THE SOLUTION OF THE EQUATIONS OF INTERNAL BALLISTICS FOR POWER LAW OF BURNING

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Abstract

Clemmow (1928, 1951) has given a method of solving the equations of Internal Ballistics for the power law of burning. In the present paper, three alternative methods have been given, and a series solution for the Isothermal Model for the specific case of a tubular propellent has been obtained.

1. Introduction

The Law of burning most commonly used in Internal Ballistics of guns and rockets is Vielles Law (1893).

Law I: $$B(p) = \beta p^{\alpha} \qquad \ldots \quad (1)$$

where the pressure index α lies between 0·9 and 1 for the high pressure in guns and between 0·3 and 0·8 for the comparatively lower pressures in rocket motors. Accordingly for convenience in analytic solutions of the equations of Internal Ballistics of guns, Law I is replaced by

Law II: $$B(p) = \beta' p, \qquad \ldots \quad (2)$$

with $$\beta' = \beta - \frac{1}{8}\beta(10\alpha+9)(1-\alpha). \qquad \ldots \quad (3)$$

Solutions for Law II have been given by Crow (1922), Sugot (1928), Hunt and Hinds (1929), Coppock (1942), Goldie (1945) and are described in detail in the H.M. S.O. publications, 'Internal Ballistics' (1951) and in Corner (1950).

It is necessary, however, to obtain the solutions for Law I, which is found more plausible on the basis of observations in order to check the accuracy of the results obtained with Law II, and to see what special alterations should be made if one wishes to use the simplified Law II instead of Law I. The study of the solutions for the power law would also be useful in the study of the Internal Ballistics of solid-fuel rockets, recoil-less guns and high-low guns where Law II would not be even a good approximation.

Clemmow (1928, 1951) solved the system of equations of Internal Ballistics for Law I in terms of the solution of an ordinary non-linear differential equation of order three, in the general case, and of order two, in the special case of a tubular charge. In the present paper, we obtain the solution in terms of a solution of a non-linear differential equation of order two, even in the general case. This is achieved by a suitable choice of the variables. In fact, the system admits of a number of such choices and we have worked them out completely.

In the final two sections, we have given the solution of the resulting differential equations under the appropriate initial conditions, for the isothermal model, for the specific case of a tubular propellent. The solution is obtained for the case of guns in the form of a series in powers of $1-\alpha$. This is warranted, because for most of propellents, $1-\alpha$ lies between $-0{\cdot}1$ and $+0{\cdot}1$, as will be seen from the following table:

Propellent	W	MD	WM	SC	HSC	A	AN	ASN	N	NQ	NFQ	NH
$1-\alpha$	0·03	0·09	−0·05	−0·04	0·03	−0·11	−0·06	−0·05	0·07	0·11	0·09	0·01

2. Equations of Internal Ballistics for the Power Law of Burning

Neglecting co-volume correction terms, the fundamental equations are (Clemmow (1951), page 117):

$$z = \zeta\xi + \tfrac{1}{2}(\gamma-1)\frac{\eta^2}{M} \quad \ldots \quad (4)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \quad \ldots \quad (5)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (6)$$

$$\eta\frac{df}{d\xi} = -\zeta^{\alpha} \quad \ldots \quad (7)$$

where the dimensionless variables ξ, η, ζ corresponding to shot-travel velocity and pressure respectively, and the central Ballistic parameter M are given by

$$\xi = 1+\frac{x}{l} \quad \ldots \quad (8)$$

$$\eta = \frac{AD}{FC\beta}\left(\frac{FC}{Al}\right)^{1-\alpha} v \quad \ldots \quad (9)$$

$$\zeta = \frac{Al}{FC}p \quad \ldots \quad (10)$$

$$M = \frac{A^2D^2}{FC\beta^2W_1}\left(\frac{FC}{Al}\right)^{2-2\alpha}. \quad \ldots \quad (11)$$

Using (5) and (7)

$$\frac{d}{df}\left[\zeta^{\alpha}\frac{d\xi}{df}\right] = -\left(\frac{M\zeta}{\eta}\right)\left(-\frac{\eta}{\zeta^{\alpha}}\right) = M\zeta^{1-\alpha}, \quad \ldots \quad (12)$$

and from (6),

$$\frac{dz}{df} = -\nu\sqrt{1-qz} \quad \ldots \quad (13)$$

where

$$\nu = 1+\theta \quad \ldots \quad (14a)$$

$$q = \frac{4\theta}{(1+\theta)^2}. \quad \ldots \quad (14b)$$

From (12) and (13)

$$(1-qz)\frac{d}{dz}\left(\zeta^{\alpha}\frac{d\xi}{dz}\right) - \frac{1}{2}q\zeta^{\alpha}\frac{d\xi}{dz} = \frac{M}{\nu^2}\zeta^{1-\alpha} \quad \ldots \quad (15)$$

Differentiating (4) and substituting from (5)

$$dz = \xi^{1-\gamma}dY, \quad \ldots \quad (16a)$$

so that

$$\frac{dY}{dz} = \xi^{\gamma-1} \quad \ldots \quad (16b)$$

where

$$Y = \zeta\xi^{\gamma}. \quad \ldots \quad (17)$$

Before proceeding further, we note the importance of variable Y introduced in (17). Since z is a monotonic increasing variable, it follows from (16a) that Y is also monotonic increasing variable throughout burning. (16b) shows that not only Y increases as z increases, but it does so at an increasing rate, since x and therefore ξ goes on increasing throughout the motion. After all-burnt, the equation of energy is obtained from (4), by putting $z = 1$; and therefore from (16a), $Y = \zeta\xi^{\gamma}$ is constant after all-burnt. These conclusions about Y can also be obtained from its physical interpretation as being proportional to e^{s}, where S is the entropy of the system which goes on increasing throughout the burning period and becomes constant after all-burnt.

