MOTION INDUCED BY A SPHERE VIBRATING ALONG THE AXIS OF ROTATION OF AN INFINITE ROTATING LIQUID

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1. INTRODUCTION

The motion of solids in inviscid rotating infinite liquid medium presents many interesting features. G. I. Taylor's (1922) problem of the motion of a sphere moving uniformly along the axis of rotation of an infinite mass of liquid first exposed peculiar difficulties in the way that the motion appeared to remain indeterminate even when (apparently sufficient) boundary conditions corresponding to the given motion were completely satisfied. Some light was thrown on this anomaly when R. R. Long (1953) showed that, when the rotating liquid was contained in a circular cylinder, if the ratio of peripheral velocity of the liquid to the velocity of the sphere exceeded a definite limit waves were produced behind the spherical body and the liquid motion remained indefinite if further condition was not imposed. This theoretical result on the production of waves was supported by some brilliant experimental work.

A different type of interesting anomalous situation was shown by K. Stewartson (1952) to arise for motion in inviscid rotating liquid. If the sphere be supposed to be set in motion at time $t = 0$ with a constant velocity V along the axis of the infinite rotating liquid, Stewartson was able to show that the linearized equations of motion possessed solution corresponding to this condition in which eventually (at large time) the original liquid motion broke up into two apparently unconnected motions in two separate regions. Inside the enveloping cylinder of the sphere, the motion relative to the sphere was as follows : zero velocity parallel to axis, a swirling velocity ranging from zero on the axis to infinite value on the inner surface of the cylinder, and zero normal velocity on the cylinder, while outside the enveloping cylinder the velocity parallel to the axis of rotation ranged from infinite value on the cylinder to V at infinity, but the normal velocity on the cylinder vanished. That inside the cylinder the sphere ultimately pushes the liquid in front with velocity V has been confirmed by observation.

Somewhat similar anomalous situations are found to arise in our present problem in which the harmonic vibration of a sphere, started at time $t = 0$, along the axis of rotation of an infinite inviscid liquid has been studied. As in the case of Stewartson, only linearized motion has been considered, the disturbance caused by vibrating sphere being supposed small. The solution corresponding to the boundary conditions taken has been exhibited in the form of integrals. The integral enables us to find a power series expansion, and an asymptotic expansion for the velocity and pressure force on the sphere in limiting cases. Two cases have been dealt with, namely, when λ ($= 2\Omega/\beta$, i.e. twice the ratio of the circular frequency of the rotating liquid to the frequency of harmonic vibration of the sphere) < 1, and when $\lambda > 1$. We have concentrated attention (as in Stewartson's paper) on the asymptotic expression for velocity distribution for large time. In the limiting case $\beta \to 0$, i.e. $\lambda \to \infty$, our results go over to those of Stewartson. A situation similar to that on the enveloping cylinder in Stewartson's problem arises in our case on two (double) cones tangent to the sphere and enveloping it, with apexes on the axis of rotation. On these cones the tangential components of relative velocity attain infinite values. The normal flux across the cones is of the order $(1/\lambda)$, and hence is vanishingly small for large λ. The entire space is then divided into eight separate zones by the cones three of which are within the volume enclosed by the two enveloping cones, and five outside them. This separation into eight zones with no communicating flow in the limiting case of large λ, and infinite velocity components on the cones seem to suggest that, under the circumstances envisaged, a solution of our problem continuous throughout the entire region of the liquid is not possible on the basis of a linearized theory.

2. Formulation of the Problem and the Solution as Integrals

Let us suppose that the liquid unlimited in all directions is rotating about the axis of z with uniform angular velocity Ω, and that a sphere of radius a oscillates along the axis of rotation of the liquid and its velocity at time t is $v(t) = U \cos \beta t$. Further, we assume that at time $t = 0$, the sphere starts to move under impulse with a velocity $V(0) = U$. The perturbation of the general velocity of rotation of the liquid due to the motion of the sphere is supposed to be sufficiently small for the squares and products of perturbation velocities to be neglected.

We choose the origin of co-ordinates to be at the centre of the sphere and adopt cylindrical co-ordinates; let us take Oz along the axis of rotation, and r, θ polar co-ordinates in a plane normal to Oz, and let the components of liquid velocity referred to instantaneously fixed axes along the directions of r, θ, and z be u, $\Omega r+v$, and $w+V(t)$ respectively. We shall take u, v, w and V to be small quantities whose products and higher powers may be neglected. Then if p be the pressure at a point of the liquid and ρ the density, then on putting

$$P = \frac{p}{\rho} - \tfrac{1}{2}\Omega^2 r^2 \qquad \text{(2.1)}$$

the equations of motion, when linearized, take the form

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial P}{\partial r} \qquad \text{(2.2)}$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0 \qquad \text{(2.3)}$$

$$\frac{\partial}{\partial t}(w+V) = -\frac{\partial P}{\partial z}. \qquad \text{(2.4)}$$

The continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \qquad \text{(2.5)}$$

since the motion is symmetrical about the axis of rotation, and hence independent of θ.

The boundary conditions to be satisfied are

$$\left.\begin{aligned} &u \to 0,\ v \to 0,\ w \to -V(t) \text{ as } z \to \infty \text{ for fixed } r \text{ and } t; \\ &ru+zw = 0 \text{ on the sphere } r^2+z^2 = a^2, \text{ for all } t; \\ &w = -V(0) = -U,\ u = v = 0 \text{ when } t = 0 \text{ for all } r, z \\ &\qquad\qquad \text{satisfying } r^2+z^2 > a^2. \end{aligned}\right\} \qquad \text{(2.6)}$$

We shall make use of Laplace transformation to solve this problem, and introduce the transform functions such as

$$\bar{w} = \int_0^\infty e^{-st} w(r, z, t)dt, \text{ etc.} \quad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

the function in the transformed space being denoted by a bar.

The equations of motion and continuity then become

$$s\bar{u} - 2\Omega\bar{v} = -\frac{\partial \bar{P}}{\partial r} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

$$s\bar{v} + 2\Omega\bar{u} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

$$s\bar{w} + \frac{s^2 U}{s^2+\beta^2} = -\frac{\partial \bar{P}}{\partial z} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

The boundary conditions would be

$$\bar{u} \to 0,\ \bar{v} \to 0,\ \bar{w} \to -\frac{Us}{s^2+\beta^2}, \text{ as } z \to \infty \text{ for fixed } r,$$

and

$$r\bar{u} + z\bar{w} = 0 \text{ on the sphere } r^2+z^2 = a^2. \quad \ldots \quad \ldots \quad (2.12)$$

Expressing $\bar{u}$, $\bar{v}$, $\bar{w}$ in terms of $\bar{P}$ by algebraic solution of (2.8), (2.9) and (2.10) we have

$$\bar{u} = -\frac{s}{s^2+4\Omega^2}\frac{\partial \bar{P}}{\partial r},\ \bar{v} = \frac{2\Omega}{s^2+4\Omega^2}\frac{\partial \bar{P}}{\partial r}, \text{ and } \bar{w} = -\frac{1}{s}\left(\frac{\partial \bar{P}}{\partial z} + \frac{s^2 U}{s^2+\beta^2}\right). \quad (2.13)$$

