ON THE ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES, ITS CONJUGATE SERIES AND THEIR DERIVED SERIES

by S. R. SINHA, *Mathematics Department, University of Allahabad*

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1.1. **Definitions.** Let Σa_n be a given infinite series, and let λ_n be a positive monotonic increasing function of n, steadily tending to infinity with n. We write

$$A_{\lambda}(w) = A_{\lambda}^{0}(w) = \sum_{\lambda_n \leqslant w} a_n,$$

$$A_{\lambda}^{r}(w) = \sum_{\lambda_n \leqslant w} (w-\lambda_n)^r a_n, \ (r \geqslant 0).$$

The series Σa_n is said to be absolutely summable (R, λ, r) or summable $|R, \lambda, r|$, $r > 0$, if $\left(A_{\lambda}^{r}(w)/w^r\right)$ is of bounded variation in (A, ∞), where A is a finite positive number (Obrechkoff, 1928, 1929).

An equivalent definition is obtained, as follows, by defining λ suitably at non-integral points and by a corresponding change of variable.

Let $\lambda = \lambda(w)$ be a continuous, differentiable and monotonic increasing function of w in (K, ∞), where K is a positive constant and let $\lambda(w) \to \infty$ with w. We write

$$c_r(w) = \sum_{n \leqslant w} \{\lambda(w)-\lambda(n)\}^r a_n, \ (r \geqslant 0).$$

Then Σa_n is said to be summable $| R, \lambda, r |$, $r > 0$, if the integral

$$\int_A^{\infty} | d[c_r(w)/\lambda(w)]^r |,$$

where A is a finite positive number, is convergent. Now for $r > 0$, $m<w<m+1$,

$$(d/dw)[c_r(w)/(\lambda(w))^r]$$
$$= [r\lambda'(w)/\{\lambda(w)\}^{r+1}] \sum_{n \leqslant w} \{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)a_n.$$

Hence summability $| R, \lambda, r |$, $r > 0$, is equivalent to the convergence of the integral

$$\int_A^{\infty} \left| \left[r\lambda'(w)/\{\lambda(w)\}^{r+1}\right] \sum_{n \leqslant w} \{\lambda(w)-\lambda(n)\}^{r-1} \lambda(n)a_n \right| dw.$$

Evidently summability $| R, \lambda, 0 |$ is equivalent to absolute convergence. For convenience we shall adopt here the alternative definition given above.

1.2. Let $f(t)$ be a periodic function with period 2π and integrable-(L) over $(-\pi, \pi)$. Without any loss of generality we assume the constant term in the Fourier series of $f(t)$ to be zero, so that the Fourier series of $f(t)$ is given by

$$\sum_{1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{1}^{\infty} A_n(t), \tag{1.2.1}$$

and

$$\int_{-\pi}^{\pi} f(t)\,dt = 0. \tag{1.2.2}$$

Then the conjugate series of the Fourier series is given by

$$\sum_{1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} B_n(t). \tag{1.2.3}$$

Throughout we shall use the following notations:—

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\},$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\},$$

$$P(t) = \sum_{i=0}^{r-1}(\theta_i t^i/i\,!),$$

where θ's are arbitrary constants,

$$g(t) = \tfrac{1}{2}[\{f(x+t)-P(t)\}+(-1)^r\{f(x-t)-P(-t)\}],$$

$$h(t) = \tfrac{1}{2}[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}],$$

$$\Phi_\sigma(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-u)^{\sigma-1}\phi(u)\,du,\ \sigma > 0,$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_\sigma(t) = \Gamma(\sigma+1)t^{-\sigma}\Phi_\sigma(t),\ \sigma > 0,$$

$\Psi_\sigma(t)$, $\psi_\sigma(t)$, $G_\sigma(t)$, $g_\sigma(t)$, $H_\sigma(t)$, $h_\sigma(t)$ have similar meanings

$$\gamma_{\alpha, r}(t) = g_{\alpha-r}(t)/t^r;\ \theta_{\alpha, r}(t) = h_{\alpha-r}(t)/t^r,$$

$$e(w) = \exp\{(\log w)^{1+\beta/\alpha}\},$$

$$E(w, t) = \sum_{n \leqslant w}\{e(w)-e(n)\}^{\alpha-1+\delta}.\,e(n)\cos nt,$$

$$E^r(w, t) = (\partial/\partial t)^r E(w, t),$$

$$\bar{E}(w, t) = \sum_{n \leqslant w}\{e(w)-e(n)\}^{\alpha-1+\delta}.\,e(n)\sin nt,$$

$$\bar{E}^r(w, t) = (\partial/\partial t)^r \bar{E}(w, t),$$

$$g(w, t) = \int_0^t [u^\alpha/\{\log(k/u)\}^\beta].\,E^\alpha(w, u)\,du,$$

$$h(w, t) = \int_t^\pi [u^\alpha/\{\log (k/u)\}^\beta] \,.\, E^\alpha(w, u)\, du,$$

$$\{F(t)\}_r = \{(\partial/\partial t)^r F(t)\}.$$

1.3. **Introduction.** The problem of the absolute Riesz summability of the rapidly increasing type $\exp\{(\log w)^{1+1/\alpha}\}$, $\alpha \geqslant 1$, of a Fourier series, its conjugate series and their derived series has been recently treated by Pati in his two papers (1954, 1957). He considers the case when α is an integer $\geqslant 1$, and establishes the following theorem (Pati, 1957, Theorem 1) for the Fourier series, in which the order is more precise than in his previous result (Pati, 1954, Theorem 1):

THEOREM A. If α is an integer $\geqslant 1$ and $\phi_\alpha(t) \log (k/t) \in BV(0, \pi)$, then the Fourier series of $f(t)$, at $t = x$, is summable $|R, \exp\{(\log w)^{1+1/\alpha}\}, \alpha+\delta|$, for every $\delta > 0$.

He has also obtained similar theorems for the conjugate series and the derived series of the Fourier series and its conjugate series.

The object of this paper is to extend these theorems of Pati by introducing a parametric constant β into the type, which now is taken to be

$$\lambda(w) = \exp\{(\log w)^{1+\beta/\alpha}\}, \quad \beta \geqslant 0.$$

It is obvious that if $\beta = 0$, $\lambda(w) = w$, and it reduces down to the case of absolute Cesàro summability considered by Bosanquet (1936) and if $\beta = 1$, $\lambda(w) = \exp\{(\log w)^{1+1/\alpha}\}$, which is the type considered by Pati (1957). Now since $|R, w, k| \sim |C, k|$ (Hyslop, 1940), it is clear that the type: $\exp\{(\log w)^{1+\beta/\alpha}\}$, where $0 \leqslant \beta \leqslant 1$, is one that covers the entire range of types from w to $\exp\{(\log w)^{1+1/\alpha}\}$, and thus provides us with a scale which, in a sense, makes it possible to bridge the gulf between absolute Cesàro summability and absolute Riesz summability of type $\exp\{(\log w)^{1+1/\alpha}\}$.

Incidentally, we have obtained, in the form of Lemma 11(a), a result on the absolute Riesz summability of the series $\Sigma(-1)^n n^\rho$, where ρ is a positive integer. This, so far as known to us, is the most general result in this direction, inasmuch as it generalizes Pati's similar result (1957, Lemma 11), which had provided an improvement over the corresponding work of Hyslop (1940).

