DENSITY FLUCTUATIONS IN TURBULENCE IN AN INVISCID COMPRESSIBLE FLUID

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ABSTRACT

The new theory of turbulence as presented by S. Chandrasekhar (1955) has been applied to the problem of density fluctuations in stationary homogeneous turbulence in an inviscid compressible fluid. On the basis of the assumptions that the fourth order correlation is related to the second order correlations in the same manner as in a joint Gaussian distribution, and that the variations in density and pressure are adiabatic, a differential equation in the density correlation is obtained and solved. It is found that each scale of the density fluctuation varies periodically with time independently of the others and is propagated through the medium with velocity $\sqrt{C^2+\frac{1}{3}\overline{u^2}}$. An invariant of the type of Loitsiansky Invariant is also deduced from the equation of continuity.

1. INTRODUCTION

Recently S. Chandrasekhar (1955) deduced a new theory of turbulence by taking the velocity fluctuations at two points and at two different times. On the basis of this theory he deduced an equation purely in terms of the defining scalar of the second order velocity correlation tensor and obtained its solution in the limiting case of zero viscosity (Chandrasekhar, 1956).

In the present paper this new theory has been applied to the fluctuations of density in stationary homogeneous turbulence in an inviscid compressible fluid. In the first section, an invariant, similar to that of Loitsiansky, has been derived from the equation of continuity. Incidentally, this invariant comes out to be exactly the same as the one obtained on the basis of the old theory of turbulence (Chandrasekhar, 1951).

In the next section, an equation in the correlation

$$\varpi\left[=\overline{(\rho'-\bar{\rho})(\rho''-\bar{\rho})}=\overline{\delta\rho'\delta\rho''}\right]$$

is obtained with the help of the equation of continuity and Stokes-Navier equation. In deducing this equation, use has been made of the following two assumptions :—

(1) The correlation of the fourth order is related to the correlation of the second order in the same manner as in a joint Gaussian distribution.
(2) The adiabatic law governs the fluctuations in density and pressure.

In another paper, S. Chandrasekhar (1951) has dealt with the problem of fluctuations of density in homogeneous turbulence in the compressible fluid on the basis of the usual method of taking correlations at two different points and at the same instant of time. There the fluid is taken to be viscous and approximations are used to justify the use of the second assumption mentioned above and to reduce the basic equation in ϖ to a simplified form.

Since the assumption of the adiabatic law is inconsistent with the retention of viscous terms in the equations of motion, we have solved the problem for the more restricted case of inviscid compressible fluid and the results are obtained in a rigorous manner. In the end, in order to compare the results of the present paper with that of S. Chandrasekhar (1951), the basic equation in ϖ is simplified to the spherical wave equation by making use of the approximation that the turbulence is fairly subsonic. The velocity of the propagation of the wave is found to be C, while on the basis of the old theory the corresponding value is $\sqrt{2}C$.

2. An Invariant for Physical Consideration

For a compressible fluid, the equation of continuity is

$$\frac{\partial \rho'}{\partial t'}+\frac{\partial}{\partial x_i'}(\rho' u_i') = 0 \quad \ldots \quad (1)$$

where the primed symbols have their values at a point x_i' and at time t'.

Multiplying equation (1) by ρ'' which is the value of the density at the point x_j'' and at time t'', and averaging, we get

$$\frac{\partial \overline{\rho'\rho''}}{\partial t'}+\frac{\partial}{\partial x_i'}(\overline{\rho''\rho' u_i'}) = 0 \quad \ldots \quad (2)$$

Let $t = t'-t''$.

Therefore, for homogeneous turbulence,

$$\left.\begin{aligned}\frac{\partial}{\partial t'} &= +\frac{\partial}{\partial t} \quad \text{if} \quad t' > t'' \\ &= -\frac{\partial}{\partial t} \quad \text{if} \quad t'' > t'\end{aligned}\right\} \quad \ldots \quad (3)$$

Let $\xi_i = x_i''-x_i'$.

Then $\dfrac{\partial}{\partial x_i'} = -\dfrac{\partial}{\partial \xi_i}$.

Also $\nabla^2 = \dfrac{\partial^2}{\partial x_1'^2}+\dfrac{\partial^2}{\partial x_2'^2}+\dfrac{\partial^2}{\partial x_3'^2} = \dfrac{\partial^2}{\partial \xi_i \partial \xi_i}$ $\quad \ldots \quad (4)$

Therefore, equation (2) transforms to

$$\pm\frac{\partial \overline{\rho'\rho''}}{\partial t} = \frac{\partial}{\partial \xi_i}(\overline{\rho''\rho' u_i'}) \quad \ldots \quad (5)$$

Let

$$\left.\begin{aligned}\varpi &= \overline{(\rho'-\bar\rho)(\rho''-\bar\rho)} \\ &= \overline{\delta\rho'\delta\rho''}\end{aligned}\right\}, \quad \ldots \quad (6)$$

where $\bar\rho$ is the mean density which is independent of r and t for homogeneous turbulence. Then,

$$\varpi = \overline{\rho'\rho''}-\bar\rho^2 \quad \ldots \quad (7)$$

and ϖ is a function of the distance r and the relative time t.

