ON SUMMABILITY FACTORS

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1.1. Given the improper Stieltjes integral of $f(x)$, with respect to $\alpha(x)$, in the infinite interval (a, ∞), viz.

$$\int_a^\infty f(x)\,d\alpha(x) = \lim_{R\to\infty}\int_a^R f(x)\,d\alpha(x), \tag{1.1}$$

its (C, k) mean is defined by the equation

$$C_k(\omega) = \int_a^\omega (1-x/\omega)^k f(x)\,d\alpha(x),$$

where $k > -1$.

The integral (1.1) is said to be summable (C, k) to the sum s if

$$\lim_{\omega\to\infty} C_k(\omega) = s \text{ (\textit{see} Hardy, 1949, p. 111)}$$

and if

$$C_k(\omega) = O(1),$$

as $\omega \to \infty$, the integral is said to be bounded (C, k).

Again, if $C_k(\omega)$ is of bounded variation in (h, ∞), where h may be any finite positive number, that is to say if

$$\int_h^\infty |\,dC_k(\omega)\,| < \infty\,,$$

we say that the integral (1.1) is absolutely summable (C, k), or simply summable $|\,C, k\,|$ (*see* Bosanquet, 1947 and 1948).

And, if

$$\int_h^\omega |\,C_{k-1}(x)-s\,|\,dx = o(\omega),$$

we say that the integral is strongly summable (C, k), or simply summable $[C, k]$ to sum s ; while if

$$\int_h^\omega |\,C_{k-1}(x)\,|\,dx = O(\omega),$$

we say that the integral is bounded $[C, k]$; $k > 0$. For $k = 0$, if the integral is summable (or bounded) $(C, 0)$ and

$$\int_h^\omega x\,|f(x)\,d\alpha(x)\,| = o(\omega) \text{ (or } O(\omega)),^*$$

we say that the integral is summable (or bounded) $[C, 0]$.

We observe that if, in particular, $\alpha(x)$ is a step function defined as follows:

$$\alpha(x) = 0, \text{ for } a \leqslant x \leqslant x_0,$$

$$= a_0 + a_1 \ldots + a_n, \text{ for } x_n < x \leqslant x_{n+1},$$

where

$$a \leqslant x_0 < x_1 \ldots < x_n \to \infty\,;$$

the integral (1.1) may become an infinite series, that is to say

$$\int_a^\infty f(x)\,d\alpha(x) = \sum_{n=0}^\infty a_n f(x_n) \text{ (Widder, 1941, p. 15).}$$

1.2. A sequence $\{\lambda_n\}$ of positive steadily increasing numbers tending to infinity, determines Riesz means $x^{-k}A_\lambda^k(x)$ of order k type λ, for a series of complex numbers Σa_n; the means being defined by the equation

$$C_\lambda^k(x) = x^{-k}A_\lambda^k(x) = \sum_{\lambda_n < \omega} (1-\lambda_n/x)^k a_n$$

$$= \int_0^x (1-s/x)^k dC_\lambda^0(s) \tag{1.2}$$

where $k \geqslant 0$.

If

$$C_\lambda^k(x) = O(1),$$

as $x \to \infty$, the series Σa_n is said to be bounded (R, λ, k); and if

$$\lim_{x\to\infty} C_\lambda^k(x) = s,$$

where s is finite, the series is said to be summable (R, λ, k) to sum s.

If $C_\lambda^k(x)$ is of bounded variation in (h, ∞), the series is said to be summable $|R, \lambda, k|$.

The definition (1.2) is still applicable for negative k, $k > -1$. With this extension, if

$$\int^\omega \left| C_\lambda^{k-1}(x) \right| dx = O(\omega),$$

the series is said to be bounded $[R, \lambda, k]$, and if

$$\int^\omega \left| C_\lambda^{k-1}(x) - s \right| dx = o(\omega),$$

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* The lower limit h may be any finite positive number. From now on in such cases we omit the lower limit.

the series Σa_n is said to be summable $[R, \lambda, k]$ to sum s, $k > 0$. If the series be convergent (or bounded) and

$$\int^{\omega} x \left| dC^{0}_{\lambda}(x) \right| = o(\omega) \text{ (or } O(\omega)),$$

then it is said to be summable (or bounded) $[R, \lambda, 0]$.