It may be mentioned here that although for Theorems 1 and 3, we do not put any restriction on β, except that it is positive or zero, for Theorems 2 and 4, we require for our analysis the restriction that $0 < \beta \leqslant 1$. It will be interesting to examine whether this condition can be removed.

2.1. We establish the following theorems:

Theorem 1. If α is an integer $\geqslant 1$, $\beta \geqslant 0$ and $\phi_\alpha(t)\, \{\log (k/t)\}^\beta$ is of bounded variation in $(0, \pi)$, then the Fourier series of $f(t)$, at $t = x$, is summable $|R, e(w), \alpha+\delta|$, for every $\delta > 0$.

Theorem 2. If α is an integer $\geqslant 1$, $0 < \beta \leqslant 1$ and if

(i) $\psi_\alpha(t)\, \{\log (k/t)\}^\beta$ is of bounded variation in $(0, \pi)$ and

(ii) $|\psi_\alpha(t)|/[t\,\{\log (k/t)\}^{1-\beta}]$ is integrable-(L) over $(0, \pi)$, then the conjugate series of Fourier series, at $t = x$, is summable $|R, e(w), \alpha+\delta|$, for every $\delta > 0$.

Theorem 3. If α is an integer $\geqslant 1$ and $\beta \geqslant 0$ and $\gamma_{\alpha, r}(t)\, \{\log (k/t)\}^\beta$ is of *bounded variation in* $(0, \pi)$, *then the* r-*th derived series of the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|R, e(w), \alpha+\delta|$, *for every integral* $\alpha > r$ *and* $\delta > 0$.

Theorem 4. If α *is an integer* > 1, $0 < \beta \leq 1$ *and if*

(i) $\theta_{\alpha, r}(t) \{\log (k/t)\}^{\beta}$ *is of bounded variation in* $(0, \pi)$ *and*

(ii) $|\theta_{\alpha, r}(t)|/[t\{\log (k/t)\}^{1-\beta}]$ *is integrable-*(L) *over* $(0, \pi)$, *then the* r-*th derived series of the conjugate series of the Fourier series of* $f(t)$, *at* $t = x$, *is summable* $|R, e(w), \alpha+\delta|$ *for every integral* $\alpha > r$ *and* $\delta > 0$.

2.2. We shall require the following lemmas for the proof of our theorems:

LEMMA 1. (Obrechkoff, 1928, 1929). If a series Σa_n is summable $|R, \lambda, r|$, $r > 0$, then it is summable $|R, \lambda, r'|$ for every $r' > r$.

LEMMA 2. The Fourier series of special functions

$$(\log |k/t|)^{-\beta}, (\log |k/t|)^{-\beta-1}, \ldots, (\log |k/t|)^{-\beta-\alpha},$$

when $\beta > 0$ and α is an integer, are all absolutely convergent at $t = 0$.

PROOF OF LEMMA 2. We treat here only the special function $(\log |k/t|)^{-\beta}$. The treatment of the remaining special functions proceeds on parallel lines.

Let

$$(\log |k/t|)^{-\beta} \sim \Sigma a_n \cos nt$$

$$a_n = (2/\pi) \int_0^{\pi} \cos nt/\{\log (k/t)\}^{\beta} dt.$$

Now by integration by parts

$$a_n = (2/\pi) [\{\log (k/t)\}^{-\beta} (\sin nt)/n]_0^{\pi}$$

$$-(2/n\pi) \int_0^{\pi} \{\log (k/t)\}^{-\beta-1} (\sin nt)/t \,.\, dt$$

$$= -(2/n\pi) \int_0^{\pi} \{\log (k/t)\}^{-\beta-1} (\sin nt)/t \,.\, dt.$$

Now

$$\int_0^{\pi} \{\log (k/t)\}^{-\beta-1} (\sin nt)/t \,.\, dt$$

$$= \left(\int_0^{n^{-1}} + \int_{n^{-1}}^{\pi}\right) \{\log (k/t)\}^{-\beta-1} (\sin nt)/t \,.\, dt$$

$$= I_1 + I_2, \text{ say.}$$

Then, since $\{\log (k/t)\}^{-\beta-1}$ is increasing in $(0, n^{-1})$ we have by Mean Value Theorem

$$I_1 = (\log kn)^{-\beta-1} \int_{\eta}^{n^{-1}} (\sin nt)/t \,.\, dt, \ (0 < \eta < n^{-1})$$

$$= O[(\log n)^{-\beta-1}].$$

Since $t^{-1}\{\log (k/t)\}^{-\beta-1}$ is decreasing in (n^{-1}, π), we have

$$I_2 = n\ (\log kn)^{-\beta-1} \int_{n^{-1}}^{\zeta} \sin nt\, dt,\ (n^{-1} < \zeta < \pi)$$
$$= O\{(\log n)^{-\beta-1}\}.$$

Thus finally

$$a_n = -(2/n\pi)(I_1+I_2) = O[n^{-1}(\log n)^{-\beta-1}].$$

Hence

$$\Sigma|a_n| < K\,\Sigma n^{-1}\ (\log n)^{-\beta-1} < \infty, \text{ since } \beta > 0,$$

which was required to be proved.

Lemma 3. (Obrechkoff, 1934; Pati, 1954). Let $C_n^{(k)}$, $S_n^{(k)}$ and $\bar{S}_n^{(k)}$ denote the n-th Cesàro mean of order k $(k \geqslant 0)$ corresponding to the series

$$\sum_1^\infty (-1)^n n^\rho,\ \sum_1^\infty (\cos nt)_\rho \text{ and } \sum_1^\infty (\sin nt)_\rho$$

respectively, then

(i) $S_n^{(k)} = O(n^{\rho+k+1})$ $(0 < t \leqslant 1/n)$,

(ii) $S_n^{(k)} = O(n^\rho t^{-k-1}) + O(n^{k-1} t^{-\rho-1})$ $(1/n < t \leqslant \pi)$,

(iii) $\bar{S}_n^{(k)} = O(n^{\rho+k+1})$ $(0 < t < 1/n)$,

(iv) $\bar{S}_n^{(k)} = O(n^\rho t^{-k-1}) + O(n^k t^{-\rho-1})$ $(1/n < t \leqslant \pi)$,

(v) when ρ is an even integer $\geqslant 2$

$$C_n^{(k)} = O\{n^{\max(\rho,\, k-1)}\}.$$

Lemma 4. (Hardy and Riesz, 1915). Let

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \leqslant x} a_n.$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n \leqslant x} (x-\lambda_n)^r a_n,\ r > 0.$$

Then, if k is a positive integer

$$A_\lambda(x) = (k\,!)^{-1}(d/dx)^k A_\lambda^k(x).$$

Lemma 5. The n-th derivative of $\{F(x)\}^m$ is a sum of constant multiples of terms of the type

$$\{F(x)\}^{m-r}\{F^{(1)}(x)\}^{\alpha_1}\{F^{(2)}(x)\}^{\alpha_2} \ldots . \{F^{(n)}(x)\}^{\alpha_n},$$

where $r < n$ and the α's are positive integers or zeros such that

$$\sum_{\nu=1}^n \alpha_\nu = r,\quad \sum_{\nu=1}^n \nu\alpha_\nu = n.$$

This is a particular case of a result, due to Faa di Bruno (de la Valleé Poussin, 1923), on the successive derivative of a function of a function.

