ON THE STABILITY OF A GRAVITATING SPHERE IN A MAGNETIC FIELD

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(Communicated by F. C. Auluck, F.N.I.)

(*Received April 22 ; approved for reading on December 6, 1957*)

1. In a recent paper Chandrasekhar and Fermi (1953) (hereafter referred to as C.F.) have considered, among various others, the problem of gravitational equilibrium of an incompressible fluid sphere with a uniform magnetic field inside and a dipole-like field outside. The fluid is supposed to be infinitely conducting and inviscid; the effects of thermal conduction and of Joule heating are neglected.

It is shown by C.F. that due to the presence of the magnetic field, the sphere is flattened at the poles by contracting along its axis of symmetry and that the resulting figure is an oblate spheroid, the ellipticity of the figure being given by

$$\frac{\epsilon}{R} = -\frac{35}{24}\frac{H^2R^4}{GM^2},$$

where $\vec{H}$ is the uniform magnetic field inside the sphere, G the gravitational constant and M the mass of the sphere of radius R. There are no volume currents inside the sphere, the field inside and outside the sphere being due to surface currents of density $(0, 0, 3H/8\pi \sin\theta)$.

Ferraro (1954) has considered the case of a sphere with currents flowing inside it. In this case no surface currents are present on the surface of the sphere, the sphere deforms into an oblate spheroid, but the ellipticity is given by

$$\frac{\epsilon}{R} = -\frac{75}{64}\frac{H_p^2}{G\pi^2\rho^2R^2},$$

where $\vec{H}_p$ is the magnetic field at the poles, ρ the density of the fluid constituting the sphere and R the radius of the sphere. An extension of this problem was considered by Auluck and Kothari (1956) ; in Ferraro's case the field $\vec{H}_1 = (0, 0, \frac{1}{3}\chi R^2)$ where χ is a constant is superposed throughout space so as to give zero field at the centre of the sphere. Further the equilibrium figure formed is a prolate spheroid and its eccentricity is given by

$$\frac{\epsilon}{R} = \frac{1}{2}H_1^2R^4/GM^2,$$

$\vec{H}_1$ being defined above.

In all the above cases the field considered was purely poloidal.

Recently J. De (1956) has considered a sphere in which there is a purely toroidal field inside the sphere and zero field outside.

In the present paper we have discussed the problem of stability of the above configurations using the ' energy method ' and we prove that all the models described above are those of stable equilibrium.

2. In this section we consider Ferraro's case. We proceed to find out the change in the magnetic energy of the sphere to the second order of smallness. To

do this we note that the magnetic field inside the sphere of radius R is given (cf. Ferraro, 1954) by

$$\left.\begin{aligned} H_r &= \chi\left(\frac{r^2}{5}-\frac{R^2}{3}\right)\cos\theta, \\ H_\theta &= \chi\left(-\frac{2r^2}{5}+\frac{R^2}{3}\right)\sin\theta, \\ H_\phi &= 0 \end{aligned}\right\} \quad \ldots \quad (1)$$

and the volume density of current by

$$\left.\begin{aligned} j_r &= 0, \\ j_\theta &= 0, \\ j_\phi &= -\frac{\chi}{4\pi} r\sin\theta \end{aligned}\right\} \quad \ldots \quad (2)$$

where χ is a constant and r, θ, ϕ spherical polar co-ordinates. Following C.F. we deform the surface of the sphere to

$$r = R+\epsilon P_l, \quad \ldots \quad (3)$$

where P_l is Legendre polynomial of order l and ϵ is small.

The disturbed field components $(H_r', H_\theta', H_\phi')$ and current density $(j_r', j_\theta', j_\phi')$ are given (cf. Auluck and Kothari, 1956) by

$$\left.\begin{aligned} H_r' &= \chi\left(\frac{r^2}{5}-\frac{R^2}{3}\right) \\ &\quad +\frac{\chi R}{15}\left(\frac{r}{R}\right)^{l-2}\epsilon\left\{\left[3(l-3)\frac{r^2}{R^2}-5(l-1)\right]P_l\cos\theta\right. \\ &\quad \left.+\left[\frac{3(2l-1)}{2}\frac{r^2}{R^2}-\frac{5(l-1)}{l}\right]P_l'\sin^2\theta\right\}, \\ H_\theta' &= \chi\left(-\frac{2r^2}{5}+\frac{R^2}{3}\right) \\ &\quad +\frac{\chi R}{15}\left(\frac{r}{R}\right)^{l-2}\epsilon\left\{3(l+2)\frac{r^2}{R^2}\left(2P_l-\frac{1}{l}P_l'\cos\theta\right)\right. \\ &\quad \left.-5l\left(P_l-\frac{1}{l}P_l'\cos\theta\right)\right\}\sin\theta, \end{aligned}\right\} \quad \ldots \quad (4)$$