Substituting these values of $\bar{u}$ and $\bar{w}$ in (2.11) we have an equation for $\bar{P}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{P}}{\partial r}\right) + \frac{s^2+4\Omega^2}{s^2}\frac{\partial^2 \bar{P}}{\partial z^2} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

with the boundary conditions derived from the two equations in (2.12) as follows :—

$$\left.\begin{aligned} &\frac{\partial \bar{P}}{\partial r},\ \frac{\partial \bar{P}}{\partial z} \to 0 \text{ as } z \to \infty \text{ for fixed } r, \\ &\text{and} \\ &\frac{s^2}{s^2+4\Omega^2} r\frac{\partial \bar{P}}{\partial r} + z\frac{\partial \bar{P}}{\partial z} = -Uz\frac{s^2}{s^2+\beta^2} \text{ on } r^2+z^2 = a^2. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.15)$$

Let us introduce a new set of co-ordinates ζ, μ defined by

$$z = \frac{2\Omega}{s}a\mu\zeta,\ r = \frac{2\Omega a}{\sqrt{s^2+4\Omega^2}}(1-\mu^2)^{\frac{1}{2}}(\zeta^2+1)^{\frac{1}{2}}. \quad \ldots \quad \ldots \quad (2.16)$$

Eliminating μ from these two equations we have

$$4\Omega^2 a^2 \zeta^4 - \zeta^2\{s^2(r^2+z^2) + 4\Omega^2(r^2-a^2)\} - s^2 z^2 = 0; \quad \ldots \quad (2.17)$$

whence $$2\zeta^2 = \frac{s^2(r^2+z^2)}{4a^2\Omega^2}+\frac{r^2-a^2}{a^2}+\frac{r^2+z^2}{a^2}\left(\frac{s^2}{4\Omega^2}+\xi_1^2\right)^{\frac{1}{2}}\left(\frac{s^2}{4\Omega^2}+\xi_2^2\right)^{\frac{1}{2}}, \quad \ldots \quad (2.18)$$

where $$\xi_1 = \frac{r(r^2+z^2-a^2)^{\frac{1}{2}}+az}{r^2+z^2}; \quad \xi_2 = \left|\frac{r(r^2+z^2-a^2)^{\frac{1}{2}}-az}{r^2+z^2}\right|. \quad \ldots \quad (2.19)$$

The other solution of equation (2.17) is not real, when s is real. We shall make ζ one valued by requiring it to be positive when s is large and positive. When $r^2+z^2 = a^2$, $\zeta = \dfrac{s}{2\Omega}$, and $\zeta \to \infty$ with z, for fixed r.

Also $$\frac{\partial \bar{P}}{\partial \zeta} = \frac{z}{\zeta}\frac{\partial \bar{P}}{\partial z}+\frac{r\zeta}{\zeta^2+1}\frac{\partial \bar{P}}{\partial r}; \quad \ldots \quad (2.20)$$

and hence from the second boundary condition in (2.15)

$$\frac{\partial \bar{P}}{\partial \zeta} = -\frac{2\Omega s}{s^2+\beta^2}a\mu U, \text{ when } r^2+z^2 = a^2, \text{ i.e. } \zeta = \frac{s}{2\Omega}. \quad \ldots \quad (2.21)$$

Equation (2.14) in terms of μ, ζ becomes

$$\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial \bar{P}}{\partial \mu}\right\}+\frac{\partial}{\partial \zeta}\left\{(\zeta^2+1)\frac{\partial \bar{P}}{\partial \zeta}\right\} = 0 \quad \ldots \quad (2.22)$$

The appropriate solution of this equation is

$$\bar{P} = A\mu\zeta+B\mu\left(\zeta\log\frac{\zeta-i}{\zeta+i}+2i\right), \quad \ldots \quad (2.23)$$

where the constants A and B are to be determined by the boundary conditions.

Since $\dfrac{\partial \bar{P}}{\partial \zeta} \to 0$, as $\zeta \to \infty$, $A = 0$; again since $\dfrac{\partial \bar{P}}{\partial \zeta} = -\dfrac{2\Omega s}{s^2+\beta^2}a\mu U$, when $\zeta = \dfrac{s}{2\Omega}$

$$B = -\frac{2a\Omega U s}{(s^2+\beta^2)\left\{\log\dfrac{s-2i\Omega}{s+2i\Omega}+\dfrac{4i\Omega s}{s^2+4\Omega^2}\right\}} \quad \ldots \quad (2.24)$$

Hence $$\bar{P} = -Uz\frac{s^2\left(\log\dfrac{\zeta-i}{\zeta+i}+\dfrac{2i}{\zeta}\right)}{(s^2+\beta^2)\left\{\log\dfrac{s-2i\Omega}{s+2i\Omega}+\dfrac{4i\Omega s}{s^2+4\Omega^2}\right\}} \quad \ldots \quad (2.25)$$

The function P in the flow space can be obtained by inverting the transformation with respect to t, and we have

$$P = -\frac{Uz}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\frac{s^2\left[\log\dfrac{\zeta-i}{\zeta+i}+\dfrac{2i}{\zeta}\right]}{(s^2+\beta^2)\left\{\log\dfrac{s-2i\Omega}{s+2i\Omega}+\dfrac{4i\Omega s}{s^2+4\Omega^2}\right\}}\,ds, \quad \ldots \quad (2.26)$$

where C is a positive constant. Equation (2.13) now gives $\bar{u}$, $\bar{v}$, $\bar{w}$, from which expressions for u, v and w can be obtained as follows

$$u = -\frac{1}{2\pi i}\times \int_{C-i\infty}^{C+i\infty} e^{st} \frac{2iUrzs^3\,ds}{(s^2+\beta^2)\zeta(\zeta^2+1)(r^2+z^2)\left(s^2+4\Omega^2\xi_1^2\right)^{\frac{1}{2}}\left(s^2+4\Omega^2\xi_2^2\right)^{\frac{1}{2}}\left\{\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right\}} \quad \ldots (2.27)$$

$$v = \frac{1}{2\pi i}\times \int_{C-i\infty}^{C+i\infty} e^{st} \frac{4\Omega Uirzs^2\,ds}{(s^2+\beta^2)\zeta(\zeta^2+1)(r^2+z^2)\left(s^2+4\Omega^2\xi_1^2\right)^{\frac{1}{2}}\left(s^2+4\Omega^2\xi_2^2\right)^{\frac{1}{2}}\left\{\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right\}} \quad \ldots (2.28)$$

$$w = -U\cos\beta t+\frac{U}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st} \frac{s\left[\log\frac{\zeta-i}{\zeta+i}+\frac{2i}{\zeta}-\frac{2iz^2s^2}{\zeta^3(r^2+z^2)\left(s^2+4\Omega^2\xi_1^2\right)^{\frac{1}{2}}\left(s^2+4\Omega^2\xi_2^2\right)^{\frac{1}{2}}}\right]}{(s^2+\beta^2)\left\{\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right\}} \quad \ldots (2.29)$$

(2.26), (2.27), (2.28) and (2.29) constitute the solution of equations (2.8), (2.9), (2.10) and (2.11), subject to the given boundary conditions.