Lemma 6. If ρ is an even integer such that $2 \leqslant \rho \leqslant \alpha-1$, then

$$\int_1^\infty [(\log w)^{\beta/\alpha}/w \;\; e^{\alpha+\delta}(w)]\,|\,E^{(\rho)}(w,\,\pi)\,|\,dw < \infty.$$

We have

$$E^{(\rho)}(w,\,\pi) = (-1)^{\rho/2} \sum_{n \leqslant w} \{e(w)-e(n)\}^{\alpha+\delta-1} e(n)\,.\,(-1)^n n^\rho.$$

Evidently it suffices to consider the convergence of

$$\int_1^\infty [(\log w)^{\beta/\alpha}/w \;\; e^{\alpha+\delta}(w)]\,|\,\Sigma_1\,|\,dw$$

and

$$\int_1^\infty [(\log w)^{\beta/\alpha}/w \;\; e^{\alpha+\delta}(w)]\,|\,\Sigma_2\,|\,dw$$

where

$$\Sigma_1 = \sum_{n \leqslant w} \{e(w)-e(n)\}^{\alpha+\delta-1} (-1)^n n^\rho,$$

and

$$\Sigma_2 = \sum_{n \leqslant w} \{e(w)-e(n)\}^{\alpha+\delta} (-1)^n n^\rho.$$

We adopt the notations

$$C(x) = \sum_{n \leqslant x} (-1)^n n^\rho$$

$$C^{(k)}(x) = \sum_{n \leqslant x} (x-n)^k\,.\,(-1)^n n^\rho,\; (k = 1,\, 2,\, \ldots .).$$

We have then (Pati, 1954, 1957)

(2.2.1) $$C^{(\alpha-1)}(x) = O(x^{\alpha-1}),$$
$$C^{\alpha}(x) = O(x^{\alpha-1}).$$

Now

$$\sum\nolimits_1 = -\int_1^w C(x)(d/dx)\,\{e(w)-e(x)\}^{\alpha+\delta-1}\,dx$$
$$= -\left(\int_1^e + \int_e^w\right) C(x)(d/dx)\,\{e(w)-e(x)\}^{\alpha+\delta-1}\,dx$$
$$= -(I_1+I_2),\text{ say.}$$

We easily have

$$e(w)\,.\,I_1 = O[e^{\alpha+\delta-1}(w)].$$

Next, using Lemma 4, and integrating $(\alpha-1)$ times by parts we have

$$I_2 = O\{e^{\alpha+\delta-2}(w)\} + O\left(\left|\int_e^w C^{\alpha-1}(x).\,(d/dx)^{\alpha}\,\{e(w)-e(x)\}^{\alpha-1+\delta}\,dx\right|\right).$$

Now by Lemma 5, the last integral is a sum of constant multiples of integrals of the type

$$I = \int_e^w C^{\alpha-1}(x)\{e(w)-e(x)\}^{\alpha+\delta-1-r}\{e^{(1)}(x)\}^{\beta_1}\cdot\ldots\cdot\{e^{(\alpha)}(x)\}^{\beta_\alpha}\,dx,$$

(2.2.2) when $$\sum_1^{\alpha}\beta_\nu = r \leqslant \alpha, \quad \sum_1^{\alpha}\nu\beta_\nu = \alpha.$$

Case (*i*). $r < \alpha$. In this case proceeding as in Pati (1954, Lemma 6, Case (i)), $r \leqslant \alpha-1$, so that

$$I = O[\{(\log w)^{\beta}/w\}\,.\,e^{\alpha+\delta-1}(w)]$$

and therefore

$$e(w)\,.\,I_2 = O[\{(\log w)^{\beta}/w\}\,.\,e^{\alpha+\delta}(w)].$$

Case (*ii*). $r = \alpha$. In this case, by subtraction, we get from the relation (2.2.2)

$$\beta_2+2\beta_3+\ldots\ldots+(\alpha-1)\beta_\alpha = 0,$$

which implies that

$$\beta_2 = \beta_3 = \ldots\ldots\ldots = \beta_\alpha = 0,\ \beta_1 = \alpha.$$

Then

$$I = \int_e^w C^{\alpha-1}(x)\{e(w)-e(x)\}^{\delta-1}\{e^{(1)}(x)\}^{\alpha}\,dx$$

$$= \left(\int_e^{w_1}+\int_{w_1}^{w}\right)C^{\alpha-1}(x)\{e(w)-e(x)\}^{\delta-1}\{e^{(1)}(x)\}^{\alpha}\,dx$$

$$= I_1+I_2, \text{ say,}$$

where

$$(\log w_1)^{1+\beta/\alpha} = (\log w)^{1+\beta/\alpha}-1.$$

Now

$$I_1 = O[\{(\log w)^{\beta}/w\}\,.\,e^{\alpha+\delta-1}(w)]. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Also using the transformation

$$u = (\log w)^{1+\beta/\alpha}-(\log x)^{1+\beta/\alpha} = w'-(\log x)^{1+\beta/\alpha},$$

we have

$$I_2 = K\int_0^1 \{e(w)-e(w)\,.\,e^{-u}\}^{\delta-1}\,e^{\alpha}(w)\,.\,e^{-\alpha u}.$$

$$\times[(w'-u)^{\beta(\alpha-1)/(\alpha+\beta)}/\exp\{(\alpha-1)(w'-u)^{\alpha/(\alpha+\beta)}\}]$$

$$\times[C^{\alpha-1}(e^{(w'-u)^{\alpha/(\alpha+\beta)}})]du, \ [\text{since } x = e^{(w'-u)^{\alpha/(\alpha+\beta)}}],$$

$$= K \int_0^{(\log w)^{\beta/\alpha}/w} + \int_{(\log w)^{\beta/\alpha}/w}^1 \text{etc.}$$

$$= K(I_{2,1}+I_{2,2}), \text{ say.}$$

$$I_{2,1} = O\left[e^{(\alpha)}(w) \,.\, (\log w)^{\beta-\beta/\alpha} e^{\delta-1}(w) \int_0^{(\log w)^{\beta/\alpha}/w} u^{\delta-1} du\right]$$

$$= O[e^{\alpha+\delta-1}(w) \,.\, (\log w)^{\beta-\beta/\alpha} . \{(\log w)^{\beta/\alpha}/w\}^{\delta}]$$

$$= O[e^{\alpha+\delta-1}(w) \,.\, (\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^{\delta}].$$

$$I_{2,2} = \int_{(\log w)^{\beta/\alpha}/w}^1 \{e(w)-e(w) \,.\, e^{-u}\}^{\delta-1} \,.\, e^{\alpha}(u) \,.\, e^{-\alpha u}$$

$$\times[(w'-u)^{\beta(\alpha-1)/(\alpha+\beta)}/\exp\{(\alpha-1)\,(w'-u)^{\alpha/(\alpha+\beta)}\}]$$

$$\times[C^{\alpha-1}\{\exp\,(w'-u)^{\alpha/(\alpha+\beta)}\}]du$$

$$= O\Bigg[e^{\alpha+\delta-1}(w) \Bigg| \int_{(\log w)^{\beta/\alpha}/w}^1 (1-e^{-u})^{\delta-1} \,.\, e^{-\alpha u}$$

$$\times \{(w'-u)^{\beta(\alpha-1)/(\alpha+\beta)}/e^{\alpha(w'-u)^{\alpha/(\alpha+\beta)}}\}$$

$$\times \{C^{\alpha-1}(e^{(w'-u)^{\alpha/(\alpha+\beta)}}) d(e^{(w'-u)^{\alpha/(\alpha+\beta)}})\Bigg|\Bigg]$$

$$= O\Bigg[e^{\alpha+\delta-1}(w) \,.\, \{(\log w)^{\beta/\alpha}/w\}^{\delta-1} \,.\, \{(\log w)^{\beta}/w^{\alpha}\}$$

$$\times \left|\int_{\xi}^{\xi'} C^{\alpha-1}(x)dx\right|\Bigg]$$

$$= O[e^{\alpha+\delta-1}(w) \,.\, \{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^{\delta}\}].$$