$$H_\phi' = 0$$

and

$$\left.\begin{aligned} j_r' &= 0, \\ j_\theta' &= 0, \\ j_\phi' &= -\frac{\chi}{4\pi} r\sin\theta \\ &\quad -\frac{\chi}{10\pi}\left(\frac{r}{R}\right)^{l-1}\epsilon\left(3+\frac{1}{l}\right)P_{l-1}'\sin\theta, \end{aligned}\right\} \quad \ldots \quad (5)$$

respectively.

Now an arbitrary deformation (like (3)) of an incompressible body can be realized by applying at each point of the body a displacement $\vec{\xi}$ which is the gradient of a scalar point function ψ satisfying Laplace's equation

$$\nabla^2\psi = 0 \qquad (6)$$

As a solution appropriate to the problem in hand one has

$$\psi = \frac{\epsilon R}{l}\left(\frac{r}{R}\right)^l P_l, \qquad (7)$$

Therefore

$$\left.\begin{aligned} \xi_r &= \epsilon\left(\frac{r}{R}\right)^{l-1} P_l, \\ \xi_\theta &= -\frac{\epsilon}{l}\left(\frac{r}{R}\right)^{l-1} P_l' \sin\theta, \\ \xi_\phi &= 0 \end{aligned}\right\} \qquad (8)$$

Our procedure would now be to find out the change in the magnetic energy to the second order of ϵ as the sphere changes its configuration to (3). We imagine the change to take place gradually as ϵ varies from 0 to ϵ. The displacement $\delta\vec{\xi}$ which must be applied to increase the amplitude of deformation from ϵ to $\epsilon+\delta\epsilon$ is given by

$$\left.\begin{aligned} \delta\xi_r &= \delta\epsilon\left(\frac{r}{R}\right)^{l-1} P_l, \\ \delta\xi_\theta &= -\frac{\delta\epsilon}{l}\left(\frac{r}{R}\right)^{l-1} P_l' \sin\theta, \\ \delta\xi_\phi &= 0 \end{aligned}\right\} \qquad (9)$$

The change in the magnetic energy $\delta(\Delta m)$ involved in the infinitesimal displacement $\delta\vec{\xi}$ can be obtained by integrating, over the sphere, the work done during the displacement (9) by the magnetic field inside the sphere.

Now the mechanical (electro-magnetic) force $\vec{F}$ at a point in an electromagnetic field is given by

$$\vec{F} = \vec{j}\times\vec{H}, \qquad (10)$$

where $\vec{H}$ is magnetic field and $\vec{j}$ current density at that point. Therefore in this case

$$\begin{aligned} \vec{F} = \vec{j}'\times\vec{H}' &= (0, 0, j_\phi')\times(H_r', H_\theta', 0) \\ &= j_\phi'(-H_\theta', H_r', 0). \end{aligned} \qquad (10)'$$

Thus

$$\begin{aligned} \delta(\Delta m) &= -2\pi\int_0^{R+\epsilon P_l}\int_0^\pi \vec{F}\cdot\delta\vec{\xi} r^2 \sin\theta \,.\, dr\,.\, d\theta \\ &= -2\pi\int_0^{R+\epsilon P_l}\int_{-1}^1 j_\phi'(-H_\theta'\delta\xi_r + H_r'\delta\xi_\theta)\, r^2 dr d\mu \end{aligned} \qquad (11)$$

We shall limit our calculation to the case $l = 2$. On substituting the values of H_r', H_θ', j_ϕ', $\delta\xi_r$ and $\delta\xi_\theta$ from (4), (5) and (9) in (11) we get

$$\delta(\Delta m) = \frac{2}{225}\chi^2 R^6 \delta\epsilon + \frac{14}{1125}\chi^2 R^5 \epsilon\delta\epsilon \, . \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Therefore, integrating $\omega . r . t . \epsilon$ from $\epsilon = 0$ to ϵ,

$$\Delta \, . \, m = \frac{2}{225}\chi^2 R^6 \epsilon + \frac{7}{1125}\chi^2 R^5 \epsilon^2 \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

to the second order of ϵ only.