2.1. Summability $|R, \lambda, k|$ is known to include summability $[R, \lambda, k]$ (Srivastava, 1957, Theorem 9) and this fact is written as

$$|R, \lambda, k| \subset [R, \lambda, k].$$

The following inclusion relations also hold

$$[R, \lambda, k] \subset (R, \lambda, k),$$

$$(R, \lambda, k-1) \subset [R, \lambda, k] \text{ (Srivastava, 1957, Theorems 1 and 2).}$$

The above two relations are true for boundedness, as well, in place of summability. Winn (1933) has given an example to show that there are series bounded $[C, k]$, summable (C, k), but not summable $[C, k]$. The same example goes to show that there are series bounded $[C, k]$ but not bounded $(C, k-1)$, that is to say that there are series bounded $[R, \lambda, k]$ but not bounded $(R, \lambda, k-1)$.

It is, further, known that there are series summable (R, λ, k) but not summable absolutely for any order whatsoever (Kogbetliantz, 1925). The summability factor problem, that naturally arises, is to find a sequence $\{f_n\}$ such that if the series Σa_n is summable (R, λ, k), then the series $\Sigma a_n f_n$ may be absolutely summable for some order, say $k+1$. Several theorems, in connection with this problem, have been obtained, both for an infinite series in general, and in case of Fourier series and also the series conjugate to it in particular. In the present paper we consider this problem for general infinite series. We obtain here two theorems on summability factors involving Stieltjes integral, and then give certain deductions from these.

2.2. While dealing with the problem of obtaining absolute summability from ordinary summability with the help of summability factors, Bosanquet (1948) stated the following theorem involving Stieltjes integral from which, as is evident from the definitions, corresponding theorem for Riesz summability of infinite series can be deduced.

Theorem A. If $\lambda \geqslant 0$ *and is an integer* (i) $k(x)$ *is continuous for* $x \geqslant 1$ *and*

$$\text{(ii)} \int_1^{\infty} x^{-1} |k(x)| \, dx < \infty,$$

$$\text{(iii)} \int_1^{\infty} x^{\lambda} \left| dk^{(\lambda)}(x) \right| < \infty,$$

and if

$$\int_1^{x} d\phi(t) = O(1)(C, \lambda),$$

as $x \to \infty$, *then*

$$\int_1^{\infty} k(x) \, d\phi(x)$$

is summable $|C, \lambda+1|$.

Recently Borwein (1954) extended this theorem to general λ. Theorem 1 of the present paper, giving a summability factor $k(x)$ such that summability $|C, \lambda|$ of the factored integral follows from summability $[C, \lambda]$ of a given integral, is similar to it; and for a large class of functions $k(x)$, as can be easily seen, Theorem A can be deduced from Theorem 1. A start in this direction, that of obtaining absolute summability from strong summability of the same order, was made by Pati (1954). He proved:

Theorem B. If Σa_n is summable $[C, 1]$, and $\{\lambda_n\}$ is a convex sequence such that the series $\Sigma n^{-1}\lambda_n$ is convergent, then $\Sigma a_n\lambda_n$ is summable $|C, 1|$.

Theorem 2 is a generalization of Theorem 1, still in another direction, in the sense that in it the hypothesis of strong boundedness is replaced by strong asymptotic estimate of the given integral. Theorems 3 to 5 concern summability factors of infinite series—Theorem 3 giving, in particular, the extension of Theorem B to general positive orders, since summabilities $[R, n, k]$ and $|R, n, k|$ are equivalent to summabilities $[C, k]$ and $|C, k|$ respectively (Boyd and Hyslop, 1952; Srivastava, 1957; Hyslop, 1936).

