MAXIMAL COVERING DOMAINS

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1. Let $\mathscr{K}$ be the class of closed, bounded and symmetrical convex domains in the plane. Let $K \in \mathscr{K}$. A lattice Λ is called a *covering lattice* for K if every point of the plane lies in at least one of the domains obtained from K by applying to it all possible lattice translations. The upper bound of the determinants $d(\Lambda)$ of the lattices Λ which are covering lattices for K is called the *covering constant* of K and is denoted by $c(K)$.

It follows immediately from the definition that if $K' \supset K$ then $c(K') \geqslant c(K)$. By analogy with the irreducible bodies of Mahler [1946] we define K to be a *covering domain maximal in* $\mathscr{K}$ if for all $K' \supset K$ (the inclusion being proper), $K' \in \mathscr{K}$ we have $c(K') > c(K)$. When there is no danger of confusion, we shall also call such bodies K *maximal in* $\mathscr{K}$ or $\mathscr{K}$*-maximal.*

In §2 we prove that parallelograms and ellipses are $\mathscr{K}$*-maximal.* We also make a conjecture about the class of $\mathscr{K}$-maximal domains. In §3 we obtain a necessary and sufficient condition for a strictly convex domain $K \in \mathscr{K}$ to be maximal in $\mathscr{K}$. In §4 we show the equivalence of the conjecture of §2 after slight modification to an interesting conjecture about ellipses. We indicate possible generalisations in §5.

If **a**, **b** are any two points we denote the length of the segment **ab** by $|\mathbf{ab}|$. If S is any set we denote its area, if it exists, by $a(S)$.

2. Let $K \in \mathscr{K}$. We write $t(K)$ for the area of the largest triangle inscribed in K. Then it is known (see Bambah [1954]) that

$$c(K) = 2t(K). \qquad \qquad (1)$$

From this we easily conclude

Theorem 1 :—*Parallelograms are* $\mathscr{K}$*-maximal.*

As a particular case of a Theorem of Sas [1939] we have the

Lemma : Let $K \in \mathscr{K}$. *Then*

$$t(K) \geqslant \frac{3\sqrt{3}}{4\pi} a(K), \qquad \qquad (2)$$

the equality sign holding if and only if K *is an ellipse.*

This lemma together with (1) easily gives

Theorem 2 :—*Ellipses are* $\mathscr{K}$*-maximal*

I conjecture that

Ellipses and parallelograms are the only domains of $\mathscr{K}$ *that are* $\mathscr{K}$*-maximal.*

3. Let $K \in \mathscr{K}$. Let **p** be a point on the boundary B of K. Let **pqr** be the largest triangle inscribed in K which has one vertex at **p**. (The existence of such

a triangle for every $\mathbf{p} \in B$ can be proved by using the properties of functions continuous on compact sets). We define $t(\mathbf{p})$ to be the area of $\mathbf{pqr}$. Then clearly

$$t(K) = \sup_{\mathbf{p} \in B} t(\mathbf{p}).$$

We now prove

THEOREM 3:—*Let $K \in \mathscr{K}$ be strictly convex. Then the necessary and sufficient condition for K to be $\mathscr{K}$-maximal is that $t(\mathbf{p})$ has the same value for all points $\mathbf{p}$ on B.*

Proof:—We first prove that the condition is sufficient.

Since $t(\mathbf{p})$ is constant for all $\mathbf{p} \in B$, it follows that

$$c(K) = 2t(K) = 2t(\mathbf{p}), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where $\mathbf{p}$ is any point on B.

Let $K \subset K' \in \mathscr{K}$, the inclusion being proper. Then there is a point $\mathbf{p} \in B$ which is an inner point of K'. Let $\mathbf{pqr}$ be the largest triangle with a vertex at $\mathbf{p}$ that can be inscribed in K. Since $\mathbf{p}$ is an inner point of K', there exists a point $\mathbf{p}' \in K'$ whose distance from $\mathbf{qr}$ is greater than that of $\mathbf{p}$ from $\mathbf{qr}$. Since $\mathbf{p}'\,\mathbf{qr}$ is inscribed in K' and since its area is bigger than that of $\mathbf{pqr}$ it follows that

$$c(K') = 2t(K') \geqslant 2a(\mathbf{p}'\mathbf{qr}) > 2t(\mathbf{p}) = 2t(K) = c(K),$$

so that the condition is sufficient.

We next prove the necessity of the condition.

Since K contains an inner point and hence a circle C, one can easily show that for every $\mathbf{p} \in B$, $t(\mathbf{p}) \geqslant t(C) = \gamma$, say. Since K is bounded, its diameter* δ is finite.