3. Pressure and Velocity Distribution on the Sphere

The resultant pressure on the sphere is along the axis of rotation and is given by

$$Z = -\int\int p\frac{z}{a}\,dS \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.1)$$

taken over the surface of the sphere. Since ζ is constant and is equal to $s/(2\Omega)$ on the sphere, on substituting for p from (2.1), we have

$$Z = \frac{U}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st} \frac{s^2\left[\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega}{s}\right]}{(s^2+\beta^2)\left\{\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right\}}\,ds\int\int\rho\frac{z^2}{a}\,dS, \quad \ldots (3.2)$$

as integration over the term Ω in (2.1) vanishes. The double integral in (3.2) is equal to $\frac{4}{3}\pi\rho a^3$, and *when Ωt is small* we may obtain a power series for Z by first expanding the logarithmic terms in a series of descending powers of s as

$$Z = -\frac{2\rho a^3 U}{3i}\int_{C-i\infty}^{C+i\infty} e^{st}\frac{s^2}{s^2+\beta^2}\left[\frac{1}{2}+\frac{3\,.\,2^2\Omega^2}{10s^2}-\frac{12\,.\,2^4\Omega^4}{175s^4}+\frac{4\,.\,2^6\Omega^6}{125s^6}-\ldots\right]ds \quad (3.3)$$

which on integration gives

$$Z = \frac{4}{3}\pi\rho a^3 U\left[-\frac{1}{2}\delta(t)+\beta\sin\beta t\left\{\frac{1}{2}-\frac{3}{10}\left(\frac{2\Omega}{\beta}\right)^2-\frac{12}{175}\left(\frac{2\Omega}{\beta}\right)^4-\cdots\right\}\right.$$
$$+\beta(2\Omega t)\left\{\frac{12}{175}\left(\frac{2\Omega}{\beta}\right)^3+\frac{4}{125}\left(\frac{2\Omega}{\beta}\right)^5+\cdots\cdots\right\}$$
$$-\beta\frac{(2\Omega t)^3}{3!}\left\{\frac{4}{125}\left(\frac{2\Omega}{\beta}\right)^3+\cdots\cdots\cdots\cdots\right\}$$
$$\left.+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\right], \quad (3.4)$$

where $\delta(t)$ is Dirac's delta function which vanishes for $t > 0$.

If we put $\lambda = 2\Omega/\beta$, then in terms of λ, (3.4) becomes

$$Z = \frac{4}{3}\pi\rho a^3 U\left[-\frac{1}{2}\delta(t)+\beta\sin\beta t\left(\frac{1}{2}-\frac{3}{10}\lambda^2-\frac{12}{175}\lambda^4-\cdots\cdots\right)\right.$$
$$\left.+\beta(2\Omega t)\left(\frac{12}{175}\lambda^3+\frac{4}{125}\lambda^5+\cdots\right)-\frac{\beta(2\Omega t)^3}{3!}\left(\frac{4}{125}\lambda^3+\cdots\right)+\cdots\right]^* \quad (3.4a)$$

When Ωt is large, the value of Z can be obtained by inserting cuts in the s-plane from $s = \pm 2i\Omega$ to infinity along lines on which the imaginary part of s is constant, and the real part decreases. The path of integration may be replaced by a path round the infinite semicircle with $R(s) < 0$ and round the two cuts, together with the contributions from the poles at $s = \pm i\beta$. This change of contour has been carried out throughout the calculations of similar integrals in this paper. The contributions from the two cuts when Ωt is large may be found as follows: We consider the integral on the upper side of the cut at $s = 2i\Omega$ and write $s = 2i\Omega - \alpha$. Then this part of the integral is

$$-\frac{4}{3}\frac{\pi\rho a^3 U}{2\pi i}\exp(2i\Omega t)\int_0^\infty e^{-\alpha t}\frac{(2i\Omega-\alpha)^2\left[\log\alpha+i\pi-\log(4i\Omega-\alpha)+\dfrac{4i\Omega}{2i\Omega-\alpha}\right]}{\{\beta^2+(2i\Omega-\alpha)^2\}\left[\log\alpha+i\pi-\log(4i\Omega-\alpha)+\dfrac{4i\Omega(2i\Omega-\alpha)}{\alpha(\alpha-4i\Omega)}\right]}d\alpha \quad (3.5)$$

$$= -\frac{2\rho a^3 U\lambda^2}{3i(\lambda^2-1)}\exp(2i\Omega t)\int_0^\infty e^{-\alpha t}\left[1-\frac{\left(\dfrac{i\alpha}{2\Omega}\right)^2+\dfrac{i\alpha}{\Omega}}{\lambda^2-1}+\cdots\right]\times$$
$$\frac{\log\alpha+i\pi-\log(4i\Omega-\alpha)+\dfrac{4i\Omega}{2i\Omega-\alpha}}{\log\alpha+i\pi-\log(4i\Omega-\alpha)+\dfrac{4i\Omega(2i\Omega-\alpha)}{\alpha(\alpha-4i\Omega)}}d\alpha$$

$$= -\frac{2\rho a^3 U\lambda^2}{3i(\lambda^2-1)}\exp(2i\Omega t)\int_0^\infty e^{-\alpha t}\left[1-\frac{\left(\dfrac{i\alpha}{2\Omega}\right)^2+\dfrac{i\alpha}{\Omega}}{\lambda^2-1}+\cdots\right]\frac{i\alpha}{2\Omega}(\log\alpha+i\pi)[1+O(\alpha\log\alpha)]\,d\alpha \quad (3.6)$$

* For the validity of the above expansion in series λ should also be small.

The contribution from the lower side of the cut at $s = 2i\Omega$ is similarly

$$\frac{2\rho a^3 U\lambda^2}{3i(\lambda^2-1)} \exp(2i\Omega t) \int_0^\infty e^{-\alpha t}\left[1-\frac{\left(\frac{i\alpha}{2\Omega}\right)^2+\frac{i\alpha}{\Omega}}{\lambda^2-1}+\cdots\right]\frac{i\alpha}{2\Omega}(\log\alpha - i\pi)[1+O(\alpha\log\alpha)]\,d\alpha \quad \ldots (3.7)$$

Hence we find that the total contribution from the cut at $s = 2i\Omega$ is

$$\frac{2\rho a^3 U\lambda^2\pi}{3i(\lambda^2-1)\Omega} \exp(2i\Omega t) \int_0^\infty e^{-\alpha t}\left[1-\frac{\left(\frac{i\alpha}{2\Omega}\right)^2+\frac{i\alpha}{\Omega}}{\lambda^2-1}+\cdots\right]\alpha[1+O(\alpha\log\alpha)]\,d\alpha$$

$$= \frac{2\pi\rho a^3 U\lambda^2}{3i(\lambda^2-1)\Omega t^2}\left[1+O\left(\frac{\log\Omega t}{\Omega t}\right)+O\left(\frac{1}{\Omega t(\lambda^2-1)}\right)\right]\exp(2i\Omega t), \quad \ldots \quad \ldots (3.8)$$

as $\Omega t \to \infty$. Similar contributions also arise from the cut at $s = -2i\Omega$