3. First Method. Fundamental Differential Equation

In this method, we take ζ as dependent and Y as independent variable. From (17)

$$\xi = \left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}}$$

and

$$\frac{d\xi}{dz} = \frac{1}{\gamma\zeta^2}(\zeta - Y\zeta'). \quad \ldots \quad (18)$$

Hence from (15), on simplification, we have

$$(1-qz)\left(\frac{Y}{\zeta}\right)^{\frac{\gamma-1}{\gamma}}[-Y\zeta\zeta''+(2-\alpha)Y\zeta'^2-(2-\alpha)\zeta\zeta'] - \tfrac{1}{2}q\zeta[\zeta - Y\zeta'] = \frac{M\gamma}{\nu^2}\zeta^{4-2\alpha} \quad (19)$$

On eliminating η between (4) and (7), we get

$$1-qz = \frac{1-q\zeta\left(\frac{Y}{\zeta}\right)^{\frac{1}{\gamma}}}{1+\frac{1}{2}\frac{\gamma-1}{M}\frac{q\nu^2}{\gamma^2}\zeta^{2\alpha-4}(\zeta-Y\zeta')^2} \quad \ldots \quad (19a)$$

so that (19) becomes

$$\left[1-qY^{\frac{1}{\gamma}}\zeta^{\frac{\gamma-1}{\gamma}}\right]\left[\left(\frac{Y}{\zeta}\right)^{\frac{\gamma-1}{\gamma}}\right][-Y\zeta\zeta''+(2-\alpha)Y\zeta'^2-(2-\alpha)\zeta\zeta']$$
$$= \left[\frac{M\gamma}{\nu^2}\zeta^{4-2\alpha} + \tfrac{1}{2}q\zeta(\zeta-Y\zeta')\right]\times\left[1+\tfrac{1}{2}\frac{\gamma-1}{M}\frac{q\nu^2}{\gamma^2}\zeta^{2\alpha-4}(\zeta-Y\zeta')^2\right] \quad \ldots \quad (20)$$

This is the Fundamental differential equation of the second order, the numerical solution of which under suitable initial conditions will give the solutions of the main problems of Internal Ballistics.

If we set

$$X = Y^{\frac{1}{\gamma}} = \zeta^{\frac{1}{\gamma}}\xi,\ Y = X^{\gamma} \quad \ldots \quad (21)$$

in (20), we get

$$\left[\zeta^{\frac{1-\gamma}{\gamma}} - qX\right][X\zeta\zeta''+(\alpha-2)X\zeta'^2+(1-\gamma\alpha+\gamma)\zeta\zeta'] + \left[\frac{M\gamma^3}{\nu^2}\zeta^{4-2\alpha} + \tfrac{1}{2}q\gamma(\gamma\zeta^2 - X\zeta\zeta')\right]$$
$$\times\left[1+\tfrac{1}{2}q\frac{\nu^2}{\gamma^4}\frac{\gamma-1}{M}\zeta^{2\alpha-4}(\gamma\zeta - X\zeta')^2\right] = 0, \quad \ldots \quad (22)$$

in which only the integral powers of the independent variable X occur.

Particular Cases :

(i) *Constant-burning surface :* Here $\theta = 0, \nu = 1, q = 0$ .. .. .. (23)

so that (22) becomes

$$X\zeta\zeta'' + (\alpha-2)X\zeta'^2 + (1-\gamma\alpha+\gamma)\zeta\zeta' = -M\gamma^3\zeta^{5-2\alpha-\frac{1}{\gamma}}. \quad .. \quad (24)$$

If we substitute

$$Z = MX, \quad .. \quad .. \quad .. \quad .. \quad .. \quad (25)$$

(24) reduces to

$$Z\zeta\frac{d^2\zeta}{dZ^2} + (\alpha-2)Z\left(\frac{d\zeta}{dZ}\right)^2 + (1-\gamma\alpha+\gamma)\zeta\frac{d\zeta}{dZ} = -\gamma^3\,\zeta^{5-2\alpha-\frac{1}{\gamma}}, \quad .. \quad (26)$$

an equation which does not contain M, and the parameters concerning the gun can enter through the boundary conditions only.

(ii) *Standard form-function. Simplified energy equation on isothermal model :*

In this case, we neglect the kinetic energy term from the energy equation and compensate for this loss by adjusting the force constant F. We can deduce the equations for this case by putting $\gamma = 1$ in (22) and replacing M by adjusted $\overline{M}$

$$[1-qX]\,[X\zeta\zeta'' + (\alpha-2)X\zeta'^2 + (2-\alpha)\zeta\zeta'] + \left[\frac{\overline{M}}{\nu^2}\zeta^{4-2\alpha} + \tfrac{1}{2}q\zeta(\zeta-\zeta'X)\right] = 0 \quad (27)$$

(iii) *Constant-burning surface. Simplified energy equation on isothermal model.*

Here $q = 0$, $\nu = 1$, $\gamma = 1$, and we get from (24) and (26)

$$X\zeta\zeta'' + (\alpha-2)X\zeta'^2 + (2-\alpha)\zeta\zeta' = -\overline{M}\zeta^{4-2\alpha} \quad .. \quad (28)$$

and

$$Z\zeta\frac{d^2\zeta}{dZ^2} + (\alpha-2)Z\left(\frac{d\zeta}{dZ}\right)^2 + (2-\alpha)\zeta\frac{d\zeta}{dZ} = -\zeta^{4-2\alpha} \quad .. \quad .. \quad (29)$$

4. Solution of the Main Problems of Internal Ballistics

We represent the circumstance of band engraving by a finite non-zero shot-start pressure, so that initially

$$x = 0,\ v = 0,\ p = p_0,\ \xi = 1,\ \eta = 0,\ \zeta = \frac{Al}{FC}p_0 = \zeta_0. \quad .. \quad (30)$$

But

$$X = \xi\,\zeta^{\frac{1}{\gamma}}$$

$\therefore$ initially

$$X = \zeta_0^{\frac{1}{\gamma}}.$$

From (4), initially $\dfrac{d\zeta}{dZ} = 0$, since $\eta = 0$ at shot-start.

