^2+\sin^2\theta d\phi^2)+(1+V)dt^2-2drdt]$$
$$g = g(x),\ x = r-t,\ V = V(r, t) \quad \ldots \quad (2.3)$$

The surviving components of the tensor $G_{\mu\nu} = R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$ are

$$-G^1_1 = e^{-g}\left[-(1+V)\left\{\frac{3}{4}\overset{*2}{g}+\frac{2\overset{*}{g}}{r}+\frac{\overset{*}{g}V'}{2(1+V)}+\frac{V'}{r(1+V)}+\frac{1}{r^2}\right\}+\overset{**}{g}+\overset{*2}{g}+\frac{2\overset{*}{g}}{r}+\frac{1}{r^2}\right]$$

$$-G^2_2 = -G^3_3 = e^{-g}\left[-(1+V)\left\{\overset{**}{g}+\frac{\overset{*2}{g}}{4}+\frac{\overset{*}{g}}{r}+\frac{\overset{*}{g}V'}{(1+V)}+\frac{V''}{2(1+V)}+\frac{V'}{r(1+V)}\right\}\right.$$
$$\left.+2\overset{**}{g}+\frac{\overset{*2}{g}}{2}+\frac{\overset{*}{g}}{r}\right] \quad \ldots \quad (2.4)$$

$$-G^4_4 = e^{-g}\left[-(1+V)\left\{\overset{**}{g}+\frac{\overset{*2}{g}}{4}+\frac{2\overset{*}{g}}{r}+\frac{\overset{*}{g}V'}{2(1+V)}+\frac{V'}{r(1+V)}+\frac{1}{r^2}\right\}+\overset{**}{g}+\overset{*2}{g}+\frac{2\overset{*}{g}}{r}+\frac{1}{r^2}\right]$$

$$-G^1_4 = e^{-g}\left[(1+V)\left\{-\overset{**}{g}+\frac{\overset{*2}{g}}{2}+\frac{\overset{*}{g}V'}{2(1+V)}+\frac{\overset{*}{g}\dot{V}}{2(1+V)}+\frac{\dot{V}}{r(1+V)}\right\}+\overset{**}{g}-\frac{\overset{*}{g}}{2}\right].$$

Here an overhead star indicates a differentiation with regard to x, an overhead dash indicates that with regard to r, and an overhead dot indicates that with regard to t.

We take the space surrounding the mass particle as occupied by radially flowing radiation of density σ together with some spherically symmetrical distribution of matter with non-zero density ρ and pressure p as demanded by O'Brien-Synge jump conditions. For such a mixture of matter and flowing radiation, we take the energy tensor of the form

$$T^{\nu}_{\mu} = (p+\rho)v_{\mu}v^{\nu}-pg^{\nu}_{\mu}+\sigma w_{\mu}w^{\nu}. \quad \ldots \quad (2.5)$$

For the line-element of the form (2.3) we can take

$$v^1 = v^2 = v^3 = 0, \qquad v_4v^4 = 1\,; \quad \ldots \quad (2.6)$$

$$w^2 = w^3 = w^4 = 0, \qquad w_{\mu}w^{\mu} = 0,\ (w^{\mu})_{\nu}w^{\nu} = 0. \quad \ldots \quad (2.7)$$

Then

$$T^1_1 = T^2_2 = T^3_3 = -p,\ T^4_4 = \rho,\ T^1_4 = -e^g\sigma(w^1)^2. \quad \ldots \quad (2.8)$$

The field equations $G_{\mu\nu} = -8\pi T_{\mu\nu}$ lead to

$$-\frac{V''}{2}+\frac{V}{r^2}=\frac{V'\overset{*}{g}}{2}+V\left(\overset{**}{g}-\frac{\overset{*2}{g}}{2}-\frac{\overset{*}{g}}{r}\right) \quad \ldots \quad (2.9)$$

$$8\pi p = G^1_1,\ 8\pi\rho = -G^4_4,\ 8\pi\sigma = \frac{e^{-g}}{(w^1)^2}G^1_4. \quad \ldots \quad (2.10)$$

3. Approximate Solutions of the Field Equations

(A) In the zeroeth approximation, we put $g = 0$. Equation (2.9) then becomes

$$-\frac{V''}{2}+\frac{V}{r^2}=0 \quad \ldots \quad (3.1)$$

The solution of (3.1) is

$$V=-\frac{2m}{r}+Ar^2 \quad \ldots \quad (3.2)$$

where m and A are arbitrary functions of t only. To remove the singularity at $r = \infty$, we put $A = 0$. Thus we get, in this approximation,

$$V=-\frac{2m}{r}. \quad \ldots \quad (3.3)$$

(2.3) will then describe the field (2.2) of an isolated radiating star.

