LAGRANGE'S BALLISTIC PROBLEM FOR UNORTHODOX [H/L, R.C.L.] GUNS AND SOLID-FUEL ROCKETS

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ABSTRACT

In the present paper, the solutions of the Lagrange's ballistic problem for the H/L gun, the R.C.L. gun and the solid-fuel rocket have been obtained. The solutions are of the same accuracy as that of the conventional solution for the orthodox gun.

1. INTRODUCTION

The first and most important of the hydrodynamic problems of Internal Ballistics is to find the distribution of pressure, density and gas velocity between the breech and the base of the shot at all times during the firing. Lagrange (1793) first studied the problem and introduced a remarkably useful approximation, in that he assumed that the velocity of the gas at any instant increases linearly with distance along the bore, from zero at the breech to the full shot velocity at the base of the shot. This approximation can be deduced from the assumption that at any instant the gas density is independent of x, i.e. $\frac{\partial \rho}{\partial x} = 0$, though the converse is not necessarily true.

Further contributions to the problem were made by Reimann and Hugoniot (1889), Gossot and Liouville (1914), and Love and Pidduck (1922). In all these theories, the following assumptions were made:

(i) across any section normal to the axis of the bore, the conditions are uniform, i.e. departure from one-dimensional theory can be neglected;
(ii) the effect of skin friction can be neglected;
(iii) the effect of heat conductivity can be neglected;
(iv) the recoil of the barrel can be neglected;
(v) the surface of the bore is hydrodynamically smooth;
(vi) the chamber, of rather greater diameter than the bore, can be replaced by a cylindrical continuation of the bore with the correct total volume;
(vii) the propellant charge is in gaseous form.

Corner (1950) has given an excellent discussion of all these theories, and has obtained rough estimates for the errors arising due to the assumptions (ii), (iii), (iv), (v) and (vii). Winter (1939) has described qualitatively the effect of (vi).

Another line of attack on the hydrodynamical problems of Internal Ballistics is through the method of characteristics. Work on this line has been done by de Haller (1946, 1948), who neglected the variation of entropy with position at a given time, and later without this approximation by Carrier (1948). An account of the method which is easily adaptable for Internal Ballistics work has been recently given by Rudinger (1955).

The only disadvantage of the method as compared with the analytical methods developed earlier is that each problem has to be worked out separately, as the method is essentially numerical and/or graphical. In spite of this disadvantage, it is desirable to examine fully the consequences of this method, since the agreement between the results of the theory as given by the analytical methods and the experimental results (Rossman, 1940, Goode and Lockett, 1944, N.D.R.C. Reports A 229, 323, O.S.R.D. Reports Nos. 2019 and 4899) is not quite satisfactory.

Lagrange's problem has been so far studied mainly with reference to the orthodox gun, and the solution for this gun has been used either with some empirical adjustments or even without them in the unorthodox guns like the Recoil-less guns (Corner, 1947, 1950) and the High-Low pressure guns (Corner, 1948, 1950). In the more elementary theories of solid-fuel rockets, the pressure-drop is sometimes altogether neglected (Crawford, 1944).

In the present paper, we have examined the consequences of Lagrange's assumption for the unorthodox guns and for solid-fuel rockets. The level of accuracy is the same as that of the 'conventional' solution of the Lagrange problem for the orthodox gun. As in that case the solutions here suggested would strictly apply to the stage, where all the charge has been converted into gas. Alternatively the solutions suggested here hold on the assumption that the unburnt charge moves with the gas, the distribution of solid along the bore being the same as the distribution of the gas.

2. Pressure-Distribution in the Second Chamber of a H/L Pressure Gun

For an orthodox gun, the velocity of the gas at breech is zero, but in a H/L gun, at the throat of the nozzle, the gas velocity is equal to the local velocity of sound, and the gas, enters the second chamber with a finite velocity. We examine here how the solution of Lagrange's problem is affected by this fact.

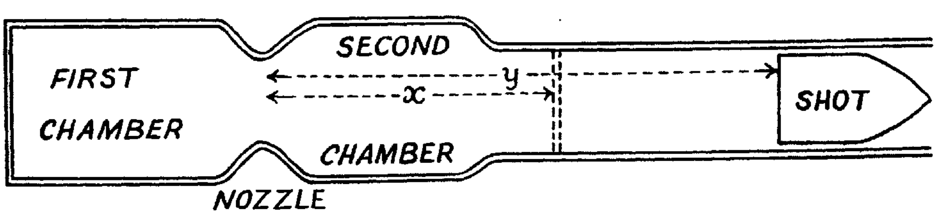

Fig. 1. A H/L gun.