Hence

$$I_2 = O[\{(\log w)^{\beta}/w\} \,.\, e^{\alpha+\delta-1}(w)] + O[e^{\alpha+\delta-1}(w) . \{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^{\delta}\}].$$

And, therefore, finally

$$e(w) \,.\, I_2 = O[\{(\log w)^{\beta}/w\} e^{\alpha+\delta}(w)] + O[e^{\alpha+\delta}(w)\{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^{\delta}\}]. \quad .. \quad (2)$$

Thus combining the results (1) and (2) we have

$$\int_1^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} e(w) . \left|\sum\nolimits_1\right| dw$$

$$< K\left\{\int_1^{\infty} (\log w)^{\beta+\beta/\alpha}/w^2 \,.\, dw\right\} + K\left\{\int_1^{\infty} (\log w)^{\beta+\beta\delta/\alpha}/w^{1+\delta} \,.\, dw\right\} < \infty.$$

Lemma 7. If ρ is zero or a positive integer, $\beta > 0$ and α an integer > 1, then

$$\sum_{n<w} \{e(w)-e(n)\}^{\alpha+\delta-1} \,.\, e(n)n^{\rho} = O[\{w^{\rho+1}/(\log w)^{\beta/\alpha}\} . \, e^{\alpha+\delta}(w)].$$

PROOF OF LEMMA 7. For $m \leqslant w \leqslant m+1$, $\alpha \geqslant 2$,

$$\sum_{n \leqslant w} \{e(w)-e(n)\}^{\alpha+\delta-1} e(n)n^{\rho}$$

$$= \sum_{1}^{m} \{e(w)-e(n)\}^{\alpha+\delta-1} . \ e(n)n^{\rho}$$

$$= O\left[e^{\delta}(w) \sum_{1}^{m} \{e(w)-e(n)\}^{\alpha-1} . \ e(n)n^{\rho}\right]$$

$$= O\left[e^{\delta}(w) \left\{ \sum_{1}^{m-1} \Delta[e(w)-e(n)]^{\alpha-1} . \sum_{1}^{n} e(\nu)\nu^{\rho} \right\} + \{e(w)-e(m)\}^{\alpha-1} \sum_{1}^{m} e(\nu)\nu^{\rho}\right]$$

$$= O\left[e^{\delta}(w) \left\{ e^{\alpha-2}(w) \sum_{1}^{m-1} [e(n+1)-e(n)] \sum_{1}^{n} e(\nu)\nu^{\rho} \right\} + \{e(w)-e(m)\}^{\alpha-1} \sum_{1}^{m} e(\nu)\nu^{\rho}\right], \ \alpha \geqslant 2,$$

$$= O\left[e^{\alpha+\delta-2}(w) \sum_{1}^{m-1} \frac{\{\log (n+1)\}^{\beta/\alpha}}{n+1} . \ e(n+1)n^{\rho} . \ \{n/(\log n)^{\beta/\alpha}\} . \ e(n)\right]$$

$$+ O\left[\left\{\frac{(\log w)^{\beta/\alpha}}{w} . \ e(w)\right\}^{\alpha-1} . \ e^{\delta}(w) . \frac{w^{\rho} . w}{(\log w)^{\beta/\alpha}} . \ e(w)\right]$$

$$= O\left[e^{\alpha+\delta-2}(w) \sum_{1}^{m-1} n^{\rho} . \ e^{2}(n+1)\right] + O[\{(\log w)^{\beta(\alpha-2)/\alpha}/w^{\alpha-1}\} . \ w^{\rho+1} . \ e^{\delta+1}(w)]$$

$$= O\left[e^{\alpha+\delta-1}(w) . \ w^{\rho} . \frac{w . e(w)}{(\log w)^{\beta/\alpha}}\right] + O[\{(\log w)^{\beta(\alpha-2)/\alpha}/w^{\alpha-1}\} w^{\rho+1} . \ e^{\alpha+\delta}(w)]$$

$$= O[\{e^{\alpha+\delta}(w) . \ w^{\rho+1}/(\log w)^{\beta/\alpha}] + O[\{(\log w)^{\beta(\alpha-2)/\alpha}/w^{\alpha-1}\} w^{\rho+1} . \ e^{\alpha+\delta}(w)]$$

$$= O[e^{\alpha+\delta}(w) . \ w^{\rho+1}/(\log w)^{\beta/\alpha}].$$

Since

$$\{e(n+1)-e(n)\} = K \int_{n}^{n+1} \{(\log w)^{\beta/\alpha}/w\} e(w) dw$$

$$= O[\{(\log (n+1))^{\beta/\alpha}/(n+1)\} e(n+1)]$$

and

$$\sum_{1}^{n} e(\nu)\nu^{\rho} = O[n^{\rho} \int_{1}^{n} \{e'(w) . \ w/(\log w)^{\beta/\alpha}\} dw$$

$$= O[n^{\rho} . \ \{n/(\log n)^{\beta/\alpha}\} . \ e(n)].$$

If $\alpha = 1$, $\sum_{1}^{m} \{e(w)-e(n)\}^{\delta} . e(n)n^{\rho}$

$$= O[e^{\delta}(w) . w^{\rho} . \sum_{1}^{m} e(n)]$$
$$= O[e^{\delta}(w) . w^{\rho} . \{we(w)/(\log w)^{\beta/\alpha}\}]$$
$$= O[e^{\delta+1}(w) . w^{\rho+1}/(\log w)^{\beta/\alpha}].$$

Thus finally the Lemma is proved.