Also the change in the gravitational potential energy is given (cf. Chandrasekhar and Fermi, 1953) by

$$\Delta \Omega = \frac{3}{25}\frac{GM^2}{R^3}\epsilon^2. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Thus change in the total energy is

$$\Delta E = \frac{2}{225}\chi^2 R^6 \epsilon + \frac{7}{1125}\chi^2 R^5 \epsilon^2 + \frac{3}{25}\frac{GR^2}{R^3}\epsilon^2. \quad \ldots \quad \ldots \quad (14)$$

Retaining only the first order term in the change in the magnetic energy and using the equation of equilibrium

$$\frac{d}{d\epsilon}\left(\Delta E\right) = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

we get the same value of ϵ/R as the one obtained by Ferraro (1954). As the coefficient of ϵ^2 in (14) is positive corresponding change in the energy is a minimum, and accordingly the equilibrium is stable.

It may however be added that the extension of Ferraro's case considered by Auluck and Kothari (1956) has also been worked through and the change in the magnetic energy due to the superposed field is found to be

$$-\frac{1}{45}\chi^2 R^6 \epsilon + \frac{3}{225}\cdot\frac{1}{2}\cdot\chi^2 R^5 \epsilon^2.$$

Therefore in this case

$$\Delta E = -\frac{3}{225}\chi^2 R^6 \epsilon + \frac{29}{2250}\chi^2 R^5 \epsilon^2 + \frac{3}{25}\frac{GM^2}{R^3}\epsilon^2. \quad \ldots \quad \ldots \quad (16)$$

It should be observed that the superposition of the field has given rise to a different configuration of equilibrium and that the model remains stable.

3. In this section we shall obtain a purely toroidal solution for the interior of the sphere, from the hydromagnetic equilibrium equations and by choosing a particularly simple solution, we find out the change in the magnetic energy as the sphere changes its configuration to (3).

Consider a sphere of radius R of an incompressible fluid of constant density ρ. Let $\vec{H}$ be the magnetic field produced by a system of currents of density $\vec{j}$ flowing in the fluid. The field equations in steady state are

$$\operatorname{div} \vec{H} = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\operatorname{curl} \vec{H} = 4\pi\vec{j}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

We infer from (17) that for axial symmetry the field $\vec{H}$ can be derived from two scalar functions U, V which are azimuth independent. Taking spherical polar co-ordinates r, θ, ϕ referred to the centre of the sphere as origin we have, since $\text{div}\,\vec{H} = 0$,

$$H_r = -\frac{1}{r^2 \sin\theta}\frac{\partial U}{\partial \theta}, \quad H_\theta = \frac{1}{r\sin\theta}\frac{\partial U}{\partial r}, \quad H_\phi = \frac{V}{r\sin\theta} \quad \ldots \quad (19)$$

where U, V are functions of r, θ only. If p be the pressure and Φ the gravitational potential, then the equation of hydromagnetic equilibrium is

$$\text{grad}\,(p-\rho\Phi) = \vec{j}\times\vec{H} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Therefore

$$\text{curl}\left(\vec{j}\times\vec{H}\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

From (19) and (21) we obtain

$$J\begin{pmatrix} U, & V \\ r, & \theta \end{pmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

and

$$J\begin{pmatrix} \Delta U, & U \\ r, & \theta \end{pmatrix} = J\begin{pmatrix} \dfrac{V}{r^2 \sin^2\theta}, & V \\ r, & \theta \end{pmatrix}$$

$$= -\frac{2}{r^2\sin^3\theta} V\frac{\partial V}{\partial z}, \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where J stands for Jacobian,

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\theta}\right)$$

and $z = r\cos\theta$.

As a purely toroidal solution, we are interested in the solutions of (22) and (23) with $V = 0$, we thus get

$$V\frac{\partial V}{\partial z} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

which implies that either $V = 0$ or V is independent of $r\cos\theta$.