(B) Next, in the first approximation, we neglect the terms $\overset{*2}{g}$, $\overset{**}{g}$ in the right hand side of (2.9) which then becomes, after substitution from (3.3),

$$-\frac{V''}{2}+\frac{V}{r^2}=\frac{3m\overset{*}{g}}{r^2}. \quad \ldots \quad (3.4)$$

The solution of (3.4) is

$$V=-\frac{2m}{r}+3m\overset{*}{g}. \quad \ldots \quad (3.5)$$

Using (3.5), (2.10) leads to

$$p=\rho=0$$

$$8\pi\sigma=\frac{e^{-2g}}{(w^1)^2}\left[\frac{-2\dot{m}\overset{*}{g}}{r}+\frac{2\dot{m}}{r^2}-\frac{m\overset{*}{g}}{r^2}\right]. \quad \ldots \quad (3.6)$$

If we take

$$\frac{\dot{m}}{m}=\frac{\overset{*}{g}}{2} \quad \ldots \quad (3.7)$$

(3.6) leads to $\sigma = 0$ in this approximation.

Hence our solution, in this approximation, with the restriction (3.7), gives McVittie's solution of a mass particle in an expanding universe.

(C) In the second approximation, we neglect the terms $\overset{*3}{g}$, $\overset{***}{g}$, $\overset{*}{g}\,\overset{**}{g}$. The field equation (2.9) then becomes, after substitution from (3.5) for V in the coefficients of $\overset{*}{g}$ and from (3.3) for V in the coefficients of $\overset{*2}{g}$ and $\overset{**}{g}$,

$$-\frac{V''}{2}+\frac{V}{r^2}=\frac{3m\overset{*}{g}}{r^2}-\frac{2m}{r}(\overset{**}{g}+\overset{*2}{g}). \quad \ldots \quad (3.8)$$

Solving this, we get

$$V = -\frac{2m}{r}+3m\overset{*}{g}-2m(\overset{**}{g}+\overset{*2}{g})r. \qquad \ldots \qquad (3.9)$$

With (3.9), (2.10) gives

$$8\pi p = e^{-g}\left[-\frac{\overset{*2}{g}}{4}\left(1-\frac{2m}{r}\right)-\overset{**}{g}\left(1+\frac{m}{r}\right)\right] \qquad \ldots \qquad (3.10)$$

$$8\pi\rho = e^{-g}\left[\frac{3}{4}\overset{*2}{g}\left(1-\frac{2m}{r}\right)+\frac{3m\overset{**}{g}}{r}\right] \qquad \ldots \qquad (3.11)$$

$$8\pi\sigma = \frac{e^{-2g}}{(\omega^1)^2}\left[\frac{2\dot{m}}{r^2}-\frac{\overset{*}{g}}{r}\left(2\dot{m}+\frac{m}{r}\right)+\overset{*2}{g}\left(\frac{\dot{m}}{2}+\frac{m}{r}\right)+\overset{**}{g}\left(2\dot{m}+\frac{m}{r}\right)\right]. \qquad (3.12)$$

Up to this order, McVittie's relation (3.7) will take the form

$$\frac{\dot{m}}{m}=\frac{\overset{*}{g}}{2}-\frac{\overset{**}{g}\,r}{2}.$$

If we, therefore, put

$$\dot{m} = m\left(\frac{\overset{*}{g}}{2}-\frac{\overset{**}{g}\,r}{2}\right)+\dot{\alpha}, \qquad \ldots \qquad (3.13)$$

$\dot{\alpha}$ being a function of t only and of the order of $\overset{*2}{g}$, equation (3.12) will give

$$8\pi\sigma = \frac{e^{-2g}}{(w^1)^2}\frac{2\dot{\alpha}}{r^2} \qquad \ldots \qquad (3.14)$$

which agrees with the density of flowing radiation as obtained in the case when the star is considered as isolated. Thus in this approximation we have the field of a radiating McVittie star. The method of approximation used here is quite general and one can work out the solution to any desired degree of approximation.

Since the effect of the radiation from a normal star does not produce any appreciable effect on the geodesics as calculated from the Schwarzschild's field of a star, one expects that the geodesics in the McVittie field of a star are not appreciably affected by regarding the star as radiating. We have verified that this is so.

4. Conclusion

The jump conditions of Sen and O'Brien and Synge provide an immediate verification of the intuitive fact that the Einstein-Straus picture of a non-radiating mass is not applicable to the field description of a radiating mass. A solution for a non-static (radiating) mass particle in an expanding universe is obtained. It is a generalization of the solution obtained by McVittie (1933).

Abstract

The McVittie problem of a mass particle in an expanding universe is solved in the case when the mass particle is a source of radiation.

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