3.1. *Theorem 1. If $\lambda \geqslant 0$ and*

$$\int^{\infty} d\alpha(x) = O(1)\ [C, \lambda],$$

then

$$\int^{\infty} k(x)\, d\alpha(x) \text{ is summable } |C, \lambda|,$$

where (i) *$k(t)$ is a continuous function,*

$$\text{(ii)} \int^{\infty} |k(t)|\, t^{-1}\, dt < \infty,$$

and (a) when λ is an integer

$$\text{(iii)} \int^{\infty} t^{\lambda}\, |dk^{(\lambda)}(t)| < \infty;$$

(b) when λ is not an integer

$$\text{(iii}') \int^{\infty} t^{[\lambda]+1}\, |dk^{([\lambda]+1)}(t)| < \infty,$$

where $[\lambda]$ is the greatest integer less than λ, and

(iv) *$|k'(t)|$ is a monotonic non-increasing function of t.*

Theorem 2. If $\lambda \geqslant 0$ and

$$\int^{\infty} d\alpha(x) = O\{\chi(x)\}\ [C, \lambda],$$

where $\chi(x) \to \infty$, as $x \to \infty$, and

$$\chi'(t) = O\{\chi(t)\, t^{-1}\},$$

then

$$\int^{\infty} k(x)\, d\alpha(x)$$

is summable $|C, \lambda|$, *where*

(i) $k(t)$ *is a continuous function,*

$$\text{(ii)} \int^{\infty} t^{-1} |k(t)\chi(t)| dt < \infty,$$

and (a) when λ *is an integer*

$$\text{(iii)} \int^{\infty} t^{\lambda} |\chi(t)\, dk^{(\lambda)}(t)| < \infty,$$

(b) when λ *is not an integer*

$$\text{(iii')} \int^{\infty} t^{[\lambda]+1} |\chi(t)\, dk^{([\lambda]+1)}(t)| < \infty,$$

and

(iv) $|k'(t)|$ *is a monotonic non-increasing function of* t.

3.2. We require the following lemmas for the proofs of Theorems 1 and 2.

*Lemma 1.** *If* $k(t)$ *is a continuous function of* t *such that*

$$\text{(i)} \int^{\infty} t^{-1} |k(t)| dt < \infty,$$

and

$$\text{(ii)} \int^{\infty} t^{n} |dk^{(n)}(t)| < \infty,$$

then

$$\int^{\infty} t^{r} |dk^{(r)}(t)| < \infty,$$

for $r = 0, 1, 2, \ldots, n-1$, *and hence also*

$$|t^{r}k^{(r)}(t)| < \infty,$$

for $r = 0, 1, 2, \ldots, n-1$.

Proof of the Lemma. From (ii) it follows that

$$\int_{c}^{\infty} dk^{(n)}(t)$$

is convergent. Therefore there is a number l such that, for $t \geqslant c$,

$$k^{(n)}(t) - l = -\int_{t}^{\infty} dk^{(n)}(t)$$
$$= o(1),$$

as $t \to \infty$. If $n = 0$, we have

$$l = \lim_{t \to \infty} k(t),$$

and hence

$$|k(t)| < \infty.$$

* *See* Borwein (1954), section 5. The Lemma is implicit in the analysis given there.

For $n \geqslant 1$,

$$l = \lim_{t\to\infty} nt^{-n} \int^{\infty} (t-u)^{n-1}k^{(n)}(u)\, du$$
$$= \lim_{t\to\infty} n!\, t^{-n}k(t).$$

In either case we deduce from the convergence of $\int^{\infty} t^{-1}|k(t)|\,dt$ that $l = 0$.

Now

$$\int^{\infty} t^{n-1}|dk^{(n-1)}(t)| = \int^{\infty} t^{n-1}|k^{(n)}(t)|\,dt$$
$$\leqslant \int^{\infty} t^{n-1}\,dt \int_t^{\infty} |dk^{(n)}(u)|$$
$$\leqslant 1/n \int^{\infty} u^{n}\,|dk^{(n)}(u)|$$
$$< \infty,$$

by hypothesis (ii). This gives the first part of the Lemma. To deduce the second part, we observe that

$$d/dt\,\{t^r k^{(r)}(t)\} = rt^{r-1}k^{(r)}(t) + t^r \frac{d}{dt}k^{(r)}(t).$$

Integrating and applying the first part of the Lemma we obtain

$$t^r k^{(r)}(t) = O(1),$$

as $t \to \infty$, for $r = 1, 2, \ldots, n-1$. That

$$k(t) = O(1),$$

follows from

$$\int^{\infty} |dk(t)| < \infty,$$

proved in the earlier part.