This gives from (17) and (21), initially

$$\frac{d\zeta}{dX} = \gamma\zeta_0^{\frac{\gamma-1}{\gamma}}.$$

Thus the initial conditions for the integration of (22) are :

$$\xi = 1,\ \eta = 0,\ \zeta = \zeta_0,\ X = \zeta_0^{\frac{1}{\gamma}},\ \frac{d\zeta}{dX} = \gamma\zeta_0^{\frac{\gamma-1}{\gamma}}. \quad .. \quad .. \quad (31)$$

If we take $Z = MX$ as independent variable, the initial conditions are:

$$Z = M\zeta_0^{\frac{1}{\gamma}},\ \frac{d\zeta}{dZ} = \frac{\gamma}{M}\zeta_0^{\frac{\gamma-1}{\gamma}} \quad \ldots \quad (32)$$

Thus we see that while by using Z instead of X, M disappears from the differential equation, it reappears in the initial conditions.

For isothermal model $\gamma = 1$, the initial conditions are:

$$\xi = 1,\ \eta = 0,\ \zeta = 0,\ X = 0,\ \frac{d\zeta}{dX} = 0,\ Z = 0,\ \frac{d\zeta}{dZ} = \frac{1}{M} \quad \ldots \quad (33)$$

Solution of the Fundamental Equation

Integrating the equation (22) or any of its particular cases, we obtain ζ and $\frac{d\zeta}{dX}$ as functions of X.

Let

$$\zeta = I(X) \quad \ldots \quad (34)$$

$$\frac{d\zeta}{dX} = J(X) \quad \ldots \quad (35)$$

then $I(X)$ and $J(X)$ can be tabulated as functions of X.

From (21)

$$\xi = X[I(X)]^{-\frac{1}{\gamma}} \equiv K(X), \quad \ldots \quad (36)$$

and from (16) and (21)

$$\frac{dX}{dz} = \frac{1}{\gamma}\zeta^{\frac{1-\gamma}{\gamma}} \quad \ldots \quad (37)$$

$$\therefore\quad z - z_0 = \gamma\int_{\zeta_0^{\frac{1}{\gamma}}}^{X}[I(X)]^{\frac{\gamma-1}{\gamma}}\,dX \equiv L(X) \quad \ldots \quad (38)$$

Using (34), (36) and (38) in (4), we have

$$\eta^2 = \frac{2M}{\gamma-1}\{z_0 + L(X) - I(X)K(X)\}. \quad \ldots \quad (39)$$

At shot-start

$$z_0 = \zeta_0 = \frac{Al}{FC}p_0 \quad \ldots \quad (40)$$

(40) determines z_0 in terms of the shot-start pressure.

Now for any given value of f, (6) determines z; then (38) determines X, (8) and (36) determine shot-travel, (10) and (34) determine pressure and finally (9) and (39) determine the velocity.

Incidentally we note that when $\gamma = 1$ (38) gives

$$z - z_0 = \int_{\zeta_0}^{X} dX = X - \zeta_0,$$

which with (21) verifies (40).

Maximum Pressure

For maximum pressure $\frac{d\zeta}{dX}$ should vanish and since X is monotonic increasing, $\frac{d\zeta}{dX}$ should change sign from positive to negative.

Let X_1 be the solution of $J(X) = 0$, then from (10) and (34), maximum pressure p_1 is given by

$$\frac{Al}{FC}p_1 = I(X_1). \quad \dots \quad (41)$$

Now

$$1+\frac{x_1}{l} = \xi_1 = K(X_1) \quad \dots \quad (42)$$

determines the shot-travel x_1 up to the instant of maximum pressure.

Again from (9) and (41), the velocity up to this instant is given by

$$\frac{A^2D^2}{F^2C^2\beta^2}v_1^2\left(\frac{FC}{Al}\right)^{2-2\alpha} = \frac{2M}{\gamma-1}\left[z_0+L(X_1)-K(X_1)I(X_1)\right] \quad \dots \quad (43)$$

All-Burnt Positions

Let suffix 2 correspond to this position, then from (38), $L(X_2) = 1-z_0$ determines z_0 and

$$1+\frac{x_2}{l} = \xi_2 = K(X_2) \quad \dots \quad (44)$$

$$\frac{Al}{FC}p_2 = \zeta_2 = I(X_2) \quad \dots \quad (45)$$

and

$$\frac{A^2D^2}{F^2C^2\beta^2}v_2^2\left(\frac{FC}{Al}\right)^{2-2\alpha} = \frac{2M}{\gamma-1}\left[z_0+L(X_2)-K(X_2)I(X_2)\right] \quad \dots \quad (46)$$

determine shot-travel, pressure and velocity up to all-burnt.

Motion after All-Burnt Muzzle Velocity

After all-burnt $z = 1$, $dz = 0$

$\therefore$ from (16) $dY = 0$ or $d(\zeta\xi^\gamma) = 0$

$$\therefore \quad \zeta\xi^\gamma = \zeta_2\xi_2^\gamma. \quad \dots \quad (47)$$

If suffix 3 denotes the muzzle position

$$\zeta_3\xi_3^\gamma = \zeta_2\xi_2^\gamma. \quad \dots \quad (48)$$

where

$$\xi_3 = 1+\frac{x_3}{l} \quad \dots \quad (49)$$

(48) determines the pressure at the muzzle. From (1), the muzzle velocity V_3 is determined from

$$\frac{A^2D^2}{F^2C^2\beta^2}v_3^2\left(\frac{FC}{Al}\right)^{2-2\alpha} = \eta_3^2 = \frac{2M}{\gamma-1}\left[1-\zeta_3\xi_3\right] = \frac{2M}{\gamma-1}\left[1-\zeta_2\xi_2^\gamma\xi_3^{1-\gamma}\right] \quad (50)$$

5. Second Method of Solving the Equations

In this method, we use ξ as dependent and Y as independent variables.