Let the temperature in the first chamber be T_o. (This will be constant, since we make, following Corner (1948, 1950), the isothermal assumption about the first chamber.)

The velocity at the throat is given by

$$v_t^2 = \frac{2\gamma RT_0}{\gamma+1} = v_0^2 \text{ (say)}. \qquad (1)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \qquad (2)$$

which gives, on account of Lagrange's approximation,

$$\frac{\partial \rho}{\partial x} = 0, \qquad (3)$$

the equation

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial v}{\partial x}=0$$

or

$$\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial \rho}{\partial t}. \qquad (4)$$

The R.H.S. is independent of x.

$$\therefore\quad v=k_1x+k_2, \qquad (5)$$

where k_1, k_2 are independent of x, but can be functions of t.

Now

$$v=v_0 \quad \text{at} \quad x=0,$$

$$v=\frac{dy}{dt} \quad \text{at} \quad x=y.$$

Therefore (5) gives

$$v=\frac{x}{y}\left(\frac{dy}{dt}-v_0\right)+v_0. \qquad (6)$$

To find the pressure-distribution, we use the equation of motion

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad (7)$$

but from (6)

$$\frac{\partial v}{\partial x}=\frac{1}{y}\left(\frac{dy}{dt}-v_0\right)$$

$$\frac{\partial v}{\partial t}=\frac{x}{y}\frac{d^2y}{dt^2}-\frac{x}{y^2}\frac{dy}{dt}\left(\frac{dy}{dt}-v_0\right).$$

Therefore (7) gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{x}{y}\frac{d^2y}{dt^2}-v_0\frac{x}{y^2}\frac{dy}{dt}+\frac{v_0}{y}\frac{dy}{dt}+v_0^2\frac{x}{y^2}-\frac{v_0^2}{y}. \qquad (8)$$

The equation of continuity for the first chamber gives

$$\frac{dN}{dt}=\frac{dz}{dt}-\frac{\psi SP}{C\sqrt{\lambda}}, \qquad (9)$$

where z is the fraction of the charge mass C burnt, N is the fraction of the charge mass remaining in the first chamber, S is the throat area, P is the pressure in the first chamber, λ is the adjusted value of the force constant and

$$\psi=\gamma^{\frac{1}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (10)$$

and it is assumed that at any instant,

$$\frac{p}{P}\leqslant\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}. \qquad (11)$$

The equation for the rate of burning gives

$$D\frac{df}{dt}=-\beta P. \qquad (12)$$

From (9) and (12)

$$\frac{dN}{dt} = \frac{dz}{dt} + \Psi \frac{df}{dt}, \quad \quad (13)$$

where

$$\Psi = \frac{\psi S}{C\sqrt{\lambda}} \frac{D}{\beta}, \quad \quad (14)$$

is the dimensionless leakage parameter.

Integrating (13) and remembering that initially

$$f = 1, z = 0, N = 0,$$

we get

$$z - N = \Psi(1-f). \quad \quad (15)$$

But $C(z-N)$ denotes at any instant, the mass in the second chamber.

$$\therefore \quad \rho A y = C\Psi(1-f). \quad \quad (16)$$

From (8)

$$\frac{\partial p}{\partial x} = -\rho[\mu x + \nu], \quad \quad (17)$$

where

$$\mu = \frac{1}{y^2}\left[y\frac{d^2y}{dt^2} - v_0\frac{dy}{dt} + v_0^2\right] \quad \quad (18a)$$

$$\nu = \frac{v_0}{y}\frac{dy}{dt} - \frac{v_0^2}{y} \quad \quad (18b)$$

$$\rho = \frac{C\Psi(1-f)}{Ay} \quad \quad (18c)$$

so that μ, ν, ρ are functions of t only.

Integrating (17)

$$p = g(t) - \rho\left[\frac{\mu x^2}{2} + \nu x\right]. \quad \quad (19)$$

Let p_B denote the pressure at the entrance of the second chamber and let p_S denote the pressure at the base of the shot. Then

$$p = p_B - \rho[\tfrac{1}{2}\mu x^2 + \nu x] \quad \quad (20)$$

The last equation shows that the pressure-distribution, at any instant, in the second chamber of a H/L gun is parabolic.