The contribution from the poles at $s = \pm i\beta$ is

$$-\frac{4}{3}\pi\rho a^3 U\beta \sin\beta t \frac{\log\frac{1-\lambda}{1+\lambda}+2\lambda}{\log\frac{1-\lambda}{1+\lambda}+\frac{2\lambda}{1-\lambda^2}} \text{ for } \lambda < 1, \quad \ldots \quad \ldots (3.9)$$

and

$$-\frac{4}{3}\pi\rho a^3 U\beta \frac{\frac{2\pi\lambda^3}{\lambda^2-1}\cos\beta t+\left[\pi^2+\left(\log\frac{\lambda+1}{\lambda-1}\right)^2-\frac{4\lambda^2}{\lambda^2-1}+\frac{2\lambda(2-\lambda^2)}{\lambda^2-1}\log\frac{\lambda+1}{\lambda-1}\right]\sin\beta t}{\pi^2+\left[\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right]^2}$$

$$\text{for } \lambda > 1 \quad \ldots (3.10)$$

Adding all these contributions we have finally

$$Z = -\frac{4}{3}\pi\rho a^3 U\left[\beta\sin\beta t\frac{\log\frac{1-\lambda}{1+\lambda}+2\lambda}{\log\frac{1-\lambda}{1+\lambda}+\frac{2\lambda}{1-\lambda^2}}+\frac{\lambda^2\sin 2\Omega t}{(1-\lambda^2)\Omega t^2}\left\{1+O\left(\frac{\log\Omega t}{\Omega t}\right)+O\left(\frac{1}{\Omega t(\lambda^2-1)}\right)\right\}\right] \text{ for } \lambda < 1,$$

and

$$-\frac{4}{3}\pi\rho a^3 U\left[\beta\frac{\frac{2\pi\lambda^3}{\lambda^2-1}\cos\beta t+\sin\beta t\left\{\pi^2+\left(\log\frac{\lambda+1}{\lambda-1}\right)^2-\frac{4\lambda^2}{\lambda^2-1}+\frac{2\lambda(2-\lambda^2)}{\lambda^2-1}\log\frac{\lambda+1}{\lambda-1}\right\}}{\pi^2+\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right)^2}\right.$$

$$\left.-\frac{\lambda^2\sin 2\Omega t}{(\lambda^2-1)\Omega t^2}\left\{1+O\left(\frac{\log\Omega t}{\Omega t}\right)+O\left(\frac{1}{\Omega t(\lambda^2-1)}\right)\right\}\right] \text{ for } \lambda > 1 \ldots \quad (3.11)$$

We note that when $\beta \to 0$ (i.e. period of vibration is infinitely large) so that $\lambda \to \infty$, the last expression for Z in (3.11) goes over to Stewartson's result, namely

$$Z = -\frac{16}{3}\rho a^3 U\Omega - \frac{4\pi\rho a^3 U}{3\Omega t^2}\sin 2\Omega t\left[1+O\left(\frac{\log \Omega t}{\Omega t}\right)\right]. \qquad .. (3.12)$$

We shall now calculate the velocity components u, v, w on the sphere. Here $r^2+z^2 = a^2$, $\xi_1 = \xi_2 = \frac{z}{a}$, and $\zeta = \frac{s}{2\Omega}$, so that

$$w = -U\cos\beta t + \frac{U}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\frac{s\left[\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega}{s}-\frac{16iz^2\Omega^3}{s(a^2s^2+4\Omega^2z^2)}\right]}{(s^2+\beta^2)\left\{\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right\}}\,ds. \quad (3.13)$$

As before the branch points at $s = \pm 2i\Omega$ give contributions which tend to zero as $\frac{1}{t^2}$ as $\Omega t \to \infty$. Only the poles at $s = \pm i\beta$, and at $s = \pm\frac{2i\Omega z}{a}$ give contributions to w.

Adding up these contributions we find that as $\Omega t \to \infty$,

$$w = -\frac{\frac{2\lambda}{1-\lambda^2}+\frac{2\lambda a^2}{\lambda^2 z^2-a^2}}{\log\frac{1-\lambda}{1+\lambda}+\frac{2\lambda}{1-\lambda^2}}U\cos\beta t + \frac{2azU\lambda^2}{a^2-\lambda^2z^2}\times\frac{\left\{\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right\}\cos\frac{2\Omega zt}{a}+\pi\sin\frac{2\Omega zt}{a}}{\pi^2+\left\{\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right\}}, \text{ for } \lambda < 1$$

and

$$\frac{U\cos\beta t\left[\frac{4\lambda^4(a^2-z^2)}{(\lambda^2z^2-a^2)(\lambda^2-1)^2}+\frac{2\lambda^3(a^2-z^2)}{(\lambda^2-1)(\lambda^2z^2-a^2)}\log\frac{\lambda+1}{\lambda-1}\right]+\pi\frac{2\lambda^3(a^2-z^2)}{(\lambda^2-1)(\lambda^2z^2-a^2)}U\sin\beta t}{\pi^2+\left[\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right]^2}$$
$$+\frac{2\lambda^2zUa}{a^2-\lambda^2z^2}\times\frac{\left\{\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right\}\cos\frac{2\Omega zt}{a}+\pi\sin\frac{2\Omega zt}{a}}{\pi^2+\left\{\log\frac{a+z}{a-z}+\frac{2az}{a^2-z^2}\right\}^2} \text{ for } \lambda > 1 \quad .. (3.14)$$

Since on the surface of the sphere $ru = -zw$, the variation of u over the sphere can be obtained by substituting in it the value of w from (3.14) as the case may be. On the surface of the sphere the variation of v is given by

$$v = \frac{32\Omega^4 Uirz}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\cdot\frac{s\,ds}{(s^2+\beta^2)(a^2s^2+4\Omega^2z^2)\left\{(s^2+4\Omega^2)\log\frac{s-2i\Omega}{s+2i\Omega}+4i\Omega s\right\}} \quad (3.15)$$

As $\Omega t \to \infty$, the only contributions arise from the poles at $s = \pm i\beta$, and at $s = \pm\frac{2i\Omega z}{a}$.

Adding up these contributions we find

$$v=\frac{2Urz\lambda^4\sin\beta t}{(\lambda^2z^2-a^2)(1-\lambda^2)\left\{\log\dfrac{1-\lambda}{1+\lambda}+\dfrac{2\lambda}{1-\lambda^2}\right\}}+\frac{4\lambda^2Ua^2rz}{(\lambda^2z^2-a^2)(a^2-z^2)}\times$$

$$\times\frac{\pi\cos\dfrac{2\Omega zt}{a}-\left\{\log\dfrac{a+z}{a-z}+\dfrac{2az}{a^2-z^2}\right\}\sin\dfrac{2\Omega zt}{a}}{\pi^2+\left\{\log\dfrac{a+z}{a-z}+\dfrac{2az}{a^2-z^2}\right\}^2}\text{ for }\lambda<1 \quad .. \quad (3.16)$$

and

$$-\frac{2Urz\lambda^4\left[\pi\cos\beta t-\left(\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right)\sin\beta t\right]}{(\lambda^2-1)(\lambda^2z^2-a^2)\left\{\pi^2+\left(\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right)^2\right\}}+\frac{4\lambda^2Ua^2rz}{(\lambda^2z^2-a^2)(a^2-z^2)}\times$$

$$\times\frac{\pi\cos\dfrac{2\Omega zt}{a}-\left\{\log\dfrac{a+z}{a-z}+\dfrac{2az}{a^2-z^2}\right\}\sin\dfrac{2\Omega zt}{a}}{\pi^2+\left\{\log\dfrac{a+z}{a-z}+\dfrac{2az}{a^2-z^2}\right\}^2}\text{ for }\lambda>1 \quad .. \quad (3.17)$$

It is clear that on the surface of the sphere u, v, w become infinite when $z=\pm a/\lambda$ for the case $\lambda>1$.

In the following section the velocity distributions at the ultimate stage, namely when $t\to\infty$, have been discussed. The distributions have been calculated only on the axis of rotation and on specific surfaces of importance. The motion automatically falls into two categories defined by $\lambda<1$, and $\lambda>1$.