LEMMA 8. If α is an integer $\geqslant 1$, then

$$E^{(\alpha-1)}(w, t) = O[\{(\log w)^{\beta}/w\} e^{\alpha+\delta}(w) t^{-\alpha-1}]$$
$$+O[\{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^{\delta}\} e^{\alpha+\delta}(w) t^{-\alpha-\delta}].$$

PROOF OF LEMMA 8. The proof proceeds along the same lines as that of Lemma 6. In the analysis $C(x)$ and $C^{\alpha-1}(x)$ will be replaced by $S(x)$ and $S^{(\alpha-1)}(x)$, where

$$S(x) = \sum_{n \leqslant x} (\cos nt)_{\alpha-1}$$
$$S^{\alpha-1}(x) = \sum_{n \leqslant w} (x-n)^{\alpha-1} (\cos nt)_{\alpha-1}$$
$$S^{\alpha}(x) = \sum_{n \leqslant w} (x-n)^{\alpha} (\cos nt)_{\alpha-1}.$$

We use the estimates [Pati, 1954, 1957],

$$S^{\alpha-1}(x) = O(x^{\alpha-1} t^{-\alpha})$$
$$S^{\alpha}(x) = O(x^{\alpha-1} t^{-\alpha-1}).$$

The integral corresponding to I_2 will be broken up into integrals over $\{O, K(\log w)^{\beta/\alpha}/wt\}$, $\{K(\log w)^{\beta/\alpha}/wt, 1\}$, where K is a suitable constant in view of the inequality $w > \tau = (k/t)\{(\log (k/t)\}^{\beta/\alpha}$ and the treatment of these will be similar to those of $I_{2,1}$ and $I_{2,2}$ respectively.

The estimates obtained in Lemmas 2 and 6, 7 and 8 have been made for special cases $\beta = 1$, $\alpha = 0$ and $\beta = 1$, α an integer $\geqslant 1$ by Mohanty (1949) and Pati (1957) respectively.

LEMMA 9. If ρ is zero or a positive integer $\leqslant \alpha - 1$, then

$$\bar{E}^{(\rho)}(w, t) = O[\{(\log w)^{\beta}/w\} . e^{\alpha+\delta}(w) . t^{-\rho-2}]$$
$$+O[w^{-\delta}(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha} . e^{\alpha+\delta}(w) . t^{-\rho-1-\delta}].$$

Application of the same technique as employed in proof of Lemmas 6 and 8 of this paper will yield the result.

LEMMA 10. If ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha-1$, then

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \,.\, |\bar{E}^{(\rho)}(w, \pi)|\, dw < \infty .$$

The result follows immediately on application of Lemma 9.

LEMMA 11. If ρ is an integer such that $1 \leqslant \rho \leqslant \alpha-1$, then the infinite series $\Sigma(-1)^n n^\rho$ is summable $|R, e(w), \alpha+\delta|$ for every $\delta > 0$.

This is essentially a combination of results of Lemmas 6 and 10.

Taking the case $\rho = \alpha-1$, in Lemma 11, we obtain

LEMMA 11 (*a*). If ρ is an integer $\geqslant 1$, then the infinite series $\Sigma(-1)^n n^\rho$ is summable $|R, \exp\{(\log w)^\Delta\}, \rho+1+\delta|$ however large $\Delta\ (> 0)$ may be.

LEMMA 12. (Chandrasekharan, 1942). If the series Σa_n is summable $|R, \lambda, r|$, $r \geqslant 0$ and μ is a logarithmico-exponential function of λ such that $\mu = O(\lambda^\Delta)$, where Δ is a constant, then the series Σa_n is summable $|R, \mu, r|$.

LEMMA 13. If $\beta > 0$, the necessary and sufficient conditions that

(i) $F(t)\{\log(k/t)\}^\beta$ be of bounded variation in $(0, \eta)$, and

(ii) $|F(t)|/[t\{\log(k/t)\}^{1-\beta}]$ be integrable-(L) in $(0, \eta)$, η being positive, are that

(2.2.3) $$\int_0^\eta \{\log(k/t)\}^\beta |dF(t)| < \infty \text{ and } F(+0) = 0.$$

PROOF OF LEMMA 13. *The conditions are sufficient.* For if (2.2.3) holds and $F(+0) = 0$, then we have

$$\begin{aligned} I &= \int_0^\eta |F(t)|/[t\{\log(k/t)\}^{1-\beta}]\, dt \\ &= \int_0^\eta \left| \int_0^t dF(u) \right| / [t\{\log(k/t)\}^{1-\beta}]\, dt \\ &\leqslant \int_0^\eta [t\{\log(k/t)\}^{1-\beta}]^{-1} dt \int_0^t |dF(u)| \\ &= \int_0^\eta |dF(u)| \int_u^\eta [t\{\log(k/t)\}^{1-\beta}]^{-1} dt \\ &= (1/\beta) \int_0^\eta [-\{\log(k/t)\}^\beta]_u^\eta |dF(u)| \\ &= (1/\beta) \int_0^\eta [\{\log(k/u)\}^\beta - \{\log(k/\eta)\}^\beta]\, |dF(u)| \\ &< \int_0^\eta \{\log(k/u)\}^\beta |dF(u)| \\ &< \infty, \end{aligned}$$

from which (ii) follows, and then (i) follows from

$$(2.2.4)\qquad \int_0^{\eta} |d[F(t)\{\log (k/t)\}^{\beta}]|$$
$$\leqslant \int_0^{\eta} \{\log (k/t)\}^{\beta} |dF(t)| + \beta \int_0^{\eta} |F(t)|/[t\{\log (k/t)\}^{1-\beta}]\, dt.$$

The conditions are necessary. For suppose that (i) and (ii) hold. We now obtain (2.2.3) from

$$(2.2.5)\qquad \int_0^{\eta} \{\log (k/t)\}^{\beta} |dF(t)|$$
$$\leqslant \int_0^{\eta} |d[F(t)\{\log (k/t)\}^{\beta}]| + \beta \int_0^{\eta} |F(t)|/[t\{\log (k/t)\}^{1-\beta}]\, dt.$$

Finally since $F(t)\{\log (k/t)\}^{\beta}$ is of bounded variation in $(0, \eta)$, $\lim_{t\to 0} [F(t)\{\log (k/t)\}^{\beta}]$ is finite. Hence

$$F(t) = O[\{\log (k/t)\}^{-\beta}] \text{ and so } F(+0) = 0.$$

This completes the proof of Lemma 13.

Lemma 14. If $F(+0) = 0$ and $\int_0^{\pi} \{\log (k/t)\}^{\beta} |dF(t)| < \infty$ then the series Σv_n, where

$$v_n = \int_0^{\pi} F(t) \sin nt\, dt = -F(\pi)\{\cos n\pi/n\} + \int_0^{\pi} (\cos nt/n)\, d\, F(t)$$

is summable $|R, \exp \{e^{\alpha_1 (\log w)^{1/\beta}}\}, 1|$ where $0 < \alpha_1 < \beta \leqslant |$.