Now, for isotropic turbulence

$$\overline{\rho''\rho' u_i'} = L(r, t)\xi_i \quad \ldots \quad (8)$$

where $L(r, t)$ is the defining scalar of the first order isotropic tensor on the left.

Putting the values in equation (5), we obtain

$$\pm\frac{\partial}{\partial t}[\varpi(r, t)+\bar\rho^2] = \frac{\partial}{\partial \xi_i}[L\xi_i]$$

or

$$\pm \frac{\partial \varpi}{\partial t} = \frac{\partial L}{\partial r} r + 3L. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Multiplying both sides of the last equation by r^2, we get

$$\pm \frac{\partial (r^2 \varpi)}{\partial t} = \frac{\partial}{\partial r} (Lr^3). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Integrating both sides with respect to r between the limits o and r, we have

$$\pm \frac{\partial}{\partial t} \int_0^r r^2 \varpi \, dr = Lr^3. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

In order to deduce an invariant of the type of the Loitsiansky Invariant, we suppose that (Chandrasekhar, 1951) $L \to 0$ faster than r^{-3} as $r \to \infty$. Hence, we get

$$\pm \frac{d}{dt} \int_0^\infty r^2 \varpi \, dr = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

or

$$\int_0^\infty r^2 \varpi(r, t) dr = \text{constant}. \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

The physical meaning of this invariant is that the large scale components of the density fluctuations are permanent features of the system.

We can also deduce the same invariant in Fourier space; the method to be followed is exactly the same as given by Chandrasekhar (1951).

2. The Equation in $\varpi(r, t)$

The equation of continuity is given by equation (1), that is

$$\frac{\partial \rho'}{\partial t'} + \frac{\partial}{\partial x_i'} (\rho' u_i') = 0$$

and the Stokes-Navier equation for an inviscid compressible fluid is

$$\frac{\partial}{\partial t'} (\rho' u_i') + \frac{\partial}{\partial x_j'} (\rho' u_i' u_j') = - \frac{\partial p'}{\partial x_i'} \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

Differentiating the two equations and eliminating the term containing $\rho' u_i'$, we obtain

$$\frac{\partial^2 \rho'}{\partial t'^2} = \frac{\partial^2}{\partial x_i' \partial x_j'} (\rho' u_i' u_j') + \frac{\partial p'}{\partial x_i' \partial x_i'}$$

or

$$\frac{\partial^2 \rho'}{\partial t'^2} = \frac{\partial^2}{\partial x_i' \partial x_j'} (\rho' u_i' u_j') + \nabla^2 p' \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

Multiplying this equation by ρ'' which is the value of the density at the point x_j'' and at the time t'', and averaging, we get

$$\frac{\partial^2 \overline{\rho' \rho''}}{\partial t'^2} = \frac{\partial^2}{\partial x_i' \partial x_j'} (\overline{\rho'' \rho' u_i' u_j'}) + \nabla^2 (\overline{\rho'' p'})$$

or
$$\frac{\partial^2 \varpi}{\partial t^2} = \frac{\partial^2}{\partial \xi_i \partial \xi_j} (\overline{\rho'' \rho' u_i' u_j'}) + \nabla_\xi^2 (\overline{\rho'' p'}). \quad \text{.. .. ..} \quad (16)$$

Now we make the assumption that the fourth order correlation is related to the second order correlations in the same manner as in a joint Gaussian distribution (Chandrasekhar, 1951).
Therefore,

$$\begin{aligned}\overline{\rho'' \rho' u_i' u_j'} &= \overline{\rho' \rho''}\ \overline{u_i' u_j'} + \overline{\rho'' u_i'}\ \overline{\rho' u_j'} \\ &\quad + \overline{\rho'' u_j'}\ \overline{\rho' u_i'} \\ &= \tfrac{1}{3} \overline{\rho' \rho''}\ \overline{u^2}\ \delta_{ij} \quad \text{.. .. .. ..} \quad (17)\end{aligned}$$