Lemma 2. If $k(t)$ is a continuous function of t such that

$$\text{(i)} \int^{\infty} t^{-1}|k(t)\chi(t)|\,dt < \infty,$$

$$\text{(ii)} \int^{\infty} t^{n}|\chi(t)dk^{(n)}(t)| < \infty,$$

$\chi(t) \to \infty$ and

$$\chi'(t) = O\{\chi(t).\, t^{-1}\},$$

as $t \to \infty$, then

$$\int^{\infty} t^r|\chi(t)dk^{(r)}(t)| < \infty,$$

and hence also

$$|t^r\chi(t)k^{(r)}(t)| < \infty,$$

for $r = 0, 1, 2, \ldots, n-1$.

Proof of Lemma 2. When $n = 0$,

$$\frac{d}{dt}\{\chi(t)k(t)\} = \chi(t)\frac{d}{dt}k(t)+k(t)\chi'(t)$$

$$= \chi(t)\frac{d}{dt}k(t)+k(t)\,O\{\chi(t).\ t^{-1}\}.$$

Integrating and applying hypotheses (i) and (ii) we get the result. For $n > 0$, since

$$\int^{\infty} t^n\,|\,\chi(t)dk^{(n)}(t)\,| < \infty,$$

and $\chi(t) \to \infty$, as $t \to \infty$,

$$\int^{\infty} dk^{(n)}(t)$$

is convergent. Also, since $\chi(t) \to \infty$, and

$$\int^{\infty} t^{-1}\,|\,\chi(t)k(t)\,|\,dt < \infty,$$

it follows that

$$\int^{\infty} t^{-1}\,|\,k(t)\,|\,dt < \infty.$$

From (ii), since $\chi(t) \to \infty$ and

$$\int^{\infty} dk^{(n)}(u)$$

is convergent, we obtain that there is a number l such that

$$k^{(n)}(t)-l = -\int_t^{\infty} dk^{(n)}(u) = o(1),$$

as $t \to \infty$. Also

$$l = \lim_{t\to\infty} nt^{-n}\int^{t}(t-u)^{n-1}k^{(n)}(u)\,du$$

$$= \lim_{t\to\infty} n\,!\,t^{-n}k(t).$$

Hence, from the convergence of

$$\int^{\infty} t^{-1}\,|\,k(t)\chi(t)\,|\,dt,$$

we deduce that $l = 0$, for $n \geqslant 1$. Now

$$\int^{\infty} t^r\,|\,\chi(t)dk^{(r)}(t)\,| \leqslant \int^{\infty} |\,t^r\chi(t)\,|\,dt \int_t^{\infty} |\,dk^{(r+1)}(u)\,|$$

$$= \int^{\infty} |\,dk^{(r+1)}(u)\,| \int^{u} |\,\chi(t)t^r\,|\,dt,$$

where

$$\int^{u} t^r|\,\chi(t)\,|\,dt = O\{u^{r+1}\chi(u)\}.$$

And therefore

$$\int^{\infty} t^r \,|\, \chi(t) dk^{(r)}(t) \,| = O \int^{\infty} u^{r+1} \,|\, \chi(u)\, dk^{(r+1)}(u) \,|$$
$$= \dots\dots$$
$$= O \int^{\infty} u^n \,|\, \chi(u) dk^{(n)}(u) \,|$$
$$< \infty,$$

for $r = 0, 1, \dots, n-1$.

For the second part we have

$$\frac{d}{dt}\{t^r\chi(t)k^{(r)}(t)\} = rt^{r-1}\chi(t)k^{(r)}(t)+t^r\chi'(t)k^{(r)}(t)+t^r\chi(t)\frac{d}{dt}\{k^{(r)}(t)\}.$$

It is given that

$$\chi'(t) = O\{\chi(t)\,.\,t^{-1}\}.$$

Hence, integrating and applying the first part of the Lemma,

$$|\, t^r\chi(t)k^{(r)}(t) \,| < \infty,$$

for $r = 0, 1, \dots, n-1$.