From (15)

$$(1-qz)\xi^{\gamma-1}\frac{d}{dY}\left[Y^\alpha\xi^{-\alpha\gamma}\frac{d\xi}{dY}\xi^{\gamma-1}\right]-\tfrac{1}{2}qY^\alpha\xi^{-\alpha\gamma}\frac{d\xi}{dY}\xi^{\gamma-1} = \frac{M}{\nu^2}Y^{1-\alpha}\xi^{-(1-\alpha)\gamma}$$

which on simplification gives

$$(1-qz)[Y\xi\xi''+(\gamma-1-\alpha\gamma)Y\xi'^2+\alpha\xi\xi'] = \frac{M}{\nu^2}\xi^{2\gamma\alpha-3\gamma+3}Y^{2-2\alpha}+\tfrac{1}{2}qY\xi^{2-\gamma}\xi' \quad (51)$$

From (19*a*)

$$1-qz = \frac{1-qY\xi^{1-\gamma}}{1+\frac{1}{2}\frac{\gamma-1}{M}q\nu^2Y^{2\alpha}\xi^{2\gamma-2-2\alpha\gamma}\xi'^2},$$

so that (51) reduces to

$$[1-qY\xi^{1-\gamma}][Y\xi\xi''+(\gamma-1-\alpha\gamma)Y\xi'^2+\alpha\xi\xi'] = \left[1+\tfrac{1}{2}\frac{\gamma-1}{M}q\nu^2Y^{2\alpha}\xi^{2\gamma-2-2\alpha\gamma}\xi'^2\right] \times\left[\frac{M}{\nu^2}Y^{2-2\alpha}\xi^{2\gamma\alpha-3\gamma+3}+\tfrac{1}{2}qY\xi^{2-\gamma}\xi'\right] \quad (52)$$

Particular Cases :

(i) *Constant-burning surface :* Here $q = 0$, $\nu = 1$ and (52) reduces to

$$Y\xi\xi''+(\gamma-1-\alpha\gamma)Y\xi'^2+\alpha\xi\xi' = MY^{2-2\alpha}\xi^{2\gamma\alpha-3\gamma+3} \quad (53)$$

(ii) *Isothermal model. Constant-burning surface :*

Putting $\gamma = 1$ and replacing M by M' we, have

$$Y\xi\xi''-\alpha Y\xi'^2+\alpha\xi\xi' = M'Y^{2-2\alpha}\xi^{2\alpha}. \quad (54)$$

Substituting in (53)

$$\xi = U^{\frac{2n}{\gamma-n}} \quad (55a)$$

$$Y = \frac{(1+n)V}{n(\gamma-n)M} \quad (55b)$$

$$\alpha = \tfrac{1}{2}\left(3-\frac{1}{n}\right), \quad (55c)$$

we get, after some simplification, Clemmow's equation

$$\frac{2uV}{1+n}\frac{d^2u}{dV^2}-\frac{2n(\gamma-1)V}{(1+n)(\gamma-n)}\left(\frac{dU}{dV}\right)^2+U\frac{dU}{dV} = 1 \quad (56)$$

Thus our equation (52) covers up the particular case of a constant-burning surface discussed by Clemmow and is more general than his equation in that it covers other cases as well.

It may be easily verified that the initial conditions for the integration of (52) or (53) are

$$\xi = 1,\ Y = \zeta_0,\ \xi' = 0. \quad (57)$$

Solution of the Equations

Subject to initial conditions (57), equation (52) can be integrated numerically, and let the solution be

$$\xi \equiv P(Y) \quad (58)$$

$$\xi' \equiv Q(Y). \quad (59)$$

then
$$\zeta = \frac{Y}{\xi\gamma} = Y[P(Y)]^{-\gamma} \equiv R(Y). \qquad \ldots \quad \ldots \quad \ldots \quad (60)$$

From (16)
$$\frac{dY}{d\xi} = \xi^{\gamma-1}$$

$$\therefore \quad z - z_0 = \int_{\zeta_0}^{Y} [P(Y)]^{1-\gamma} dY \equiv S(Y) \qquad \ldots \quad \ldots \quad (61)$$

and then (4) gives
$$\eta^2 = \frac{2M}{\gamma - 1}[z_0 + S(Y) - P(Y)R(Y)] \equiv [T(Y)]^2. \qquad \ldots \quad \ldots \quad (62)$$

Maximum Pressure

For maximum pressure, $\frac{d\zeta}{dY} = 0$

from (60)
$$\frac{1}{Y} - \gamma \frac{P'(Y)}{P(Y)} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

or
$$\frac{1}{Y} - \gamma \frac{Q(Y)}{P(Y)} = 0$$

Let Y_1 be the solution of (63) at which $\frac{d\zeta}{dY}$ changes its sign from positive to negative and then $P(Y_1)$, $R(Y_1)$, $T(Y_1)$ determine the shot-travel pressure and velocity up to the instant of maximum pressure.

All-Burnt Position

Let Y_2 be the solution of (61) when $z = 1$ then $P(Y_2)$, $R(Y_2)$ and $T(Y_2)$ determine the shot-travel pressure and velocity up to the all-burnt position.

Motion after All-Burnt. Muzzle Velocity

This is discussed in exactly the same way as in Method I.

6. General Form-Function

The two methods developed above, as well as Clemmow's method, apply when the charge has the standard form-function.
$$z = (1-f)(1+\theta f) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

We now develop a method for the case when the charge has the more general form-function
$$z = \phi(f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

We take Y as dependent and f as independent variable.