4. The Ultimate Velocity Distribution

If we replace the contour of integration of equation (2.29) by an infinite semicircle and contours round the branch points at $s=\pm 2i\Omega$, $s=\pm 2i\Omega\xi_1$, $s=\pm 2i\Omega\xi_2$, then we may show that the only contribution to w which does not tend to zero as $\Omega t\to\infty$ is that from the poles at $s=\pm i\beta$. In this section the ultimate velocity distributions for u, v and w have been studied for the two cases, namely for $\lambda<1$ and $\lambda>1$. In the latter case, when $\lambda\to\infty$ (i.e. by making $\beta\to 0$) all of them go over to Stewartson's result.

Only two cases for which the results appear in simplified forms have been discussed, namely (*a*) on $r=0$, i.e. on the axis of rotation of the liquid, and (*b*) on $r=a$, i.e. on the cylinder circumscribing the sphere.

(*a*) Velocity on $r=0$ (Axis of Rotation)

In this case $\xi_1=\xi_2=a/z$, $\zeta=\dfrac{sz}{2a\Omega}$. The integrals for u and v now are both identically zero, and so there is only an axial velocity w. The integral for w then becomes

$$w=-U\cos\beta t+\frac{U}{2\pi i}\int_{C-i\infty}^{C+i\infty}e^{st}\frac{s\log\dfrac{sz-2ia\Omega}{sz+2ia\Omega}+\dfrac{4ia\Omega}{z}-\dfrac{16ia^3\Omega^3}{z^3\left(s^2+\dfrac{4\Omega^2a^2}{z^2}\right)}}{(s^2+\beta^2)\left\{\log\dfrac{s-2i\Omega}{s+2i\Omega}+\dfrac{4is\Omega}{s^2+4\Omega^2}\right\}}\,ds. \quad (4.1)$$

The important contributions to w when Ωt is large arise from the poles at $s = \pm i\beta$ and at $s = \pm \dfrac{2ia\Omega}{z}$.

Hence collecting the residues from the poles we have

$$w = \frac{\log\dfrac{(z-a\lambda)(1+\lambda)}{(z+a\lambda)(1-\lambda)} - \dfrac{2\lambda(z-a)(z+a\lambda^2)}{(1-\lambda^2)(z^2-a^2\lambda^2)}}{\log\dfrac{1-\lambda}{1+\lambda}+\dfrac{2\lambda}{1-\lambda^2}}\, U\cos\beta t + \frac{2Ua^2\lambda^2}{z^2-a^2\lambda^2}\times$$

$$\times\frac{\left(\log\dfrac{z+a}{z-a}+\dfrac{2az}{z^2-a^2}\right)\cos\dfrac{2\Omega at}{z}+\pi\sin\dfrac{2\Omega at}{z}}{\pi^2+\left\{\log\dfrac{z+a}{z-a}+\dfrac{2az}{z^2-a^2}\right\}^2}\quad \text{for } \lambda < 1 \quad .. \quad (4.2)$$

and

$$\frac{U\left\{\log\dfrac{(a\lambda+z)(\lambda-1)}{(a\lambda-z)(\lambda+1)}+2\lambda\left(\dfrac{az}{a^2\lambda^2-z^2}-\dfrac{1}{\lambda^2-1}\right)\right\}}{\pi^2+\left\{\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right\}^2}\left[\pi\sin\beta t+\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right)\cos\beta t\right]$$

$$-\frac{2Ua^2\lambda^2}{a^2\lambda^2-z^2}\times\frac{\left(\log\dfrac{z+a}{z-a}+\dfrac{2az}{z^2-a^2}\right)\cos\dfrac{2\Omega at}{z}+\pi\sin\dfrac{2\Omega at}{z}}{\pi^2+\left\{\log\dfrac{z+a}{z-a}+\dfrac{2az}{z^2-a^2}\right\}^2}\ \text{for } \lambda > 1. \quad .. \quad (4.3)$$

For $\lambda > 1$, w becomes infinite at $z = \pm a\lambda$.

(b) Velocity on $r = a$ (Enveloping Cylinder)

(i) *Calculation of* w.—In this case $\xi_2 = 0$, and $\xi_1 = \dfrac{2az}{a^2+z^2}$. $s = 0$ now becomes a branch point of the integrand for w, whose contribution is given below. In the contour we insert a cut from $s = 0$ to $s = -\infty$ along the negative real axis of the s-plane. We first consider the integral, *when Ωt is large*, on the upper side of the cut at $s = 0$, and write $s = \alpha e^{i\pi}$. Then this part of the integral is

$$\frac{Ue^{i\pi/2}}{2\pi i}\int_0^\infty e^{-\alpha t}\,d\alpha\,\frac{\sqrt{\alpha}\left[\dfrac{1}{\sqrt{\left(\dfrac{a^2+z^2}{4\Omega a^2}\right)}\left\{-\dfrac{\alpha}{2\Omega}+\left(\dfrac{\alpha^2}{4\Omega^2}+\dfrac{4a^2z^2}{(a^2+z^2)^2}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}}}-\dfrac{z^2(4\Omega a^2)^{3/2}}{(a^2+z^2)^{5/2}\left\{\alpha^2+\dfrac{16\Omega^2a^2z^2}{(a^2+z^2)^2}\right\}^{\frac{1}{2}}\left[-\dfrac{\alpha}{2\Omega}+\left(\dfrac{\alpha^2}{4\Omega^2}+\dfrac{4a^2z^2}{(a^2+z^2)^2}\right)^{\frac{1}{2}}\right]^{3/2}}\right]}{(\alpha^2+\beta^2)\left\{\pi/2+\tan^{-1}\dfrac{\alpha}{2\Omega}+\dfrac{(\alpha/2\Omega)}{1+(\alpha/2\Omega)^2}\right\}} \quad .. \quad (4{\cdot}4)$$

$$=\frac{U\Omega}{\pi^2}\left(\frac{a}{z}\right)^{\frac{1}{2}}\int_0^\infty e^{-\alpha t}\,d\alpha\,\frac{\left[\left(\dfrac{\alpha}{2\Omega}\right)^{\frac{1}{2}}-\left(\dfrac{4}{\pi}+\dfrac{a^2+z^2}{4az}\right)\left(\dfrac{\alpha}{2\Omega}\right)^{3/2}+\left(\dfrac{16}{\pi^2}+\dfrac{a^2+z^2}{\pi az}-\dfrac{3(a^2+z^2)^2}{32a^2z^2}\right)\left(\dfrac{\alpha}{2\Omega}\right)^{5/2}+\cdots\right]}{\alpha^2+\beta^2} \quad .. \quad (4.5)$$