We shall require the following order estimates for the proof of the lemma. Writing

$$\lambda(m) = e^{g(m)} = \exp\{e^{h(m)}\}, \; [h(m) = \alpha_1(\log m)^{1/\beta}],$$

and

$$J(w, t) = \sum_{\lambda_n \leqslant w} (\lambda_n \cos nt)/n$$

we have

$$(2.2.6)\qquad \text{(i) } |J(w, t)| = O[w(\log w)^{-1} (\log\log w)^{\beta-1}] \text{ and}$$

$$(2.2.7)\qquad \text{(ii) } |J(w, t)| = O[w \,.\, \exp (\alpha_1^{-1} \log\log w)^{-\beta} \,.\, t^{-1}].$$

Proof of (2.2.6). We have

$$|J(w, t)| \leqslant \sum_{\lambda_n \leqslant w} (\lambda_n/n) = \sum_{1}^{m} (\lambda_n/n), \text{ where } \lambda_m \leqslant w < \lambda_{m+1},$$
$$= O\left[\int_1^m \{\lambda(x)/x\}\, dx\right]$$

$$= O\left[\int_1^m \lambda'(x)\{xg'(x)\}^{-1}dx\right]$$
$$= O[\lambda(m)/mg'(m)]$$
$$= O[\lambda(m)/\{(m+1)g'(m+1)\}].$$

But we know that $\lambda_m \leqslant w$, and

$$(m+1)g'(m+1) = (m+1)e^{h(m+1)}.\,h'(m+1)$$
$$= g(m+1)\,.\,(\alpha_1/\beta)[\{\log\,(m+1)\}^{1/\beta}]^{1-\beta}$$

Hence
$$|J(w,t)| = O(w/[g(m+1)\{[\log\,(m+1)]^{1/\beta}\}^{1-\beta}])$$
$$= O[w\,.\,(\log w)^{-1}\,.\,(\log\log w)^{\beta-1}].$$

Since $\lambda_{m+1} \geqslant w$

(2.2.8)
$$\begin{cases} g(m+1) > \log w, \\ \{\log\,(m+1)\}^{1/\beta} > (1/\alpha_1)\,(\log\log w). \end{cases}$$

PROOF OF (2.2.7). Next we have by Abel's lemma

$$|J(w,t)| = |\sum_{\lambda_n \leqslant w} (\lambda_n \cos nt)/n|$$
$$\leqslant (\lambda_m/m)\max|\sum_{\nu}^{m} \cos nt|, \quad (1 \leqslant \nu \leqslant m),$$
$$= O[t^{-1}\lambda_m/m]$$
$$= O[wt^{-1}/(m+1)].$$

But
$$\lambda_{m+1} = \exp\left(\exp\,\{\alpha_1(\log \overline{m+1})^{1/\beta}\}\right) > w.$$

Hence
$$\log\,(m+1) > (\alpha_1^{-1}\log\log w)^{\beta}$$

i.e.
$$(m+1) > \exp\,(\alpha_1^{-1}\log\log w)^{\beta}.$$

Hence we have finally
$$|J(w,t)| = O[w\,.\,\exp\,\{-(\alpha_1^{-1}\log\log w)^{\beta}\}t^{-1}].$$

PROOF OF LEMMA 14. The series Σv_n is summable $|R, \exp\{e^{\alpha_1 (\log w)^{1/\beta}}\}, 1|$ if

$$I = \int_e^{\infty} w^{-2}|\sum_{\lambda_n \leqslant w} \lambda_n v_n|\,dw < \infty.$$

Integrating by parts and using the fact that $F(+0) = 0$, we have

$$v_n = \int_0^{\pi} F(t)\sin nt\,dt$$
$$= -F(\pi)\,(\cos n\pi/n) + \int_0^{\pi} (\cos nt/n)\,dF(t).$$

We have then

$$I \leqslant |F(\pi)| \int_e^\infty w^{-2} |J(w, \pi)|\, dw + \int_0^\pi |dF(t)| \int_e^\infty w^{-2} |J(w, t)|\, dt$$

$$= I_1 + I_2, \text{ say.}$$

By using (2.2.7) we have

$$I_1 \leqslant A \int_e^\infty w^{-1} \exp\{-\alpha_1^{-\beta} (\log\log w)^\beta\}\, dw$$

$$< \int_e^\infty w^{-1} . (\log w)^{-1-\epsilon}\, dw < \infty, \ \epsilon > 0.$$

Hence it will be sufficient to prove that $I_2 < \infty$.

We have

$$I_2 = \int_0^\pi q(t) |dF(t)|$$

where

$$q(t) = \int_e^\infty w^{-2} |J(w, t)|\, dw.$$

Writing

$$\tau = \exp\{(k/t)^{\alpha_1/(\beta-\alpha_1)}\}, \text{ let}$$

$$q(t) = \int_e^\infty = \int_e^\tau + \int_\tau^\infty = q_1(t) + q_2(t), \text{ say.}$$

The fact that $q_2(t)$ is bounded for $0 < t < \pi$ follows at once from (2.2.7) if we assume that $\alpha_1 < \beta$. Using (2.2.6) we have

$$q_1(t) = \int_e^\tau w^{-2} |J(w, t)|\, dw$$

$$= \int_e^\tau w^{-1} (\log w)^{-1} (\log\log w)^{\beta-1}\, dw$$

$$= O[\{\log (k/t)\}^\beta], \quad \text{for } 0 < t < \pi.$$

Thus it follows that

$$q(t) = O[\{\log (k/t)\}^\beta], \quad \text{for } 0 < t < \pi.$$

Hence finally

$$I_2 \leqslant A \int_0^\pi \{\log (k/t)\}^\beta |dF(t)| < \infty,$$

and this completes the proof of Lemma 14.

Special cases of Lemmas 9, 10 and 11 have been given by Pati (1954, 1957) and those of Lemmas 13 and 14 are due to Mohanty (1949, 1951).

3.1. Proof of Theorem 1

Since

$$A_n(x) = (2/\pi) \int_0^\pi \phi(t) \cos nt \, dt$$

proceeding as in Pati (1954, Proof of Theorem 1) we have only to prove that

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_0^\pi \phi(t) E(w, t) \, dt \right| dw < \infty.$$

Integrating $\left[\int_0^\pi \phi(t) E(w, t) \, dt\right]$ by parts and proceeding as in Pati (1957, Proof of Theorem 1), we need only prove the following:

$$(I) \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid E^{(\rho)}(w, \pi) \mid dw < \infty$$

where ρ is an even integer such that $2 \leqslant \rho \leqslant \alpha - 1$.

$$(II) \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid g(w, \pi) \mid dw < \infty \text{ and}$$

$$(III) \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid g(w, t) \mid dw = O(1), \text{ for } 0 < t < \pi.$$

Since result (I) has been established in Lemma 6 and

$$g(w, t) = g(w, \pi) - h(w, t),$$

Theorem 1 will be established if

$$I_1 = \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid g(w, \pi) \mid dw < \infty, \tag{3.1.1}$$

$$I_2 = \int_1^\tau \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid g(w, t) \mid dw = O(1), \; 0 < t < \pi, \tag{3.1.2}$$

$$I_3 = \int_\tau^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \mid h(w, t) \mid dw = O(1), \; 0 < t < \pi. \tag{3.1.3}$$

Proof of (3.1.1). The proof of this is parallel to that of (3.1.1) of Pati (1954).