From (16a) and (65)
$$Y' = \frac{dY}{df} = \phi'(f)\xi^{\gamma-1}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

and hence

$$\frac{Y''}{Y'} = \frac{\phi''}{\phi'} + (\gamma-1)\frac{\xi'}{\xi} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

Substituting from (66) and (67) in (12), we get

$$\frac{d}{df}\left\{ Y^{\alpha}\left(\frac{Y'}{\phi'}\right)^{\frac{1-\alpha\gamma}{\gamma-1}}\left(\frac{Y''}{Y'}-\frac{\phi''}{\phi'}\right)\right\} = M Y^{1-\alpha}\left(\frac{Y'}{\phi'}\right)^{-\frac{\gamma(1-\alpha)}{\gamma-1}} . \qquad \ldots \quad (68)$$

This is the fundamental differential equation of this case. It may be pointed out that it is of the third order. The initial conditions are

$$f = f_0,\ Y = \zeta_0,\ Y' = \phi'(f_0),\ Y'' = \phi''(f_0), \qquad \ldots \quad \ldots \quad (69)$$

where f_0 is determined by

$$z_0 = \zeta_0 = \phi(f_0) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (70)$$

The numerical solution of (68) subject to (69) would give

$$Y \equiv \lambda(f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (71)$$

$$Y' \equiv \mu(f) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

$$Y'' \equiv \nu(f), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (73)$$

then

$$\xi = \left[\frac{\mu(f)}{\phi'(f)}\right]^{\frac{1}{\gamma-1}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

$$\zeta = \lambda(f)\left[\frac{\mu(f)}{\phi'(f)}\right]^{-\frac{\gamma}{\gamma-1}} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (75)$$

$$\eta^2 = \frac{2M}{\gamma-1}\left[\phi(f) - \lambda(f)\left[\frac{\mu(f)}{\phi'(f)}\right]^{-1}\right] \qquad \ldots \quad \ldots \quad \ldots \quad (76)$$

Maximum pressure, all-burnt position and muzzle velocity are obtained as in the other two methods.

7. Some Remarks on the Choice of Variables

The choice of variables for forming the differential equations is restricted by the following considerations:

(i) ξ is not a desirable independent variable, as the derivative of any other variable with respect to ξ tends to infinity at shot-start making numerical integration difficult.

(ii) ζ is not a desirable independent variable as (*a*) it is not monotonic and (*b*) derivative of any other variable with respect to ζ tends to infinity at the instant of maximum pressure.

(iii) Y is used as independent variable in the first two methods and as dependent variable in our third method and in Clemmow's method. Its significant physical interpretation in terms of entropy explains why it is so useful.

(iv) z and f are both monotonic and are respectively used as independent variables in Clemmow's and in our third method.

Besides it is not always possible to eliminate other variables to get the differential equations between any two variables. Our three methods together with Clemmow's almost exhaust all fruitful possibilities.

8. Isothermal Model. First Method of Solution

We have to solve the differential equation

$$Y\zeta\zeta'' - (2-\alpha)Y\zeta'^2 + (2-\alpha)\zeta\zeta' = -\bar{M}\zeta^{4-2\alpha} \quad \ldots \quad \ldots \quad (28)$$

subject to initial conditions

$$\xi = 1, \quad \eta = 0, \quad \zeta = \zeta_0, \quad Y = \zeta_0, \quad \frac{d\zeta}{dY} = 1. \quad \ldots \quad \ldots \quad (77)$$

For $\alpha = 1$ (28) reduces to

$$Y\zeta\zeta'' - Y\zeta'^2 + \zeta\zeta' + \bar{M}\zeta^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (78)$$

We shall first integrate (78) subject to initial conditions (77).

Let

$$\zeta = e^{-\bar{M}Y} H(Y), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79)$$

then (78) gives

$$YH(Y)H''(Y) - Y[H'(Y)]^2 + H(Y)H'(Y) = 0. \quad \ldots \quad \ldots \quad (80)$$

Let

$$H(Y) = e^{G(Y)}, \; Y = e^{\tau}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (81)$$

then (80) reduces to

$$\frac{d^2G}{d\tau^2} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

Integrating (82) subject to initial conditions determined by (77), (79) and (81), we get as the solution of (78)

$$\zeta = e^{-\bar{M}(Y-\zeta_0)} Y \left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} = P_0(Y) \quad \text{(say)}. \quad \ldots \quad \ldots \quad (83)$$

Now put

$$\alpha = 1 - B, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (84)$$

so that (28) reduces to

$$Y\zeta\zeta'' - (1+B)Y\zeta'^2 + (1+B)\zeta\zeta' + \bar{M}\zeta^{2+2B} = 0 \quad \ldots \quad \ldots \quad (85)$$

Let the solution of (85) be

$$\zeta = P_0(Y) + BV_1(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (86)$$

Substituting in (85) and neglecting squares and higher powers of B, we get on simplification

$$YP_0V_1'' + V_1'[P_0 - 2P_0'Y] + \frac{V_1}{P_0}[YP_0P_0'' + P_0P_0' + 2\bar{M}P_0^2]$$

$$= YP_0'^2 - P_0P_0' - 2\bar{M}P_0^2 \log P_0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (87a)$$

Since $P_0(Y)$ satisfies (78), the equation simplifies to

$$Y^2V_1'' + V_1'[Y - 2Y(1+\bar{M}\zeta_0 - \bar{M}Y)] + V_1[\bar{M}^2Y^2 + Y(1+\bar{M}\zeta_0 - \bar{M}Y^2)]$$

$$= e^{-\bar{M}(Y-\zeta_0)} \left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} Y\phi(Y), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (87b)$$

where

$$\phi(Y) = \bar{M}^2(Y-\zeta_0)^2 - \bar{M}(Y-\zeta_0) + 2\bar{M}^2Y(Y-\zeta_0)$$

$$-2\bar{M}Y(1+\bar{M}\zeta_0)\log Y + 2\bar{M}^2Y\zeta_0 \log \zeta_0 \quad \ldots \quad \ldots \quad (88)$$