The contribution from the lower side of the cut is also the same as (4.5). Hence the total contribution from the cut at $s = 0$ is

$$\frac{2\Omega U}{\pi^2}\left(\frac{a}{z}\right)^{\frac{1}{2}}\int_0^\infty e^{-\alpha t}\,d\alpha\,\frac{\left(\frac{\alpha}{2\Omega}\right)^{\frac{1}{2}}-\left(\frac{4}{\pi}+\frac{a^2+z^2}{4az}\right)\left(\frac{\alpha}{2\Omega}\right)^{3/2}+\left(\frac{16}{\pi^2}+\frac{a^2+z^2}{\pi az}-\frac{3(a^2+z^2)^2}{32a^2z^2}\right)\left(\frac{\alpha}{2\Omega}\right)^{5/2}+\cdots\cdots}{\alpha^2+\beta^2} \quad \ldots \quad (4.6)$$

$$=\frac{U\lambda^2}{\pi^2}\left(\frac{a}{z}\right)^{\frac{1}{2}}\int_0^\infty e^{-2\Omega tx}\,dx\left\{x^{\frac{1}{2}}-\left(\frac{4}{\pi}+\frac{a^2+z^2}{4az}\right)x^{3/2}+\left(\frac{16}{\pi^2}+\frac{a^2+z^2}{\pi az}-\frac{3(a^2+z^2)^2}{32a^2z^2}-\lambda^2\right)x^{5/2}+\cdots\right\}$$

$$\text{for } \lambda < 1 \quad \ldots \quad (4.7)$$

and

$$=\frac{U}{\pi^2}\left(\frac{a}{z}\right)^{\frac{1}{2}}\int_0^\infty e^{-2\Omega tx}\,dx.\,\frac{x^{\frac{1}{2}}-\left(\frac{4}{\pi}+\frac{a^2+z^2}{4az}\right)x^{3/2}+\left(\frac{16}{\pi^2}+\frac{a^2+z^2}{\pi az}-\frac{3(a^2+z^2)^2}{32a^2z^2}\right)x^{5/2}+\cdots\cdots}{x^2+\frac{1}{\lambda^2}}$$

$$\text{for } \lambda > 1. \quad \ldots \quad \ldots \quad \ldots \quad (4.8)$$

When $\lambda > 1$, to calculate the residues at $s = \pm i\beta$ we observe that when $s = i\beta$, ζ becomes

$$\zeta_1 = e^{i\pi/4}\sqrt{\left(\frac{a^2+z^2}{2a^2\lambda^2}\right)\left[i+\left\{\left(\frac{2az\lambda}{a^2+z^2}\right)^2-1\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}}$$

and when $s = -i\beta$, ζ becomes $\zeta_2 = \zeta_1{}^* =$ complex conjugate of ζ_1. Hence calculating the sum of the residue at $s = \pm i\beta$ we have

$$\frac{U}{2\pi i}\times\frac{1}{\pi^2+\left\{\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right\}^2}\left[-2\left(\frac{a\lambda}{z}\right)^{\frac{1}{2}}\left\{\pi\left(\cos(\theta/2+\pi/4-\beta t)-\frac{1}{2\cos\theta}\cos(3\theta/2+\pi/4-\beta t)\right)\right.\right.$$

$$\left.+\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right)\left(\sin(\theta/2+\pi/4-\beta t)-\frac{1}{2\cos\theta}\sin(3\theta/2+\pi/4-\beta t)\right)\right\}$$

$$+\frac{1}{2}\left(\cos\beta t\left\{\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right\}+\pi\sin\beta t\right)\log\frac{\frac{z}{a\lambda}+1+2\sqrt{\frac{z}{a\lambda}}\sin(\theta/2+\pi/4)}{\frac{z}{a\lambda}+1-2\sqrt{\frac{z}{a\lambda}}\sin(\theta/2+\pi/4)}$$

$$\left.+\tan^{-1}\frac{2\sqrt{\frac{z}{a\lambda}}\cos(\theta/2+\pi/4)}{\frac{z}{a\lambda}-1}\times\left\{\pi\cos\beta t-\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right)\sin\beta t\right\}\right] \quad \ldots \quad (4.9)$$

where

$$\sin\theta = \frac{a^2+z^2}{2az\lambda} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

Hence for $\lambda > 1$,

$$w = -U\cos\beta t - U\left(\frac{a\lambda}{z}\right)^{\frac{1}{2}}\left[2\,\frac{\pi\left\{\cos(\theta/2+\pi/4-\beta t)-\dfrac{\cos(3\theta/2+\pi/4-\beta t)}{2\cos\theta}\right\}+\left(\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right)\left\{\sin(\theta/2+\pi/4-\beta t)-\dfrac{\sin(3\theta/2+\pi/4-\beta t)}{2\cos\theta}\right\}}{\pi^2+\left\{\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right\}^2}\right.$$

$$-\frac{\frac{1}{2}\log\dfrac{z/a\lambda+1+2\sqrt{z/a\lambda}\sin(\theta/2+\pi/4)}{z/a\lambda+1-2(z/a\lambda)^{\frac{1}{2}}\sin(\theta/2+\pi/4)}\left[\left(\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right)\cos\beta t+\pi\sin\beta t\right]+\left\{\pi\cos\beta t-\left(\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right)\sin\beta t\right\}\tan^{-1}\dfrac{2\sqrt{\dfrac{z}{a\lambda}}\cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)}{z/a\lambda-1}}{\left[\pi^2+\left\{\log\dfrac{\lambda+1}{\lambda-1}+\dfrac{2\lambda}{\lambda^2-1}\right\}^2\right]\left(\dfrac{a\lambda}{z}\right)^{\frac{1}{2}}}$$

$$\left.-\frac{1}{\pi^2\sqrt{\lambda}}\int_0^\infty e^{-2\Omega tx}\,dx.\,\frac{x^{1/2}-\left(\dfrac{4}{\pi}+\dfrac{a^2+z^2}{4az}\right)x^{3/2}+\left(\dfrac{16}{\pi^2}+\dfrac{a^2+z^2}{\pi az}-\dfrac{3(a^2+z^2)^2}{32a^2z^2}\right)x^{5/2}+\ldots}{x^2+\dfrac{1}{\lambda^2}}\right] \quad \ldots (4.11)$$

When $\lambda \to \infty$ (i.e. $\beta \to 0$), $\theta \to 0$, and then the third term in (4.11) reduces to

$$\frac{U}{\pi^2}\pi\tan^{-1}\left(\frac{0}{-1}\right) = \frac{U}{\pi^2}\pi.\pi = U,$$

which cancels the first term in (4.11) in the limit $\beta \to 0$.

The second and fourth terms in (4.11) reduce to

$$\lim_{\beta\to 0} -U\left(\frac{a}{z}\right)^{1/2}\left[\sqrt{2\Omega}\,\frac{\cos(\beta t-\pi/4)}{\pi\sqrt{\beta}}\right.$$

$$\left.-\frac{1}{\pi^2}\int_0^\infty e^{-2\Omega tx}\,dx\left\{x^{-3/2}-\left(\frac{4}{\pi}+\frac{a^2+z^2}{4az}\right)x^{-1/2}+O\left(x^{1/2}\right)\right\}\right]$$

$$= \lim_{\beta\to 0} -\frac{U}{\pi}\left(\frac{2\Omega a}{z}\right)^{1/2}\left[\frac{\cos(\beta t-\pi/4)}{\sqrt{\beta}}-\frac{1}{\pi}\int_0^\infty e^{-\alpha t}\frac{d\alpha}{\alpha^{3/2}}\right.$$

$$\left.+\frac{\dfrac{4}{\pi}+\dfrac{a^2+z^2}{4az}}{2\Omega\pi}\int_0^\infty e^{-\alpha t}\frac{d\alpha}{a^{1/2}}+O\left(\int_0^\infty e^{-\alpha t}\sqrt{\alpha}\,d\alpha\right)\right]$$

$$= -\frac{U}{\pi}\left(\frac{2\Omega a}{z}\right)^{1/2}\left[\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{c^{st}ds}{s^{3/2}}+\frac{\dfrac{4}{\pi}+\dfrac{a^2+z^2}{4az}}{2\Omega\pi}\sqrt{\frac{\pi}{t}}+O\left(t^{-3/2}\right)\right]$$

$$= -\frac{U}{\pi}\left(\frac{2\Omega a}{z}\right)^{1/2}\left[2\sqrt{\frac{t}{\pi}}+O\left(\frac{1}{\Omega\sqrt{\pi t}}\right)\right] = -\frac{2U}{\pi}\left(\frac{2\Omega at}{\pi z}\right)^{1/2}\left[1+O\left(\frac{1}{\Omega t}\right)\right]$$

Hence *for* $\beta \to 0$, and *large* Ωt, (4.11) reduces to

$$w = -\frac{2U}{\pi}\left(\frac{2\Omega a t}{\pi z}\right)^{1/2}\left[1+O\left(\frac{1}{\Omega t}\right)\right], \qquad (4.12)$$

which is in agreement with Stewartson's result.