Since V is independent of the azimuth angle ϕ, we conclude that V is a function of $r \sin\theta$, which result was also arrived at by J. De. In the following we shall only be interested in the solution

$$V = H_0 r^2 \sin^2\theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

where H_0 is a constant.

The field inside the sphere is given by

$$\vec{H} = (0, 0, H_0 r\sin\theta) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

and current density by

$$\vec{j} = \left(\frac{H_0}{2\pi}\cos\theta, \ -\frac{H_0}{2\pi}\sin\theta, 0\right) \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

We now deform the surface of the sphere to

$$r = R + \epsilon P_l$$

This deformation corresponds to the displacement vector given by (8).

The change in the magnetic field is given by

$$\delta\vec{H} = \text{curl}\left(\vec{\xi}\times\vec{H}\right) \qquad (28)$$

Substituting from (8) and (26) we get

$$\delta\vec{H} = 0, \qquad (29)$$

to the first order in ϵ.

Accordingly

$$\delta\vec{j} = 0, \qquad (30)$$

to the same order in ϵ.

The mechanical force in the interior of the fluid is

$$\vec{F} = \left(\vec{j}\times\vec{H}\right) = \left(-\frac{H_0^2}{2\pi}r\sin^2\theta,\ -\frac{H_0^2}{2\pi}r\sin\theta\cos\theta,\ 0\right) \qquad (31)$$

Now change in the magnetic energy will be evaluated partly as work done by forces inside the fluid and partly as work done by surface forces. Change in the magnetic energy Δm_1 corresponding to volume currents is

$$\Delta m_1 = -\frac{2}{15}H_0^2R^4\epsilon+\frac{1}{15}H_0^2R^3\epsilon^2 \qquad (32)$$

for $l = 2$.

We now proceed to find the contribution to the change in the magnetic energy due to surface currents.

The average field $\vec{H}''$ on the surface is

$$\vec{H}'' = (0,\ 0,\ \tfrac{1}{2}HR\sin\theta) \qquad (33)$$

Due to discontinuity of the field on the surface, surface currents exist and are given by

$$\vec{j}_s = \left(-\frac{H_0}{4\pi}\epsilon P_l'\ R\sin^2\theta,\ \frac{H_0}{4\pi}R\sin\theta,\ 0\right) \qquad (34)$$

The surface force is

$$\vec{F} = \left(\frac{H_0^2}{8\pi}R^2\sin^2\theta,\ \frac{1}{8\pi}H_0^2\ \epsilon RP_l'\sin^3\theta,\ 0\right) \qquad (35)$$

The contribution to the change in the magnetic energy Δm_2 due to surface forces comes to

$$\Delta m_2 = +\frac{1}{15}H_0^2R^4\epsilon+\frac{1}{70}H_0^2R^3\epsilon^2 \qquad (36)$$

Also

$$\Delta\Omega = \frac{3}{25}\frac{GM^2}{R^3}\epsilon^2 \qquad (13)$$

Therefore

$$\Delta E = -\frac{1}{15}H_0^2R^4\epsilon+\frac{17}{210}H_0^2R^3\epsilon^2+\frac{3}{25}\frac{GM^2}{R^3}\epsilon^2. \qquad (37)$$

For equilibrium we have

$$\frac{\epsilon}{R}=\frac{5}{18}\frac{H_0^2 R^5}{GM^2}\,*,$$

retaining only the first order term in the magnetic energy, and since the coefficient of ϵ^2 is positive the equilibrium is stable. In the end it may be stated that the change in the magnetic energy in C.F. case has also been evaluated and the result is

$$\Delta E=\frac{7}{20}H^2R^2\epsilon+\frac{523}{1400}H^2R\epsilon^2+\frac{3}{25}\frac{GM^2}{R^3}\epsilon^2. \qquad \ldots \qquad \ldots \quad (38)$$

As before the equilibrium is found to be stable.

Acknowledgements

The author is grateful to Prof. D. S. Kothari and Prof. F. C. Auluck for suggesting the problem.

Summary

The stability of a gravitating sphere under the action of different magnetic fields for models considered by earlier authors has been discussed. Employing the 'Energy method' it is found that under a P_2 deformation the equilibrium configurations arrived at by them are stable.

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* Note: This result does not agree with J. De's because of his using wrong expression for the gravitational potential on the deformed surface.

Issued January 4, 1958.