In terms of the operator $D \equiv \frac{d}{dY}$ (87), becomes

$$(YD+\bar{M}Y-1-\bar{M}\zeta_0)(YD+\bar{M}Y-1-\bar{M}\zeta_0)V_1 = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} Y\phi(Y) \quad .. \quad (89)$$

Let

$$(YD+\bar{M}Y-1-\bar{M}\zeta_0)V_1 = U, \quad .. \quad .. \quad .. \quad (90)$$

then since

$$V_1(\zeta_0) = 0, \quad V_1'(\zeta_0) = 0, \quad .. \quad .. \quad .. \quad (91)$$

we get

$$U(\zeta_0) = 0. \quad .. \quad .. \quad .. \quad .. \quad (92)$$

(89) now gives

$$\frac{dU}{dY}+\left(\bar{M}-\frac{1+\bar{M}\zeta_0}{Y}\right)U = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\phi(Y). \quad .. \quad .. \quad (93)$$

Integrating (93) subject to (92)

$$U(Y) = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} Y\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY$$

or

$$\frac{dV_1}{dY}+\left(\bar{M}-\frac{1+\bar{M}\zeta_0}{Y}\right)V_1 = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY. \quad .. \quad (94)$$

Integrating (94) subject to (91), we get

$$V_1(Y) = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} Y\int_{\zeta_0}^{Y}\frac{1}{Y}\,dY\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY. \quad .. \quad (95)$$

Substituting for $\phi(Y)$ from (88) in (94) and (95) and integrating we get

$$U(Y) = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0} Y\left\{\begin{array}{l}(1+\bar{M}\zeta_0)\bar{M}\zeta_0\log\frac{Y}{\zeta_0}-(\bar{M})(2\bar{M}\zeta_0+1)(Y-\zeta_0)\\ +\bar{M}^2(Y-\zeta_0)^2-2\bar{M}(1+\bar{M}\zeta_0)(Y\log Y-Y-\zeta_0\log\zeta_0+\zeta_0)\\ +2\bar{M}^2\zeta_0\log\zeta_0(Y-\zeta_0)\end{array}\right\} \quad (96)$$

and

$$V_1(Y) = P_0(Y)F(Y),$$

where

$$F(Y) = \left\{\begin{array}{l}\frac{1}{2}\bar{M}\zeta_0(1+\bar{M}\zeta_0)[(\log Y)^2-(\log\zeta_0)^2]\\ -(1+\bar{M}\zeta_0)\bar{M}\zeta_0\log\zeta_0\log\frac{Y}{\zeta_0}-\bar{M}(2\bar{M}\zeta_0+1)(Y-\zeta_0)\\ +\bar{M}\zeta_0(2\bar{M}\zeta_0+1)\log\frac{Y}{\zeta_0}+\frac{1}{2}\bar{M}^2(Y^2-\zeta_0^2)\\ -2\bar{M}^2\zeta_0(Y-\zeta_0)+\bar{M}^2\zeta_0^2\log\frac{Y}{\zeta_0}-2\bar{M}(1+\bar{M}\zeta_0)(Y\log Y-\zeta_0\log\zeta_0)\\ +4\bar{M}(1+\bar{M}\zeta_0)(Y-\zeta_0)-2\bar{M}\zeta_0(1+\bar{M}\zeta_0)(1-\log\zeta_0)(Y-\zeta_0)\\ +2\bar{M}^2\zeta_0\log\zeta_0(Y-\zeta_0)-2\bar{M}^2\zeta_0^2\log\zeta_0\log\frac{Y}{\zeta_0}\end{array}\right\} \quad (97)$$

MAXIMUM PRESSURE

If $\alpha = 1$, maximum pressure occurs when

$$Y = \frac{1}{\overline{M}}+\zeta_0+BK \quad \ldots \quad (98)$$

provided $\frac{1}{\overline{M}}+\zeta_0 < 1$; if $\frac{1}{\overline{M}}+\zeta_0 \geqslant 1$, the maximum pressure occurs at all-burnt.

When $\alpha \neq 1$, let maximum pressure occur when

$$Y = \frac{1}{\overline{M}}+\zeta_0+BK, \quad \ldots \quad (99)$$

then

$$P_0'\left[\frac{1}{\overline{M}}+\zeta_0+BK\right]+BV_1'\left[\frac{1}{\overline{M}}+\zeta_0+BK\right] = 0.$$

Neglecting squares and higher powers of B and simplifying we get

$$K = \frac{2}{\overline{M}} - \frac{(1+\overline{M}\zeta_0)(2+\overline{M}\zeta_0)}{\overline{M}}\log\frac{1+\overline{M}\zeta_0}{\overline{M}}+(3+\overline{M}\zeta_0)\zeta_0\log\zeta_0 \quad \ldots \quad (100)$$

Also up to the first power of B, maximum pressure is given by

$$\zeta_{max} = P_0\left[\frac{1}{\overline{M}}+\zeta_0+BK\right]+BV_1\left[\frac{1}{\overline{M}}+\zeta_0+BK\right]$$

$$= P_0\left[\frac{1}{\overline{M}}+\zeta_0\right]+BKP_0'\left[\frac{1}{\overline{M}}+\zeta_0\right]+BV_1\left[\frac{1}{\overline{M}}+\zeta_0\right]$$

$$= \frac{1}{e}\left(\frac{1+\overline{M}\zeta_0}{\overline{M}\zeta_0}\right)^{\overline{M}\zeta_0}\left(\frac{1+\overline{M}\zeta_0}{\overline{M}}\right)+BV_1\left(\frac{1}{\overline{M}}+\zeta_0\right) \quad \ldots \quad (101)$$

since

$$P_0'\left(\frac{1}{\overline{M}}+\zeta_0\right) = 0.$$

In (101), the value of $V_1\left(\frac{1}{\overline{M}}+\zeta_0\right)$ is obtained from (97).