Also

$$p_S = p_B - \frac{C\Psi}{A}(1-f)[\tfrac{1}{2}\mu y + \nu]. \quad \quad (21)$$

From (20), the average pressure $\bar{p}$ in the second chamber is given by

$$\bar{p} = p_B - \frac{C\Psi(1-f)}{A}[\tfrac{1}{6}\mu y + \tfrac{1}{2}\nu]. \quad \quad (22)$$

From (18a) and (18b)

$$\mu y = \frac{d^2y}{dt^2} - \nu. \quad \quad (23)$$

Also from the equation of motion of the shot

$$w_1 \frac{d^2y}{dt^2} = Ap_S, \qquad (24)$$

where w_1 is the effective mass of the shot.

From (23) and (24)

$$\mu w_1 y = Ap_S - \nu w_1. \qquad (25)$$

From (22) and (25)

$$\bar{p} = p_B - \frac{C\Psi(1-f)}{A}\left[\frac{Ap_S}{6w_1}+\frac{\nu}{3}\right]. \qquad (26)$$

From (21) and (25)

$$p_S = p_B - \frac{C\Psi(1-f)}{A}\left[\frac{Ap_S}{2w_1}+\frac{\nu}{2}\right]. \qquad (27)$$

Eliminating ν

$$3\bar{p} = p_B + p_S\left[2+\frac{C\Psi(1-f)}{2w_1}\right]. \qquad (28)$$

If, as in conventional solution, we take the mass of the gas to be the mass at all-burnt, (28) reduces to

$$3\bar{p} = p_B + p_S\left[2+\frac{C\Psi}{2w_1}\right]. \qquad (29)$$

Now for orthodox guns,

$$\bar{p} = p_S\left[1+\frac{C}{3w_1}\right], \; p_B = p_S\left[1+\frac{C}{2w_1}\right]$$

giving

$$3\bar{p} = p_B + p_S\left[2+\tfrac{1}{2}\frac{C}{w_1}\right]. \qquad (30)$$

Thus the relation (29) for the second chamber of the high-low pressure gun is the same as for the orthodox gun, only C has to be replaced by $C\Psi$. This is, of course, consistent with the fact that mass of the gas at burnt in the second chamber is $C\Psi$ as against a gas mass C in the orthodox gun.

It may also be noted that if the nozzle ends just at the throat,

$$p_B = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} P_1 \qquad (31)$$

where P_1 is the pressure in the first chamber.

By putting $v_0 = 0$ or $\gamma = 0$, we deduce the corresponding results for the orthodox gun.

3. Pressure Distribution in a R.C.L. Gun

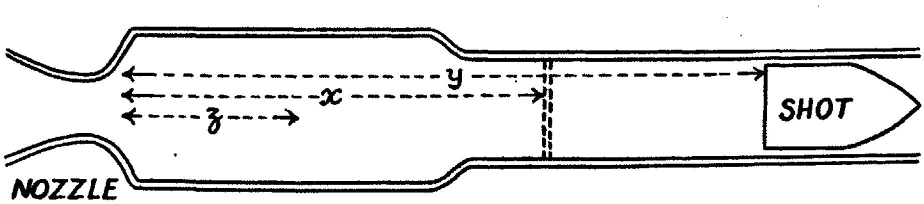

Fig. 2. A R.C.L. gun.

In this case, the gas moves both forwards to push the shot forward and backwards to make its way outside the nozzle. The stagnation point (with zero gas velocity) is somewhere in between and may go on changing its position. At time, t, let it be at a distance z from the beginning of the chamber.

We assume that, at any instant, P is constant inside the gun; then from an analysis of the gas equations, we get, as for (5)

$$v = k_1' x + k_2'. \qquad \ldots \qquad (32)$$

Now when

$$x = y, \quad v = \frac{dy}{dt};$$

$$x = z, \quad v = 0.$$

$$\therefore \quad v = \frac{x-z}{y-z}\frac{dy}{dt}. \qquad \ldots \qquad (33)$$

Therefore from (6)

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{x-z}{y-z}\frac{d^2y}{dt^2} + \frac{x-y}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}$$

$$\therefore \quad p = -\rho\left\{\frac{1}{y-z}\frac{d^2y}{dt^2} + \frac{1}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right\}\frac{x^2}{2}$$

$$+\rho\left\{\frac{z}{y-z}\frac{d^2y}{dt^2} + \frac{y}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right\} + \phi(t). \qquad \ldots \qquad (34)$$

Thus at any instant, the distribution of pressure inside the gun is parabolic.