Proof of (3.1.2).

$$\begin{aligned} g(w, t) &= \int_0^t [u^\alpha/\{\log(k/u)\}^\beta] E^{(\alpha)}(w, u) \, du \\ &= [t^\alpha/\{\log(k/t)\}^\beta] \int_\eta^t (\partial/\partial u) \, E^{(\alpha-1)}(w, u) \, du, \; 0 < \eta < t, \\ &= O[\{t^\alpha/(\log(k/t))^\beta\} \sum_{n \leqslant w} \{e(w) - e(n)\}^{\alpha+\delta-1} e(n) \, . \, n^{\alpha-1} \\ &= O[\{t^\alpha/(\log(k/t))^\beta\} \, w^\alpha \, \{e^{\alpha+\delta}(w)/(\log w)^{\beta/\alpha}\}], \text{ (by Lemma 7).} \end{aligned}$$

Therefore

$$I_2 = O\left[\int_1^\tau \{(\log w)^{\beta/\alpha}/w \,.\, e^{\alpha+\delta}(w)\} \,.\, \{t^\alpha/(\log (k/t))^\beta\} \times \{w^\alpha/(\log w)^{\beta/\alpha}\} \,.\, e^{\alpha+\delta}(w)\, dw\right]$$

$$= O\left[\{t^\alpha/(\log (k/t))^\beta\} \int_1^\tau w^{\alpha-1} dw\right]$$

$$= O(1), \qquad \text{for } 0 < t < \pi.$$

PROOF OF (3.1.3)

$$h(w, t) = |E^{(\alpha-1)}(w, \pi)| + O[\{t^\alpha/(\log (k/t))^\beta\} \, |E^{(\alpha-1)}(w, t)| + O\left[\int_t^\pi \{u^{\alpha-1}/(\log (k/u))^\beta\} \, |E^{(\alpha-1)}(w, u)| \, du\right.$$

$$= O[|E^{(\alpha-1)}(w, \pi)|] + O[t^{-1}\{\log (k/t)\}^{-\beta} \,.\, \{(\log w)^\beta/w\} \,.\, e^{\alpha+\delta}(w)]$$
$$+ O[\{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^\delta\} \,.\, e^{\alpha+\delta}(w) \,.\, t^{-\delta}\{\log (k/t)\}^{-\beta}]$$
$$+ O[\{(\log w)^\beta/w\} \,.\, e^{\alpha+\delta}(w) \int_t^\pi u^{-2}\{\log (k/u)\}^{-\beta} du]$$
$$+ O[\{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^\delta\} \,.\, e^{\alpha+\delta}(w) \int_t^\pi u^{-1-\delta}\{(\log (k/u))\}^{-\beta}],$$

(by Lemma 8)

$$= O\{|E^{(\alpha-1)}(w, \pi)|\} + O[t^{-1}\{\log (k/t)\}^{-\beta} \,.\, \{(\log w)^\beta/w\} \,.\, e^{\alpha+\delta}(w)]$$
$$+ O[t^{-\delta}\{\log (k/t)\}^{-\beta} \,.\, w^{-\delta} \,.\, (\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha} \,.\, e^{\alpha+\delta}(w)].$$

Now observing that $E^{(\alpha-1)}(w, \pi) = 0$, when α is even, and applying Lemmas 6 and 8, when α is odd, we obtain

$$I_3 = O(1) + O\left[t^{-1}\{\log (k/t)\}^{-\beta} \int_\tau^\infty \{(\log w)^{\beta+\beta/\alpha}/w^2\}\, dw\right]$$
$$+ O\left[t^{-\delta}\{\log (k/t)\}^{-\beta} \int_\tau^\infty \{(\log w)^{\beta+\beta\delta/\alpha}/w^{1+\delta}\}\, dw\right]$$
$$= O(1), \quad \text{for } 0 < t < \pi.$$

This completes the proof of Theorem 1.

3.2. PROOF OF THEOREM 2

In view of Lemma 13, Theorem 2 can be put in the following equivalent form:

THEOREM 2*a*. If α is an integer $\geqslant 1$ and if

$$\text{(i)} \int_0^\pi \{\log (k/t)\}^\beta |d\psi_\alpha(t)| < \infty \text{ and (ii) } \psi_\alpha(+0) = 0,$$

then the conjugate series of the Fourier series of $f(t)$, at $t = x$, is summable $|R, e(w), \alpha+\delta|$, for every $\delta > 0$, $0 < \beta \leqslant 1$.

We proceed to prove Theorem 2*a*.

Since

$$B_n(x) = (2/\pi)\int_0^\pi \psi(t)\sin nt\,dt,$$

we have only to show that, under the hypothesis of the theorem, the integral

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\,.\left|\int_0^\pi \psi(t)\bar{E}(w,t)dt\right|dw$$

is convergent. Integrating $\int_0^\pi \psi(t)\bar{E}(w,t)dt$ by parts α-times and proceeding as in the proof of Theorem 2 (Pati, 1954), it will suffice for the proof of the theorem to establish the following:

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\,|\bar{E}^{(\rho)}(w,\pi)|\,dw < \infty \tag{3.2.1}$$

where ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha-1$.

$$J = \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\left|\int_0^\pi d\psi_\alpha(t)\,.\,\Lambda(t)\right|dw < \infty \tag{3.2.2}$$

where

$$\Lambda(t) = t^\alpha \bar{E}^{(\alpha-1)}(w,t) - \alpha t^{\alpha-1}\bar{E}^{(\alpha-2)}(w,t) + \dots$$
$$+(-1)^{\alpha-1}\,.\,\alpha(\alpha-1)\dots 2t\,.\,\bar{E}(w,t)\text{; and}$$

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\left|\sum_{n\leqslant w}\{e(w)-e(n)\}^{\alpha-1+\delta}\,.\,e(n)\right.$$
$$\left.\times\left\{-\psi_\alpha(\pi)(\cos n\pi/n) + \int_0^\pi (\cos nt/n)d\psi_\alpha(t)\right\}\right|dw < \infty. \tag{3.2.3}$$

Proof of (3.2.1). The result has already been established in Lemma 10.

Proof of (3.2.2).

$$J < \left(\int_1^\tau + \int_\tau^\infty\right)\{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\int_0^\pi |d\psi_\alpha(t)|\,|\Lambda(t)|\,dw$$
$$= J_1 + J_2, \text{ say},$$

where $\tau = (k/t)\{\log(k/t)\}^{\beta/\alpha}$.

Now for the proof of the convergence of J_1 it is sufficient to show that

$$J' = \int_0^\pi |d\psi_\alpha(t)|\int_1^\tau \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\,t^{\rho+1}|\bar{E}^{(\rho)}(w,t)|\,dw < \infty,$$

where ρ is a positive integer $\leqslant \alpha-1$. Now

$$J' = O\left[\int_0^\pi |d\psi_\alpha(t)| \int_1^\tau \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\, t^{\rho+1} \times \left\{\sum_{n \leqslant w} [e(w)-e(n)]^{\alpha-1+\delta} e(n) n^\rho\right\} dw\right],$$

$$= O\left[\int_0^\pi \{\log(k/t)\}^\beta |d\psi_\alpha(t)|\, t^{\rho+1} \{\log(k/t)\}^{-\beta} \int_1^\tau w^\rho\, dw, \text{ (by Lemma 1)}\right.$$

$$= O(1),$$

since

$$t^{\rho+1}.\{\log(k/t)\}^{-\beta} \int_1^\tau w^\rho\, dw = O(1), \text{ for } 0 < t < \pi.$$

To prove the convergence of J_2 we observe that

$$\Lambda(t) = O[\{(\log w)^\beta/w\}.e^{\alpha+\delta}(w).t^{-1}] + O[\{(\log w)^{\beta+\beta\delta/\alpha-\beta/\alpha}/w^\delta\}\, e^{\alpha+\delta}(w).t^{-\delta}].$$