ALL-BURNT POSITION. MOTION AFTER ALL-BURNT. MUZZLE VELOCITY

All-burnt position occurs when $z = 1$ or $Y = 1$.
The pressure in this position is given by

$$\zeta(1) = P_0(1)+BV_1(1) \quad \ldots \quad (102)$$

and the shot-travel up to this instant is given by

$$\xi(1) = \frac{1}{\zeta(1)} = \frac{1}{P_0(1)} - B\frac{V(1)}{[P_0(1)]^2}. \quad \ldots \quad (103)$$

After all-burnt

$$\zeta \xi^{\gamma} = \text{const.} = \zeta(1)[\xi(1)]^{\gamma} = [\zeta(1)]^{1-\gamma}$$

$$= \left[\frac{1}{P_0(1)}\right]^{\gamma-1} + B(\gamma-1)\frac{V_1(1)}{[P_0(1)]^{\gamma}} = \lambda\,[\text{say}] \quad \ldots \quad \ldots (104)$$

The muzzle velocity is given by

$$\eta^2 = \frac{2M}{\gamma-1}[1-\zeta_3\xi_3] = \frac{2M}{\gamma-1}\left[1-\xi_3\frac{\lambda}{\xi_3^{\gamma}}\right]$$

$$= \frac{2M}{\gamma-1}\left[1-\xi_3^{1-\gamma}\left[\frac{1}{(P_0(1))^{\gamma-1}}+\frac{B(\gamma-1)V_1(1)}{[P_0(1)]^{\gamma}}\right]\right] \quad \ldots (105)$$

9. Isothermal Model: Second Method of Solution. Relation Between the Two Solutions

We have to integrate (54) subject to initial conditions (57) when $\alpha = 1$, (54) gives

$$Y\xi\xi'' - Y\xi'^2 + \xi\xi' = -\overline{M}\xi^2 \quad \ldots \quad \ldots \quad \ldots (106)$$

As in section (8), its solution subject to (57) is

$$\xi = e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0} = Q_0(Y) \quad [\text{say}] \quad \ldots \quad \ldots (107)$$

Let the solution of (54), i.e. of

$$Y\xi\xi'' - (1-B)Y\xi'^2 + (1-B)\xi\xi' = \overline{M}Y^{2B}\xi^{2-2B} \quad \ldots \quad \ldots (108)$$

be

$$\xi = Q_0(Y) + BW_1(Y). \quad \ldots \quad \ldots \quad \ldots \quad \ldots (109)$$

Neglecting squares and higher powers of B, we get on simplification

$$Y^2W_1'' + YW_1'[1-2\overline{M}Y+2\overline{M}\zeta_0] + W_1[\overline{M}^2(Y-\zeta_0)^2-\overline{M}Y]$$

$$= -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\phi(Y) \quad \ldots (110)$$

or

$$(YD-\overline{M}Y+\overline{M}\zeta_0)(YD-\overline{M}Y+\overline{M}\zeta_0)W_1 = -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\phi(Y) \quad \ldots (111)$$

Let

$$(YD-\overline{M}Y+\overline{M}\zeta_0)W_1 = R_1(Y),$$

then since

$$W_1(\zeta_0) = 0, \quad W_1'(\zeta_0) = 0, \quad \ldots \quad \ldots \quad \ldots (112)$$

we get

$$R_1(\zeta_0) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (113)$$

(111) gives

$$\frac{dR}{dY} - \overline{M}\left(1-\frac{\zeta_0}{Y}\right)R = -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\frac{\phi(Y)}{Y}. \quad \ldots \quad \ldots (114)$$

Integrating (114) subject to (113)

$$R(Y) = -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY \quad \ldots \quad \ldots (115)$$

or

$$\frac{dW_1}{dY} - \overline{M}\left(1-\frac{\zeta_0}{Y}\right)W_1 = -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\frac{1}{Y}\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY \quad \ldots (116)$$

Integrating (116) subject to (112)

$$W_1(Y) = -e^{\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-\overline{M}\zeta_0}\int_{\zeta_0}^{Y}\frac{1}{Y}\,dY\int_{\zeta_0}^{Y}\frac{\phi(Y)}{Y}\,dY \quad \ldots (117)$$

From (95) and (117)

$$\frac{W_1(Y)}{V_1(Y)} = -e^{2\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{-2\overline{M}\zeta_0}\frac{1}{Y}$$

$$= -\frac{Y}{e^{-2\overline{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{2\overline{M}\zeta_0}Y^2} = -\frac{Y}{[P_0(Y)]^2}. \quad (118)$$

This relation between the two solutions can be deduced alternatively from the definitions.

Since for our model

$$Y = \zeta\xi = [P_0(Y)+BV_1(Y)][Q_0(Y)+BW_1(Y)] \quad \ldots \quad \ldots (119)$$

But from (83) and (107)

$$P_0(Y)Q_0(Y) = Y \quad \ldots \quad \ldots \quad \ldots \quad \ldots (120)$$

$\therefore$ neglecting B^2

$$P_0(Y)W_1(Y)+Q_0(Y)V_1(Y) = 0$$

or

$$W_1(Y) = -\frac{Q_0(Y)}{P_0(Y)}V_1(Y) \quad \ldots \quad \ldots \quad \ldots \quad \ldots (121)$$

(120) shows that (118) and (121) are identical.

The shot-travel up to the instant of maximum pressure is given by, using (99) and (101)

$$\xi = \frac{Y}{\zeta}\,\frac{\frac{1}{\overline{M}}+\zeta_0+BK}{\frac{1}{e}\left(\frac{1+\overline{M}\zeta_0}{\overline{M}\zeta_0}\right)^{\overline{M}\zeta_0}\left(\frac{1+\overline{M}\zeta_0}{\overline{M}}\right)+BV_1\left(\frac{1}{\overline{M}}+\zeta_0\right)}, \quad \ldots \quad \ldots (122)$$

where K is obtained from (100) and $V_1\left(\frac{1}{\overline{M}}+\zeta_0\right)$ is obtained from (97).