We first prove that $t(\mathbf{p})$ is a continuous function of $\mathbf{p}$. Let $\mathbf{pqr}$ be the triangle with largest area inscribed in K and having a vertex at $\mathbf{p}$. Since $a\,(\mathbf{pqr}) = t(\mathbf{p}) \geqslant \gamma$, and since the length $|\mathbf{qr}| \leqslant \delta$ it follows that the distance ρ of $\mathbf{p}$ from $\mathbf{qr}$ satisfies the relation

$$\rho \geqslant 2\gamma/\delta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Let $\mathbf{p}'$ be a point of B with $|\mathbf{pp}'| < \epsilon$, where $\epsilon > 0$ is any small fixed number. Let $\mathbf{p}'\mathbf{q}'\mathbf{r}'$ be the largest triangle with a vertex at $\mathbf{p}'$ inscribed in K.

Consider the triangle $\mathbf{p}'\mathbf{qr}$. If ρ' is the distance of $\mathbf{p}'$ from $\mathbf{qr}$, then

$$\rho' > \rho - \epsilon,$$

and

$$\begin{aligned} a(\mathbf{p}'\mathbf{qr}) &= \frac{1}{2}\rho'\,|\mathbf{qr}| \\ &> \frac{1}{2}(\rho-\epsilon)\,|\mathbf{qr}| \\ &= a(\mathbf{pqr})\left(1-\frac{\epsilon}{\rho}\right) = t(\mathbf{p})\left(1-\frac{\epsilon}{\rho}\right), \end{aligned}$$

so that

$$\begin{aligned} t(\mathbf{p}') \geqslant a(\mathbf{p}'\mathbf{qr}) &> t(\mathbf{p})\left(1-\frac{\epsilon}{\rho}\right) \\ &\geqslant t(\mathbf{p})\left(1-\frac{\epsilon\delta}{2\gamma}\right), \end{aligned}$$

because of (5).

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* $\delta = \sup_{\mathbf{p},\,\mathbf{p}' \in K} |\mathbf{pp}'|$.

By considering $\mathbf{pq'r'}$ we deduce in a similar way that

$$t(\mathbf{p}) > t(\mathbf{p}')\left(1-\frac{\epsilon\delta}{2\gamma}\right),$$

so that

$$t(\mathbf{p})\left(1-\frac{\epsilon\delta}{2\gamma}\right) < t(\mathbf{p}') < t(\mathbf{p})\Big/\left(1-\frac{\epsilon\delta}{2\gamma}\right), \qquad (6)$$

and it follows that as $\epsilon \to 0$, $t(\mathbf{p}') \to t(\mathbf{p})$, which proves the continuity of $t(\mathbf{p})$.

In the rest of the proof we shall write t for $t(K)$.

Suppose $t(\mathbf{p})$ is not constant for all $\mathbf{p} \in B$. Then there exists a $\mathbf{p}$ on B such that

$$t(\mathbf{p}) < t,$$

i.e. $t(\mathbf{p}) = t(1-2\alpha)$ with $0 < 2\alpha < 1$. $\qquad$ (7)

Because of the continuity of $t(\mathbf{p})$, if $\epsilon > 0$ is sufficiently small, then for all $\mathbf{p}' \in B$ with $|\mathbf{pp}'| < \epsilon$, we have

$$t(\mathbf{p}') < t(1-\alpha). \qquad (8)$$

If ϵ is sufficiently small, the circle $C(\mathbf{p})$ with centre $\mathbf{p}$ and radius ϵ meets B in two points $\mathbf{p}_1$, $\mathbf{p}_2$. Let $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ be taclines to B at $\mathbf{p}_1$, $\mathbf{p}_2$. Then

either (i) $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ meet in $C(\mathbf{p})$,

or (ii) $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ meet the boundary of $C(p)$ at $\mathbf{u}_1$, $\mathbf{u}_2$ before meeting each other at a point outside $C(p)$.

Let L denote the region bounded by the arc $\mathbf{p}_1\mathbf{p}_2$ of B, and the lines $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ in the first case and that bounded by the arcs $\mathbf{p}_1\mathbf{p}_2$ of B, $\mathbf{u}_1\mathbf{u}_2$ of $C(p)$ and the lines $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ in the second case. Let L' be the image of L in the centre of K.