Lemma 3. Let $\lambda > 0$, then

$$\frac{d}{d\omega}\left\{\omega^{-\lambda}\int_0^{\omega}(\omega-t)^{\lambda}k(t)d\alpha(t)\right\}$$

$$= \lambda\omega^{-\lambda-1}k(\omega)\overline{C}_{\lambda}(\omega) + \frac{\Gamma(\lambda+1)}{\Gamma(h+1)\Gamma(\lambda-h-1)}\omega^{-\lambda-1}\int_0^{\omega}\overline{C}_{h+1}(t)\{k(t)-k(\omega)\} \times$$

$$(\omega-t)^{\lambda-h-2}\,dt + \frac{\omega^{-\lambda-1}}{\Gamma(h+2)}\sum_{r=1}^{h+1}(-1)^{h+1-r}\binom{h+1}{r}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-h+r-1)} \times$$

$$\int_0^{\omega}\overline{C}_{h+1}(t)k^{(r)}(t)(\omega-t)^{\lambda-h+r-2}\,dt,$$

where

$$\overline{C}_r(\omega) = \int^{\omega}(\omega-t)^{r-1}\,t d\alpha(t),$$

h denotes the greatest integer less than λ, *for* $0 < \lambda \leqslant 1$, *h is defined to be zero, and when* λ *is an integer the second term of the expression on the right is zero.*

Proof of Lemma 3.

$$\frac{d}{d\omega}\left[\omega^{-\lambda}\int_0^{\omega}(\omega-t)^{\lambda}k(t)d\alpha(t)\right]$$

$$= \lambda\omega^{-\lambda-1}\int_0^{\omega}(\omega-t)^{\lambda-1}\,tk(t)d\alpha(t)$$

$$= \lambda\omega^{-\lambda-1}k(\omega)\bar{C}_\lambda(\omega) + \lambda\omega^{-\lambda-1}\int_0^\omega \{k(t)-k(\omega)\}(\omega-t)^{\lambda-1}\, t d\alpha(t)$$

$$= \lambda\omega^{-\lambda-1}k(\omega)\bar{C}_\lambda(\omega) + (-1)^{h+1}\frac{\lambda}{\Gamma(h+1)}\omega^{-\lambda-1}\times$$

$$\int_0^\omega \bar{C}_{h+1}(t) D^{h+1}[\{k(t)-k(\omega)\}(\omega-t)^{\lambda-1}]dt,$$

by partial integration $h+1$ times. Now

$$D^{h+1}[\{k(t)-k(\omega)\}(\omega-t)^{\lambda-1}] = (-1)^{h+1}\frac{\Gamma(\lambda)}{\Gamma(\lambda-h-1)}(\omega-t)^{\lambda-h-2}\{k(t)-k(\omega)\}$$

$$+\sum_{r=1}^{h+1}(-1)^{h+1-r}\binom{h+1}{r}\frac{\Gamma(\lambda)}{\Gamma(\lambda-h+r-1)}k^{(r)}(t)(\omega-t)^{k-h+r-2},$$

the first term of the expression on the right being zero when λ is an integer. The Lemma follows upon substituting this expression for

$$D^{h+1}[\{k(t)-k(\omega)\}(\omega-t)^{\lambda-1}]$$

in the integrand.

Lemma 4. (i) *If*

$$\int^\infty d\alpha(x) = O(1)[C,\lambda],$$

then

$$\int^X \frac{|\bar{C}_l(x)|}{x^l}dx = O(x),$$

as $X\to\infty$, *for* $l \geqslant \lambda$; (ii) if

$$\int^\infty d\alpha(x) = O\{\chi(x)\}[C,\lambda],$$

then

$$\int^X \frac{|\bar{C}_l(x)|}{x^l}dx = O\{X\chi(X)\},$$

as $X\to\infty$, *for* $l \geqslant \lambda$, *where* $x\,\chi(x)$ *or* $x^k\,\chi(x)$, *according as* $k \gtrless 1$, *tends to infinity with* x *and*

$$\chi'(x) = O\{\chi(x)\,.\,x^{-1}\}\,.$$

The first part of the above Lemma has been proved elsewhere (Srivastava, 1957; §5), the second part can be obtained in a similar fashion.

3.3. *Proof of Theorem 1.* Firstly, we observe that

$$x^{-l}\bar{C}_l(x) = x\frac{d}{dx}\{C_l(x)\};$$

therefore, by Lemma 4 (i), under the hypothesis,

$$\int^X xdx\left|\frac{d}{dx}C_l(x)\right| = O(x),$$

as $X\to\infty$, where $l > \lambda > 0$.