10. The Series Solution

We have so far neglected squares and higher powers of B. When B is not very small, we may have to consider coefficients of B^2, B^3, ... in the solution of (85). Let the series solution of (85) be

$$\zeta = P_0(Y)+BV_1(V)+B^2V_2(Y)+B^3V_3(Y)+\ \ldots\ +B^rV_r(Y)+\ldots. \quad (123)$$

Substituting in (85) and equating the coefficients of various powers of B, we get

$$YP_0P_0''-YP_0'+P_0P_0'+\bar{M}P_0^2=0 \quad \ldots \quad (124)$$

$$YP_0V_1''+V_1'[-2VP_0'+P_0]+V_1[YP_0''+P_0'+2\bar{M}P_0]$$
$$=YP_0'^2-P_0P_0'-2\bar{M}P_0^2\log P_0 \quad \ldots \quad (125)$$

$$YP_0V_2''+V_2'[-2YP_0'+P_0]+V_2[YP_0''+P_0'+2\bar{M}P_0]$$
$$=-YV_1V_1''+YV_1'^2+2YP_0'V_1'-V_1V_1'-P_0V_1' \quad \ldots \quad (126)$$
$$-P_0'V_1-\bar{M}[2P_0^2\log P_0+2V_1P_0+2P_0V_1\log P_0+V_1^2]$$

$$YP_0V_3''+V_3'[-2YP_0'+P_0]+V_3[YP_0''+P_0'+2\bar{M}P_0]$$
$$=-YV_1V_2''-YV_2V_1''+2YV_1'P_0'+2YP_0'V_2'+YV_1'-V_1V_2'-Y_2V_1'$$
$$-P_0V_2'-V_1V_1'-V_2P_0'-\bar{M}\left[\frac{4P_0^2}{3}(\log P_0)^3+2P_0V_1\left(2(\log P_0)^2+\frac{2V_1}{P_0}\right)\right.$$
$$\left.+2(P_0V_2+V_1^2)\log P_0+4P_0V_1\log V_1+2V_1V_2\right] \quad \ldots \quad (127)$$

and in general

$$YP_0V_r''+V_r'[-2YP_0'+P_0]+V_r[YP_0''+P_0'+2\bar{M}P_0]$$
$$=\phi[Y, P_0, P_0', V_1, V_2, \ldots, V_{r-1}, V_1, V_2', \ldots, V_{r-1}';$$
$$V_1'', V_2'', \ldots, V_{r-1}''] \quad \ldots \quad (128)$$

Here the function ϕ is homogeneous of the second degree in P_0, P_0', V_i, V_i', V_i'' except for powers of $\log P_0$.

Equations (124) and (125) are respectively the same as equations (78) and (87*a*) and have already been solved giving

$$P_0(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}Y,$$

$$P_0'(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}[1+\bar{M}\zeta_0-\bar{M}Y],$$

$$V_1(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\psi_1(Y),$$

$$V_1'(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\left[\psi_1'(Y)+\frac{\bar{M}\zeta_0\psi_1(Y)}{Y}-\bar{M}\psi_1(Y)\right],$$

$$V_1''(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta^0}\left\{\psi_1''(Y)+\frac{\bar{M}\zeta_0\psi_1'(Y)}{Y}-\frac{\bar{M}\zeta_0\psi_1(Y)}{Y^2}-\bar{M}\psi_1'(Y)\right.$$
$$\left.+\left(\frac{\bar{M}\zeta_0}{Y}-\bar{M}\right)\psi_1'(Y)+\frac{\bar{M}\zeta_0\psi_1(Y)}{Y}-\bar{M}\psi_1(Y)\right\}$$

where $\Psi_1(Y)$ is determined from (97) and contains only integral powers of $\log Y$ and Y. Substituting in (126), we get a differential equation of exactly the same type as (87*a*), which can be integrated in the form

$$V_2(Y)=e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\psi_2(Y) \quad \ldots \quad (129)$$

Similarly

$$V_r(Y) = e^{-\bar{M}(Y-\zeta_0)}\left(\frac{Y}{\zeta_0}\right)^{\bar{M}\zeta_0}\psi_r(Y)[r = 1, 2, \ldots\ldots] \quad \ldots \quad \ldots \quad (130)$$

Now the functions $\psi_r(Y)$ contain only integral powers of Y and log Y and, at any stage, can be obtained as closed expressions not requiring any further quadratures.

The process can be carried to any stage we like, though, in practice, the calculations of terms beyond the third would obviously be tedious and we may prefer to integrate (85) numerically. At the same time, we may remember that when B is small the convergence of the series solution is expected to be quite rapid and very often even the first and the second terms may give satisfactory results.

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References

Clemmow, C. A. (1928). A theory of internal ballistics based on a pressure-index law of burning for propellents. *Phil. Trans. Roy. Soc.*, **227A**, 345.

———(1951). Chapter X. 'Internal Ballistics'. H.M.S.O., London.

Corner, J. (1950). Theory of Internal Ballistics of Guns. John Wiley and Sons, New York.

Crow, A. D. (1922). A Short Method of Calculation of Internal Ballistics and its Application to Gun Design. H.M.S.O., London.

Coppock, S. W. (1942). The method of internal ballistics calculation used in Research Department. *Arm. Res. Deptt. Bal. Rep.*, **82**, 42.

Goldie, A. W. (1945). A system of internal ballistics. *Arm. Res. Deptt. Bal. Rep.*, **10**, 45.

Hunt, F. R. W., and Hinds, G. H. (1929). A System of Internal Ballistics. H.C.S. G.M. II. (revised in 1945 by C. J. Tranter).

Sugot, G. (1928). Balistique Interreure. Paris.

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