Hence

$$J_2 = O\left[\int_0^\pi |d\psi_\alpha(t)|\, t^{-1} \int_\tau^\infty \{(\log w)^{\beta+\beta/\alpha}/w^2\}\, dw\right] + O\left[\int_0^\pi |d\psi_\alpha(t)|\, t^{-\delta} \int_0^\infty \{(\log w)^{\beta+\beta/\alpha}/w^{1+\delta}\}\, dw\right]$$

$$= O\left[\int_0^\pi \{\log(k/t)\}^\beta |d\psi_\alpha(t)| \left\{t[\log(k/t)]^{-\beta} \int_\tau^\infty (\log w)^{\beta+\beta/\alpha}/w^2.dw\right\}\right]$$

$$+ O\left[\int_0^\pi \{\log(k/t)\}^\beta |d\psi_\alpha(t)|. \left\{t^\delta[\log(k/t)]^{-\beta}. \int_\tau^\infty (\log w)^{\beta+\beta\delta/\alpha}/w^{1+\delta}\, dw\right\}\right]$$

$$= O(1).$$

Proof of (3.2.3). Proving (3.2.3) is the same thing as proving that the series Σu_n, where

$$u_n = -\psi_\alpha(\pi)(\cos n\pi/n) + \int_0^\pi (\cos nt/n)\, d\psi_\alpha(t)$$

is summable $|R, e(w), \alpha+\delta|$. By Lemma 14 we conclude that Σu_n is summable $|R, \exp[\exp\{\alpha_1(\log w)^{1/\beta}\}], 1|$, $(0 < \alpha_1 < \beta \leqslant 1)$ and therefore by Lemmas 12 and 1 it is summable $|R, e(w), \alpha+\delta|$.

This completes the proof of Theorem 2.

3.3. Proof of Theorem 3

Let r be even. Then we have to show that, under the hypothesis of the theorem, the integral

$$\int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_0^\pi \phi(t)E^{(r)}(w, t)\, dt \right| dw$$

is convergent. Now

$$\int_0^\pi \phi(t)E^{(r)}(w, t)\, dt = \tfrac{1}{2}\int_0^\pi \{P(t)+P(-t)\}\, E^{(r)}(w, t)\, dt + \int_0^\pi g(t)E^{(r)}(w, t)\, dt.$$

Thus it is sufficient for our purpose to show that

$$(3.3.1) \quad \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}. \left| \int_0^\pi \tfrac{1}{2}\{P(t)+P(-t)\}E^{(r)}(w, t)\, dt \right| dw < \infty,$$

and

$$(3.3.2) \quad \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_0^\pi g(t)E^{(r)}(w, t)\, dt \right| dw < \infty.$$

Proving (3.3.1) is the same thing as proving the summability $|R, e(w), \alpha+\delta|$, $\delta > 0$, of $\Sigma n^r p_n$, where p_n is the Fourier cosine constant of the even function $\frac{1}{2}\{P(t)+P(-t)\}$. This can be easily proved by using Lemma 11.

To prove (3.3.2) for $\alpha > r$, integrating $\int_0^\pi g(t)E^{(r)}(w, t)\, dt$ by parts $(\alpha-r)$—times, we have

$$\int_0^\pi g(t)E^{(r)}(w, t)\, dt = \left[\sum_1^{\alpha-r} (-1)^{\rho-1}G_\rho(t)E^{(r+\rho-1)}(w, t)\right]_0^\pi$$
$$+\{(-1)^{\alpha-r}/\Gamma(\alpha-r+1)\}\gamma_{\alpha, r}(\pi)\{\log (k/\pi)\}^\beta\, g(w, \pi)$$
$$+\{(-1)^{\alpha-r+1}/\Gamma(\alpha-r+1)\}\int_0^\pi d[\gamma_{\alpha, r}(t)\{\log (k/t)\}^\beta] \,.\, g(w, t).$$

Also if $\alpha = r$,

$$\int_0^\pi g(t)E^{(r)}(w, t)\, dt = \gamma_{r, r}(\pi)\{\log (k/\pi)\}^\beta g(w, \pi)$$
$$- \int_0^\pi d[\gamma_{r, r}(t)\{\log (k/t)\}^\beta]\, g(w, t).$$

Hence, as in the proof of Theorem 1, it is sufficient for our purpose to establish only the following:

$$(3.3.3) \qquad \int_1^\infty \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\, |g(w, \pi)|\, dw < \infty,$$

$$(3.3.4) \qquad \int_1^\tau \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\, |g(w, t)|\, dw = O(1), \text{ for } 0 < t < \pi,$$

and

$$\text{(3.3.5)} \qquad \int_{\tau}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\, |h(w, t)|\, dw = O(1), \text{ for } 0 < t < \pi.$$

All these have been proved in 3.1. The case in which r is odd can be treated similarly.

This completes the proof of Theorem 3.

3.4. Proof of Theorem 4

Let r be even. Then we have to show that, under the hypothesis of the theorem, the integral

$$\int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_{0}^{\pi} \psi(t)\bar{E}^{(r)}(w, t)\, dt \right| dw$$

is convergent. Now

$$\int_{0}^{\pi} \psi(t)\bar{E}^{(r)}(w, t)\, dt = \int_{0}^{\pi} \tfrac{1}{2}\{P(t)-P(-t)\}\bar{E}^{(r)}(w, t)\, dt + \int_{0}^{\pi} h(t)\bar{E}^{(r)}(w, t)\, dt.$$

Thus we need only prove the following:

$$\text{(3.4.1)} \quad \int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_{0}^{\pi} \tfrac{1}{2}\{P(t)-P(-t)\}\bar{E}^{(r)}(w, t)\, dt \right| dw < \infty,$$

$$\text{(3.4.2)} \qquad \int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_{0}^{t} h(t)\bar{E}^{(r)}(w, t)\, dt \right| dw < \infty.$$

Proving (3.4.1) is the same thing as proving the summability $|R, e(w), \alpha+\delta|$, $\delta > 0$, of $\Sigma n^r q_n$, where q_n is the Fourier sine constant of the odd function $\frac{1}{2}\{P(t)-P(-t)\}$. This can be easily proved like (3.3.1).

For the proof of (3.4.2) proceeding as in the proof of (3.3.2) above we need only show that

$$\text{(3.4.3)} \qquad \int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\}\ |\bar{E}^{(\rho)}(w, \pi)|\, dw < \infty$$

where ρ is an odd integer such that $1 \leqslant \rho \leqslant \alpha-1$,

$$\text{(3.4.4)} \qquad \int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \left| \int_{0}^{\pi} d\theta_{\alpha, r}(t).\, A(t) \right| dw < \infty, \text{ and}$$

$$\text{(3.4.5)} \qquad \int_{1}^{\infty} \{(\log w)^{\beta/\alpha}/we^{\alpha+\delta}(w)\} \Bigg| \sum_{n \leqslant w} \{e(w)-e(n)\}^{\alpha-1+\delta} e(n)$$

$$\times \left[-\theta_{\alpha, r}(\pi)\,(\cos n\pi/n) + \int_{0}^{\pi} (\cos nt/n)\, d\theta_{\alpha, r}(t) \right] \Bigg|\, dw < \infty.$$

In the arguments used in the proof of Theorem 2 we have only to replace $\psi_\alpha(t)$ by $\theta_{\alpha, r}(t)$ to establish these results. The case in which r is odd can be treated similarly. This completes the proof of Theorem 4.

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