It is to be proved that

$$\int^{\infty} |dF(x)| < \infty,$$

that is to say that

$$\lim_{X\to\infty} \int^{X} |dF(x)| < \infty,$$

where

$$F(\omega) = \int_0^{\omega} \left(1-\frac{x}{\omega}\right)^{\lambda} k(x)d\alpha(x).$$

When $\lambda = 0$, the required result reduces to

$$I = \int^{X} |k(x)d\alpha(x)| = O(1),$$

as $X \to \infty$. Integration by parts gives

$$I = \int^{X} \frac{|k(x)|}{x} |xd\alpha(x)|$$

$$= [O(x) \,.\, k(x)/x]^{X} + O\int^{X} x\left|\frac{k(x)}{x} - \frac{k(x)}{x^2}\right| dx$$

$$= O(1) + O\int^{X} |k'(x)|\,dx + O\int^{X} x^{-1}|k(x)|\,dx$$

$$= O(1),$$

using hypotheses (ii) and (iii).

For $\lambda > 0$, by Lemma 3,

$$\int^{X} |dF(\omega)| = \int^{X} \left|\sum_{p=1}^{h+3} I_p(\omega)\right| d\omega,$$

say, where

$$I_1(\omega) = \lambda\omega^{-\lambda-1}k(\omega)\overline{C}_{\lambda}(\omega),$$

$$I_2(\omega) = A\omega^{-\lambda-1}\int_0^{\omega} \overline{C}_{h+1}(t)\{k(t)-k(\omega)\}(\omega-t)^{\lambda-h-2}\,dt$$

$$I_{p+2}(\omega) = A\omega^{-\lambda-1}\int_0^{\omega} \overline{C}_{h+1}(t)(\omega-t)^{\lambda-h+p-2}k^{(p)}(t)\,dt,$$

$p = 1, 2, \ldots, h+1$; I_2 being zero when λ is an integer, A is a constant may be different at different places. It follows that it is sufficient to prove

$$\int^{X} |I_p(\omega)|\,d\omega = O(1), \tag{3.3.1}$$

as $X \to \infty$, for each $p = 1, 2, \ldots, h+3$.

Integrating by parts and making use of Lemma 3, we have

$$\int^X |I_1(\omega)|\,d\omega = \lambda \int^X \frac{|\bar{C}_\lambda(\omega)|}{\omega^\lambda}\frac{|k(\omega)|}{\omega}\,d\omega$$

$$= \left[O(x)\,.\,\frac{k(x)}{x}\right]^X + O\int^X \frac{|k(\omega)|}{\omega}\,d\omega + O\int^X |k'(\omega)|\,d\omega$$

(3.3.2) $$= O(1),$$

as $X \to \infty$, by hypotheses (ii) and (iii) and Lemma 1.

Now, we consider the integrals involving $I_{p+2}(\omega)$, where $p = 1, 2, \ldots, h+1$.

$$\int^X |I_{p+2}(\omega)|\,d\omega = A\int^X \omega^{-\lambda-1}d\omega \left|\int^\omega \bar{C}_{h+1}(t)(\omega-t)^{\lambda-h+p-2}k^{(p)}(t)\,dt\right|$$

$$\leqslant A\int^X \omega^{-\lambda-1}d\omega \int^\omega |\bar{C}_{h+1}(t)|\,|k^{(p)}(t)|\,(\omega-t)^{\lambda-h+p-2}\,dt.$$

Changing the order of integration we get

$$\int^X I_{p+2}(\omega)|\,d\omega \leqslant A\int^X |\bar{C}_{h+1}(t)k^{(p)}(t)|\,dt \int_t^\infty \frac{(\omega-t)^{\lambda-h+p-2}}{\omega^{\lambda+1}}\,d\omega$$

$$\leqslant A\int^X |\bar{C}_{h+1}(t)k^{(p)}(t)|\,t^{-h+p-2}\,dt$$

$$= A\int^X \frac{|\bar{C}_{h+1}(t)|}{t^{h+1}}\,t^{p-1}|k^{(p)}(t)|\,dt.$$

By hypothesis (iii) or (iii′), according as λ is or is not an integer, and Lemmas 1 and 4.

$$\int^X |I_{p+2}(\omega)|\,d\omega = [O(x)x^{p-1}k^{(p)}(x)]X + O\int^X t^p\,|dk^{(p)}(t)|$$

$$+O\int^X t^{p-1}|k^{(p)}(t)|\,dt$$

(3.3.3) $$= O(1),$$

as $X \to \infty$.