From equations (16) and (17), we get

$$\frac{\partial^2 \varpi}{\partial t^2} = \nabla_\xi^2 (\overline{\rho'' p'}) + \frac{\partial^2}{\partial \xi_i \partial \xi_j} (\tfrac{1}{3} \overline{\rho' \rho''}\ \overline{u^2}\ \delta_{ij}) \quad \text{.. .. ..} \quad (18)$$

or
$$\frac{\partial^2 \varpi}{\partial t^2} = \nabla_\xi^2 [\overline{\rho'' p'} + \tfrac{1}{3} \overline{u^2}\ \varpi]. \quad \text{.. .. ..} \quad (19)$$

In order to solve this equation for ϖ and, as we are considering an inviscid fluid, we make the second assumption that there are adiabatic variations in pressure and density at every point in the turbulent motion of the fluid, that is

$$\frac{\delta p'}{\overline{p'}} = \gamma \frac{\delta \rho'}{\overline{\rho'}} \quad \text{.. .. .. ..} \quad (20)$$

where γ is the ratio of the specific heats.

Replacing p' (the pressure at x_i') by $\delta p'$ (the fluctuations in p') in $\overline{\rho'' p'}$, we obtain

$$\begin{aligned}\overline{\rho'' p'} &= \overline{\rho'' \delta p'} \\ &= \gamma \frac{\overline{p'}}{\overline{\rho'}} \overline{\rho'' \delta \rho'} \\ &= C^2\ \overline{\delta \rho' (\bar{\rho} + \delta \rho'')} \\ &= C^2\ \varpi \quad \text{.. .. ..} \quad (21)\end{aligned}$$

where C is the velocity of sound appropriate for the mean pressure and the mean density.

Using equation (21), we get from equation (19)

$$\frac{\partial^2 \varpi}{\partial t^2} = \nabla_\xi^2 [C^2 \varpi + \tfrac{1}{3} \overline{u^2}\ \varpi] \quad \text{.. .. .. ..} \quad (22)$$

which is the basic equation in $\varpi(r, t)$. A similar equation for an inviscid fluid on the basis of the old theory (equation 4.37, Chandrasekhar, 1951) will be

$$\frac{\partial^2 \varpi}{\partial t^2} = 2(C^2 + \tfrac{1}{3} \overline{u^2}) \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varpi}{\partial r} \right) - 2 \frac{\partial^2}{\partial \xi_i \partial \xi_j} [\overline{\rho \rho'} \delta_{ij} - X^2 \xi_i \xi_j] \quad \text{..} \quad (23)$$

Our equation (22) is much simplified in comparison to the last equation (23).

But
$$\nabla_\xi^2 \varpi = \frac{\partial^2 \varpi}{\partial r^2} + \frac{2}{r} \frac{\partial \varpi}{\partial r} \quad \text{.. .. .. ..} \quad (24)$$

Therefore, equation (22) gives

$$\frac{\partial^2 \varpi}{\partial t^2} = (C^2+\tfrac{1}{3}\overline{u^2})\left(\frac{\partial^2 \varpi}{\partial r^2}+\frac{2}{r}\frac{\partial \varpi}{\partial r}\right) \quad \ldots \quad (25)$$

The solution of this equation is given by

$$\left.\begin{aligned} \varpi(r, t) &= \frac{A}{r}\, e^{i\sigma\left(t+\frac{r}{k}\right)} \\ \text{where} \quad k^2 &= C^2+\tfrac{1}{3}\overline{u^2} \end{aligned}\right\} \quad \ldots \quad (26)$$

Therefore, each scale of the density fluctuations varies periodically with time independently of the others and is propagated through the medium with velocity $\sqrt{C^2+\frac{1}{3}\overline{u^2}}$. (It may be pointed out that the combined occurrence of the terms C^2 and $(\frac{1}{3})\overline{u^2}$ in the solution of the present problem on the basis of old as well as new theory of turbulence may have some significant meaning.)

The result thus obtained does not depend on the subsonic or supersonic character of the turbulence and it gives the solution of the problem with only two restrictions in the form of two assumptions.

However, if the turbulence be subsonic sufficiently,

$$\overline{u^2} \ll C^2$$

Therefore, equation (23) reduces to

$$\frac{\partial^2 \varpi}{\partial t^2} = C^2\left(\frac{\partial^2 \varpi}{\partial r^2}+\frac{2}{r}\frac{\partial \varpi}{\partial r}\right) \quad \ldots \quad (27)$$

which is the spherical wave equation in ϖ. Thus, the density correlation ϖ may be expressed in terms of spherical waves of the form

$$\psi(r, t) = \frac{B}{r}\, e^{i\sigma\left(t+\frac{r}{C}\right)} \quad \ldots \quad (28)$$

The speed of propagation of these waves is C. The corresponding result obtained by S. Chandrasekhar (1951) on a less rigorous basis is $\sqrt{2}C$.

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