From (34)

$$p_B = \phi(t).$$

$$\bar{p} = -\rho\left[\frac{1}{y-z}\frac{d^2y}{dt^2} + \frac{1}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right]\frac{y^2}{6}$$

$$+\rho\left[\frac{z}{y-z}\frac{d^2y}{dt^2} + \frac{y}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right]\frac{y}{z} + p_B \qquad \ldots \qquad (35)$$

and

$$p_S = -\rho\left[\frac{1}{y-z}\frac{d^2y}{dt^2} + \frac{1}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right]\frac{y^2}{6}$$

$$+\rho y\left[\frac{z}{y-z}\frac{d^2y}{dt^2} + \frac{y}{(y-z)^2}\frac{dy}{dt}\frac{dz}{dt}\right] + p_B \qquad \ldots \qquad (36)$$

Eliminating $\frac{dy}{dt}\frac{dz}{dt}$ from (35) and (36)

$$3\bar{p} - 2p_S - p_B = \frac{1}{2}\rho y\frac{d^2y}{dt^2}.$$

But

$$w_1\frac{d^2y}{dt^2} = Ap_S \qquad \ldots \qquad (37)$$

and

$$\rho Ay = CN$$

$$3\bar{p}-2p_S-p_B = \tfrac{1}{2}CN\frac{p_S}{w_1} \quad \ldots \quad (38)$$

or

$$3\bar{p} = p_B+p_S\left[2+\tfrac{1}{2}\frac{CN}{w_1}\right]. \quad \ldots \quad (39)$$

Equation (39) is similar to (28) for H/L gun and to (30) for orthodox guns.

4. Pressure-Distribution in a Rocket Motor

From the equation of continuity

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v) = 0$$

and the assumption

$$\frac{\partial\rho}{\partial x} = 0$$

we get

$$v = k_1'' x+k_2''$$

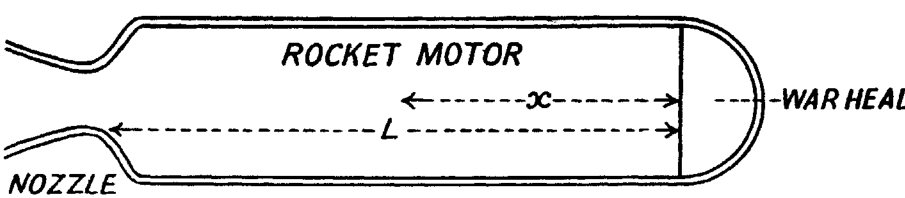

Fig. 3. A rocket.

when $x = 0$, $v = 0$

$$v = k_1'' x \quad \ldots \quad (40)$$

where k_1'' is a function of t only.

From the equation of motion

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}$$

$$= \left(k_1''^2+\frac{dk_1''}{dt}\right)x.$$

Again

$$\rho AL = CN$$

$$\therefore \quad p = \phi(t)-\frac{CN}{2AL}\left(k_1''^2+\frac{dk_1''}{dt}\right)x^2, \quad \ldots \quad (41)$$

where $\phi(t)$, N, k_1'' are functions of t only.

Thus the distribution of pressure inside a rocket motor is also parabolic.

$$\therefore \quad p_0 = \phi(t) \quad \ldots \quad (42)$$

$$p_L = \phi(t) - \frac{CN}{2AL}\left(k_1''^2 + \frac{dk_1''}{dt}\right)L^2 \quad \ldots \quad (43)$$

Also

$$\bar{p} = \phi(t) - \frac{CN}{6AL}\left(k_1''^2 + \frac{dk_1''}{dt}\right)L^2. \quad \ldots \quad (44)$$

From (42), (43) and (44)

$$\frac{p_L - p_0}{\bar{p} - p_0} = 3 \quad \ldots \quad (45)$$

or

$$3\bar{p} = p_L + 2p_0. \quad \ldots \quad (45)$$

Also from (42), (43)

$$\frac{p_0 - p_L}{p_0} = \frac{CNL\left(k_1''^2 + \frac{dk_1''}{dt}\right)}{2A\phi(t)}. \quad \ldots \quad (46)$$

Now it is known (Wimpress (1950), page 63) that for small values of $\frac{A_t}{A_p}$

$$\frac{p_0 - p_L}{p_0} = 2\phi\left(\frac{A_t}{A_p}\right)^2, \quad \ldots \quad (47)$$

where

$$p = \gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} = \psi^2 \quad \ldots \quad (48)$$

and A_t, A_p denote the throat and port areas respectively.

From (41), (42), (46) and (47)

$$p = p_0\left[1 - 2\phi\left(\frac{At}{Ap}\right)^2 \frac{x^2}{h^2}\right] \quad \ldots \quad (49)$$

and

$$\bar{p} = p_0\left[1 - \tfrac{2}{3}\phi\left(\frac{At}{Ap}\right)^2\right] \quad \ldots \quad (50)$$

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