(3.3.1), (3.3.2) and (3.3.3) show the Theorem to be true for an integer. When λ is not an integer, we have to consider one more integral involving $I_2(\omega)$. We obtain

$$\int^X |I_2(\omega)|\,d\omega$$

$$= A\int^X \omega^{-\lambda-1}\,d\omega \left|\int^\omega \bar{C}_{h+1}(t)\{k(t)-k(\omega)\}(\omega-t)^{\lambda-h-2}\,dt\right|$$

$$\leqslant A\int^X \omega^{-\lambda-1}\,d\omega \left|\int^\omega |\bar{C}_{h+1}(t)|\frac{|k(t)-k(\omega)|}{\omega-t}(\omega-t)^{\lambda-h-1}\,dt\right.$$

$$< \int^X \omega^{-\lambda-1}\,d\omega \int^\omega |\bar{C}_{h+1}(t)|\,|k'(t)|\,(\omega-t)^{\lambda-h-1}\,dt,$$

by condition (iv). Changing the order of integration, using Lemma 1 and condition (iii′), we get

$$\int^X |I_2(\omega)|\,d\omega < A\int^X |\bar{C}_{h+1}(t)||k'(t)|\,dt\int_t^\infty \frac{(\omega-t)^{\lambda-h-1}}{\omega^{\lambda+1}}\,d\omega$$

$$= O\int^X |\bar{C}_{h+1}(t)||k'(t)|\,t^{-h-1}\,dt$$

$$= O\int^X |\bar{C}_{h+1}(t)/t^{h+1}||k'(t)|\,dt$$

$$= [O(x)\,.\,k'(x)]^X + O\int^X t\,|dk'(t)|$$

$$= O(1).$$

This completes the proof of Theorem 1.

3.4. *Proof of Theorem 2.* Using Lemma 2, instead of Lemma 1, and Lemma 4 (ii), in place of Lemma 4 (i), Theorem 2 follows exactly by the procedure of the proof of Theorem 1.

3.5. We now obtain the following theorems.

Theorem 3. If Σa_n be bounded $[R, \lambda, k]$ and $\phi(t)$ satisfies the same conditions as $k(t)$ in Theorem 1 (for $\lambda_n = n$, $\phi(t)$ may satisfy the following conditions

$$\text{(i)}\quad \sum^{\infty} n^{-1}\,|\,\phi(n)\,| < \infty,$$

$$\text{(ii)}\quad \sum^{\infty} n^{1+h}\,|\,\Delta^{h+2}\phi(n)\,| < \infty,$$

where h is the greatest integer less than k, and when k is non-integral, also $|\,\Delta\phi(n)\,|$ is non-increasing), then the series $\Sigma a_n\phi(\lambda_n)$ is summable $|R, \lambda, k\,|$, for $k \geqslant 0$.

Theorem 4. If

$$\Sigma a_n = O\{\chi(n)\}\ [R, \lambda, k],$$

where $\chi(n) \to \infty$ with n, and $\chi(t)$, $\phi(t)$ satisfy the same conditions as $\chi(t)$, $k(t)$ in Theorem 2 (for $\lambda_n = n$, $\chi(t)$ and $\phi(t)$ may be such that

$$\Delta\,\chi(n) = O\{\chi(n)\,.\,n^{-1}\},$$

and the sequence $\{\phi(n)\}$ satisfies

$$\text{(i)}\quad \sum^{\infty} n^{-1}\,|\,\chi(n)\phi(n)\,| < \infty,$$

$$\text{(ii)}\quad \sum^{\infty} n^{1+h}\,|\,\chi(n)\,\Delta^{h+2}\phi(n)\,| < \infty,$$

where h is the greatest integer less than k, and when k is non-integral $|\,\Delta\phi(n)\,|$ is non-increasing), then the series $\Sigma a_n\phi(\lambda_n)$ is summable $|\,R, \lambda, k\,|$, $k > 0$.