It is clear from (4.11) that w becomes infinite when $\cos\theta = 0$, i.e. at the points $z = \pm a(\lambda \pm \sqrt{\lambda^2-1})$ on the cylinder $r = a$.

For $\lambda < 1$, the contribution from the branch point at $s = 0$ tends to zero for large Ωt, which is obvious from (4.7). In this case when Ωt is large, collecting the contributions from the poles at $s = \pm i\beta$ we have

$$w = U\cos\beta t\left[\frac{\log\dfrac{k-1}{k+1}+\dfrac{2}{k}+\dfrac{2z^2}{k^3(a^2+z^2)\left\{1-\left(\dfrac{2a\lambda z}{a^2+z^2}\right)^2\right\}^{1/2}}}{\log\dfrac{1-\lambda}{1+\lambda}+\dfrac{2\lambda}{1-\lambda^2}} - 1\right], \qquad (4.13)$$

where

$$k = \sqrt{\frac{a^2+z^2}{2a^2\lambda^2}}\left[1+\left\{1-\left(\frac{2a\lambda z}{a^2+z^2}\right)^2\right\}^{1/2}\right]^{1/2} \qquad (4.14)$$

(ii) *Calculation of u and v.* $s = 0$ is a branch point of the integrands for u and v. Its contribution for large Ωt tends to zero when $\lambda < 1$. In this case non-zero contributions come only from the poles at $s = \pm i\beta$; these give

$$u = \frac{2Uaz\cos\beta t}{(a^2+z^2)k(k^2-1)\left\{1-\left(\dfrac{2a\lambda z}{a^2+z^2}\right)^2\right\}^{1/2}\left\{\log\dfrac{1-\lambda}{1+\lambda}+\dfrac{2\lambda}{1-\lambda^2}\right\}}$$

$$(\lambda<1) \qquad (4.15)$$

$$v = -\frac{2\lambda Uaz\sin\beta t}{(a^2+z^2)k(k^2-1)\left\{1-\left(\dfrac{2a\lambda z}{a^2+z^2}\right)^2\right\}^{1/2}\left\{\log\dfrac{1-\lambda}{1+\lambda}+\dfrac{2\lambda}{1-\lambda^2}\right\}}$$

where k has the same meaning as (4.14).

For $\lambda > 1$, the branch point $s = 0$ gives contributions to u and v for large Ωt. Proceeding exactly in the same way as for w, we have

$$u = \frac{U}{\cos\theta}\sqrt{\frac{a}{\lambda z}}\,\frac{B\cos(\beta t-\theta/2+\pi/4)-C\sin(\beta t-\theta/2+\pi/4)}{B^2+C^2} + \frac{U}{\pi^2}\sqrt{\frac{a}{z}}\int_0^\infty \frac{e^{-2\Omega t x}}{x^2+\dfrac{1}{\lambda^2}}\{x^{3/2}+Ax^{5/2}+O(x^{7/2})\}\,dx \qquad (4.16)$$

$$v = \frac{U}{\cos\theta}\sqrt{\frac{a\lambda}{z}}\,\frac{C\sin(\beta t-\theta/2-\pi/4)-B\cos(\beta t-\theta/2-\pi/4)}{B^2+C^2} + \frac{U}{\pi^2}\sqrt{\frac{a}{z}}\int_0^\infty \frac{e^{-2\Omega t x}}{x^2+\dfrac{1}{\lambda^2}}\{x^{\frac{1}{2}}+Ax^{3/2}+O(x^{5/2})\}\,dx \qquad (4.17)$$

$$\left.\begin{aligned} \text{where } A &= \frac{z}{a}+\frac{a^2+z^2}{4az}-\frac{4}{\pi};\ B=\pi\left(1-\frac{z}{a\lambda}\sin\theta\right)+\frac{z}{a\lambda}\cos\theta\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right); \\ C &= \left(1-\frac{z}{a\lambda}\sin\theta\right)\left(\log\frac{\lambda+1}{\lambda-1}+\frac{2\lambda}{\lambda^2-1}\right)-\frac{\pi z}{a\lambda}\cos\theta, \end{aligned}\right\} \quad (4.18)$$

and θ having the same meaning as (4.10).

As $\lambda\rightarrow\infty$, $u\rightarrow 0$, $v\rightarrow -\frac{2U}{\pi}\left(\frac{2\Omega at}{\pi z}\right)^{\frac{1}{2}}$. Like w, u and v become infinite on $r=a$ at the points $z=\pm a(\lambda\pm\sqrt{\lambda^2-1})$.

For $r>a$, $r<a$, the components of velocity are given by (2.27), (2.28) and (2.29). The residues at the poles at $s=\pm i\beta$ of each of the integrand for u, v and w will contain the factor $(\lambda^2\xi_1{}^2-1)^{-\frac{1}{2}}$ $(\lambda^2\xi_2{}^2-1)^{-\frac{1}{2}}$, which becomes infinite only when $\lambda\xi_1=\pm 1$, $\lambda\xi_2=1$, for the case $\lambda>1$. In the rz-plane $(\lambda\xi_1-1)(\lambda\xi_2-1)=0$ gives the lines on which the velocity components become infinite as follows :—

$$r=\pm\frac{z+a\lambda}{\sqrt{\lambda^2-1}} \text{ and } r=\pm\frac{z-a\lambda}{\sqrt{\lambda^2-1}} \quad \ldots \quad \ldots \quad \ldots \quad (4.19)$$