Because of the convexity of K, if ϵ is small, $K' = K \cup L$, $\cup L'$ is a member of $\mathscr{K}$. Since K is strictly convex, the taclines $\mathbf{p}_1\mathbf{u}_1$, $\mathbf{p}_2\mathbf{u}_2$ have only the points $\mathbf{p}_1$, $\mathbf{p}_2$ in common with K and K is properly included in K'. Our object now is to show that if $\epsilon > 0$ is sufficiently small, then $t(K') = t(K)$. This will obviously complete the proof of our theorem. Thus all we have to show is that for any triangle $\mathbf{abc}$ inscribed in K' the area $a(\mathbf{abc}) \leqslant t$.

Because of symmetry of K, K', it is enough to consider the following cases:—

(1) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ all lie in K,
(2) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ all lie in L,
(3) two of the points say $\mathbf{a}$, $\mathbf{b}$ lie in L while the third is outside L,
(4) one of the points, say $\mathbf{a}$, lies in L, another, say $\mathbf{b}$, in K while $\mathbf{c}$ lies in K or L'.

In the first case since $\mathbf{abc}$ is inscribed in K, its area is not greater than t and we have nothing to prove.

In the second case, since $\mathbf{abc}$ lies in $C(p)$ we have

$$a(\mathbf{abc}) \leqslant \frac{3\sqrt{3}}{4\pi}\,.\,\pi\epsilon^2 = \frac{3\sqrt{3}}{4}\epsilon^2. \qquad (9)$$

If δ' is the diameter of K', we easily deduce that

$$\delta' \leqslant \delta+2\epsilon. \qquad (10)$$

Therefore in the third case

$$a(\mathbf{abc}) \leqslant \frac{1}{2}|\mathbf{ab}\,\delta|' \leqslant \epsilon(\delta+2\epsilon). \qquad (11)$$

In the fourth case the segment **ac** meets the part of B that bounds L in a point **a**$'$. In case **c** is in L', let **c**$'$ be the point where **ac** meets the part of B that bounds L'. If **c** is in K, for uniformity of treatment call it also **c**$'$. Then

$$a(\mathbf{a}'\mathbf{b}\mathbf{c}') \leqslant t(\mathbf{a}') < t(1-\alpha), \text{ (by (8))},$$

so that

$$\begin{aligned} a(\mathbf{abc}) &= a(\mathbf{a}'\mathbf{b}\mathbf{c}')\,|\,\mathbf{ac}\,|\,/\,|\,\mathbf{a}'\mathbf{c}'\,| \\ &\leqslant a(\mathbf{a}'\mathbf{b}\mathbf{c}')\,\{\,|\,\mathbf{a}'\mathbf{c}'\,|+4\epsilon\}\,/\,|\,\mathbf{a}'\mathbf{c}'\,| \\ &\leqslant a(\mathbf{a}'\mathbf{b}\mathbf{c}')+2\epsilon\delta, \end{aligned}$$

since $a(\mathbf{a}'\mathbf{b}\mathbf{c}') = \frac{1}{2}\,|\,\mathbf{a}'\mathbf{c}'\,|\,\rho \leqslant \frac{1}{2}\,|\,\mathbf{a}'\mathbf{c}'\,|\,\delta$, where ρ is the distance of **b** from **a**$'$**c**$'$, and we get

$$a(\mathbf{abc}) \leqslant t(1-\alpha)+2\epsilon\delta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Therefore, by (9), (11) and (12), for all triangle **abc** inscribed in K' we have

$$\begin{aligned} a(\mathbf{abc}) &\leqslant \max.\left\{t,\ \frac{3\sqrt{3}}{4}\epsilon^2,\ \epsilon(\delta+2\epsilon),\ t(1-\alpha)+2\epsilon\delta\right\} \\ &= t(K), \end{aligned}$$

if ϵ is sufficiently small. This completes the proof.

Remark: It may be noted that the strict convexity of K is used only in the necessity part, so that the condition is sufficient for *all* $K \in \mathscr{K}$. The theorems of §2 obviously follow from theorem **3** as corollaries.

4. Because of theorem 3, if we modify the conjecture of §2 to say that the only strictly convex domains of $\mathscr{K}$ that are $\mathscr{K}$-maximal are ellipses then it will be equivalent to the following interesting conjecture.

Conjecture: Let $K \in \mathscr{K}$ be strictly convex and let $t(\mathbf{p})$ have the same value for all $\mathbf{p} \in B$. Then K is an ellipse.

The above may even be true if $t(\mathbf{p})$ is replaced by the areas of the largest polygons with n sides for any fixed n.

5. It may be remarked finally that the notion of maximal covering domains can be extended in the obvious way to classes more general than $\mathscr{K}$ in the plane or to bodies in higher dimensional spaces.

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