Theorem 5. If Σa_n is bounded $[R, \lambda, k]$, then $\Sigma a_n\lambda_n^{-k-\epsilon}$, where $\epsilon > 0$, is summable $|\,R, \mu, k\,|$, where $\mu_n = \exp\,(\lambda_n)$. The factor $(\lambda_n)^{-k-\epsilon}$ may, also, be replaced by $\phi(\lambda_n)/(\lambda_n)^k$, where $\{\phi(t)\}$ satisfies the conditions of Theorem 3.

Theorems 3 and 4 follow directly from Theorems 1 and 2. Theorem 5 is obtained from Theorem 3 by an application of a Theorem due to Tatchell (1954), which we state below as a Lemma.

Lemma 5. If Σa_n *is summable* $|R, \lambda, k|$, *then* $\Sigma a_n \lambda_n^{-k}$ *is summable* $|R, \mu, k|$, *where* $\mu_n = \exp(\lambda_n)$, $k \geqslant 0$.

3.6. It may be pointed out that, as is evident from the proof of Theorem 1 given, the hypothesis in Theorems 3, 4 and 5 concerning the boundedness $[R, \lambda, k]$ of the series Σa_n can be replaced by less restrictive conditions of boundedness $[R, \lambda, k+1]$ of the sequence $\{\lambda_n a_n\}$, $-(R, \lambda, k)$ mean of the sequence $\{\lambda_n a_n\}$ is supposed to be given by

$$\sum_{\lambda_n < \omega} (\omega - \lambda_n)^{k-1} \lambda_n a_n.$$

It may also be observed here that in case $\phi(n)$ is a logarithmico exponential function of n, or simply an L-function, the conditions on $\phi(n)$ in the preceding Theorems 3 and 5 reduce to the single condition of convergence of $\Sigma n^{-1}\phi(n)$. This follows immediately from the Lemma given below.

Lemma 6. If $\Sigma n^{-1}\phi(n)$ *is convergent and* $\phi(n)$ *is an L-function of n, then*

$$n^r \Delta^r \phi(n) \to 0,$$

and

$$\Sigma n^r \,|\, \Delta^{r+1} \phi(n) \,|$$

is convergent.

Proof of Lemma 6. Since $\phi(n)$ is an L-function, $\Delta^r\phi(n)$ is monotonic for all values of r from a certain value of n onwards, we may take $\Delta^r\phi(n)$ to be monotonic for all values of n, and hence the convergence of

$$\Sigma n^r \,|\, \Delta^{r+1} \phi(n) \,|$$

reduces to the convergence of

$$\Sigma n^r \Delta^{r+1} \phi(n).$$

We suppose that the required result is true for $r = k$, that is

$$n^k \Delta^k \phi(n) \to 0,$$

and the series

$$\Sigma n^k \Delta^{k+1} \phi(n)$$

is convergent. Now since the terms of the convergent series have the same sign

$$n^{k+1} \Delta^{k+1} \phi(n) \to 0,$$

as $n \to \infty$. Also

$$\begin{aligned} \sum_{n=1}^{N} n^{k+1} \Delta^{k+2} \phi(n) &= \sum_{n=1}^{N} n^{k+1} \{ \Delta^{k+1}\phi(n) - \Delta^{k+1}\phi(n+1) \} \\ &= \sum^{N} \Delta^{k+1}\phi(n) \{ n^{k+1} - (n-1)^{k+1} \} \\ &\qquad - N^{k+1} \Delta^{k+1} \phi(N+1) \\ &\sim n^k \Delta^{k+1}\phi(n) - (N+1)^{k+1} \Delta^{k+1} \phi(N+1). \end{aligned}$$

Therefore

$$\Sigma n^{k+1} \Delta^{k+2}\phi(n)$$

is convergent. That the required result is true for $k = 0$ follows from the convergence of $\Sigma n^{-1}\phi(n)$.

And this completes the proof of the Lemma.

Further, if $\chi(n)$ be also supposed to be an L-function, then the conditions of Theorem 4 reduce to the single condition that $\Sigma n^{-1}\chi(n)\phi(n)$ is convergent.

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