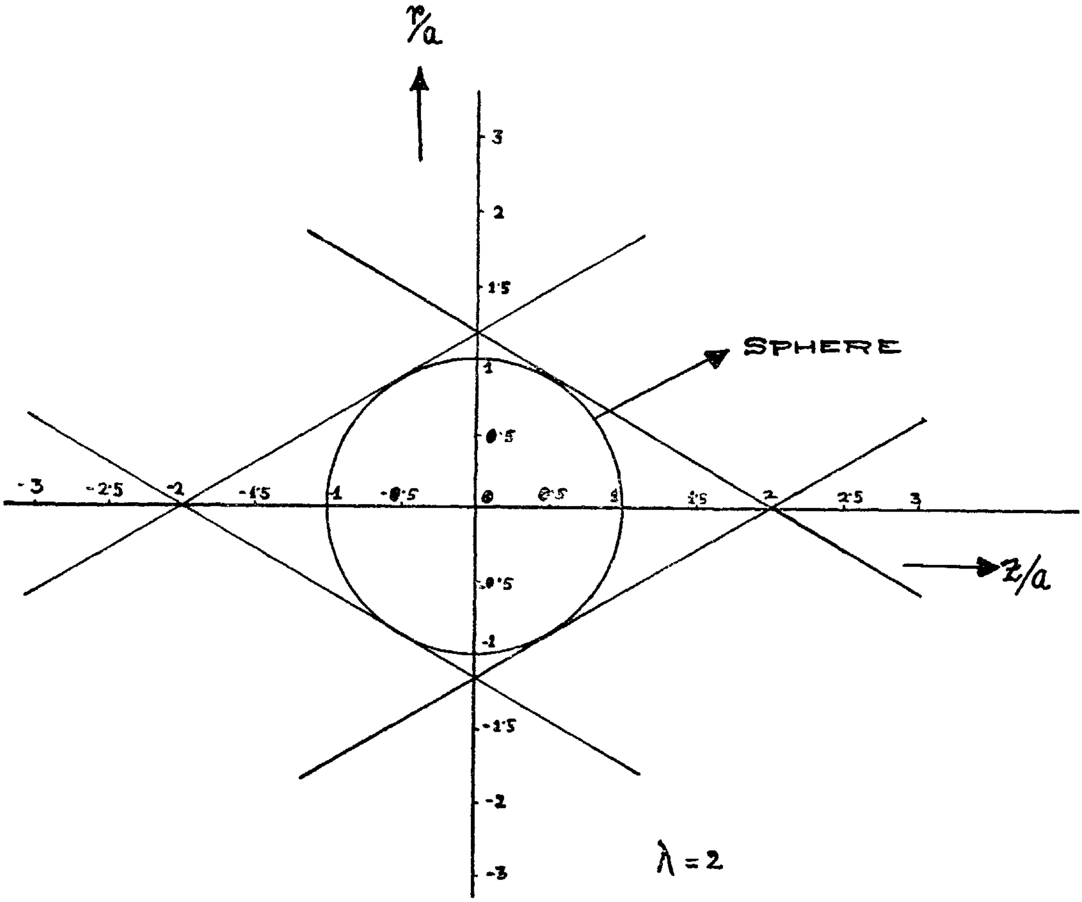

FIG. 1.

Fig. 1 shows these lines for the case $\lambda=2$.

These lines will generate two double cones whose vertices lie on the axis of symmetry of the motion, namely z-axis at distances $z = \pm a\lambda$ from the centre of the sphere. These two cones with vertices at $z = a\lambda$, and at $z = -a\lambda$ are tangent to the sphere and enclose it. These cones split the space of liquid flow into eight zones; three of them are confined in the region enclosed by the two enveloping cones and containing the sphere, and the remaining five are outside this enclosed region. We shall show that in the limit $\lambda \to \infty$ these eight compartments are watertight, there being no flow across the boundary of any two of them.

The normal velocity across the line $r = \dfrac{z-a\lambda}{\sqrt{\lambda^2-1}}$ is given by

$$F = \lim_{r \to \frac{z-a\lambda}{\sqrt{\lambda^2-1}}} \left(\frac{w}{\lambda} + \frac{\sqrt{\lambda^2-1}}{\lambda} u\right)$$

$$= -\frac{U\cos\beta t}{\lambda} + \frac{U}{2\pi i\lambda} \lim_{r \to \frac{z-a\lambda}{\sqrt{\lambda^2-1}}} \int_{C-i\infty}^{C+i\infty} e^{st} \frac{\left[s\log\dfrac{\zeta-i}{\zeta+i} + \dfrac{2is}{\zeta} - \dfrac{2izs^3\left\{z(\zeta^2+1) + r\sqrt{\lambda^2-1}\,\zeta^2\right\}}{\zeta^3(\zeta^2+1)(r^2+z^2)\left(s^2+4\Omega^2\xi_1^2\right)^{1/2}\left(s^2+4\Omega^2\xi_2^2\right)^{1/2}}\right]}{(s^2+\beta^2)\left\{\log\dfrac{s-2i\Omega}{s+2i\Omega} + \dfrac{4i\Omega s}{s^2+4\Omega^2}\right\}}\, ds$$

When Ωt is large the only contribution to F arises from the poles at $s = \pm i\beta$. Adding up these contributions we have finally on the line $r = \dfrac{z-a\lambda}{\sqrt{\lambda^2-1}}$

$$F = -\frac{U\cos\beta t}{\lambda} + \frac{U\left[\left(\log\dfrac{\lambda+1}{\lambda-1} + \dfrac{2\lambda}{\lambda^2-1}\right)\cos\beta t + \pi\sin\beta t\right]}{\left(\lambda - \dfrac{z}{a}\right)\left(\dfrac{z}{a\lambda}\right)^{1/2}\left\{\pi^2 + \left(\log\dfrac{\lambda+1}{\lambda-1} + \dfrac{2\lambda}{\lambda^2-1}\right)^2\right\}}$$

$$-U\frac{\pi\sin\beta t\left\{\log\dfrac{\lambda+1}{\lambda-1} + \dfrac{2\lambda}{\lambda^2-1} + 2\sqrt{\dfrac{a\lambda}{z}} - \log\dfrac{1+\sqrt{\dfrac{z}{a\lambda}}}{1-\sqrt{\dfrac{z}{a\lambda}}}\right\} + \cos\beta t\left\{\left(\log\dfrac{\lambda+1}{\lambda-1} + \dfrac{2\lambda}{\lambda^2-1}\right)\left(2\sqrt{\dfrac{a\lambda}{z}} - \log\dfrac{1+\sqrt{\dfrac{z}{a\lambda}}}{1-\sqrt{\dfrac{z}{a\lambda}}}\right) - \pi^2\right\}}{\lambda\left\{\pi^2 + \left(\log\dfrac{\lambda+1}{\lambda-1} + \dfrac{2\lambda}{\lambda^2-1}\right)^2\right\}} \tag{4.20}$$

It is clear from (4.20) that $F = O(1/\lambda)$, which becomes vanishingly small for large values of λ. When $\lambda \to \infty$, the enveloping cones become the enveloping cylinder of the moving sphere in Stewartson problem and consequently the vanishing of the normal velocity across the enveloping cones is in agreement with Stewartson's result. We now have a complete picture of the peculiarities of the motion mentioned in the introduction.

Other features of the secondary motion (u, v, w) given by the solution may be described thus. All such motion is referred to axes at the centre of the vibrating sphere.

The secondary motion at a point of the axis consists of a synchronous vibration in the same phase as of the sphere (but of different amplitude), together with another vibration represented by the term $\cos\left(\frac{2\Omega at}{z}\right) = \cos\left(\lambda\frac{a}{z}\right)\beta t$, which originates from a sort of interaction between the rotational frequency of the entire liquid and the vibration frequency of the sphere.

The components u, v of the secondary motion worked out show exactly similar feature, namely, of vibration in two frequencies, namely, the frequency of the vibrating sphere and a second one of the interaction type mentioned above. But now one notices a difference according as the vibrational frequency of the sphere is dominant ($\lambda < 1$), or the rotational frequency is dominant ($\lambda > 1$).

In the first case vibration in the frequency of the sphere has the same phase as that of the sphere, while in the second the phase of this vibration is altered.

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Abstract

K. Stewartson has investigated the linearized motion in an infinite inviscid liquid rotating about an axis when a sphere is set in uniform motion along the axis of rotation at time $t = 0$. The ultimate motion that would be set up under this condition was shown to be discontinuous by Stewartson. In the present paper a similar problem, namely that of a sphere set vibrating at time $t = 0$ along the axis of rotation of an infinite inviscid liquid, has been considered. The linearized equations have been used in the study of the problem. As in Stewartson's case discontinuities appear in the ultimate motion. In a limiting case our results reduce to those of Stewartson.

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