THE SUMMABILITY FACTORS OF A FOURIER SERIES AND THE SERIES CONJUGATE TO IT

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1.1. Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. The Fourier series of $f(t)$ is given by

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$
$$= \sum_{n=0}^{\infty} A_n(t).$$

Then the conjugate series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt)$$
$$= \sum_{n=1}^{\infty} B_n(t).$$

We write

$$\phi(t) = (f(x+t) + f(x-t) - 2s)/2,$$
$$\psi(t) = (f(x+t) - f(x-t))/2.$$

We also write

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du, \ \alpha > 0,$$
$$\Phi_0(t) = \phi(t),$$

and

$$\phi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Phi_\alpha(t).$$

We employ $\Psi_\alpha(t)$ and $\psi_\alpha(t)$ with similar meanings.

1.2. In our earlier paper entitled 'On summability factors'* we have investigated summability factors for Stieltjes integrals and general infinite series. The object of the present paper, which forms a sequel to the same, is to consider similar problem for the Fourier series and the series conjugate to it. We also make use of the notations and definitions of Paper 1.

For the absolute summability of the Fourier series it is known that if $\phi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the Fourier series $\Sigma A_n(t)$, at $t = x$, is summable

* We refer to this paper as Paper 1.

$|C, \beta|$, $\beta > \alpha$ $(\beta \neq \alpha)$ (Bosanquet, 1936*a*, *b*). However, for $\beta = \alpha$, Cheng (1948*b*) obtained the following theorem.

THEOREM A. *If* $\phi_\alpha(t)$ *is of bounded variation in* $(0, \pi)$, $0 \leqslant \alpha \leqslant 1$, *then* $\Sigma A_n(x)(\log n)^{-1-\epsilon}$, $\epsilon > 0$, *is summable* $|C, \alpha|$.

The result corresponding to the above for the conjugate series $\Sigma B_n(t)$, established by Wang (1949), is as follows.

THEOREM B. *For* $0 \leqslant \alpha \leqslant 1$, *if* (i) $\Psi_\alpha(+0) = 0$ *and* (ii) $\int_0^\pi t^{-\alpha} |d\Psi_\alpha(t)| < \infty$, *then* $\Sigma B_n(x)(\log n)^{-1-\epsilon}$, $\epsilon > 0$ *is summable* $|C, \alpha|$.

Extension of Theorem A to the case $\alpha > 1$ has been recently given by Sunouchi (1954). We obtain in Theorem 1 of this paper a similar extension of Theorem B with a further improvement, viz. that the multiplying factor $(\log n)^{-1-\epsilon}$ is replaced by a wider class of functions. Also, it is pointed out that the same improvement is admissible in case of Fourier series.

Two other summability factor theorems proved by Cheng are as given below.

THEOREM C (Cheng, 1948*a*). *The Fourier series* $\Sigma A_n(t)$ *and its conjugate series* $\Sigma B_n(t)$, *when multiplied by one of the factors*

(i)
$$\left\{\frac{1}{(\log n)^{1+\epsilon}}\right\}, \left\{\frac{1}{\log n\ (\log\log n)^{1+\epsilon}}\right\}, \ldots\ldots,$$
$$\left\{\frac{1}{\log n\,.\log\log n\ldots\ldots(\log\log\ldots.\log n)^{1+\epsilon}}\right\}, \epsilon > 0,$$

are summable $|C, \alpha|$, $\alpha > 1$, *at the point* $t = x$, *whenever*

$$\int_0^t |\phi(u)|\,du = o(t)$$

and

$$\int_0^t |\psi(u)|\,du = o(t),$$

as $t \to 0$, *hold respectively.*

THEOREM D (Cheng, 1947). *If*

$$\int_0^t |\phi(u)|\,du = O\{t(\log 1/t)^\beta\},$$

$\beta \geqslant 0$, *then the series* $\Sigma A_n(x)\lambda^n$ *is summable* $|C, 1|$, *where* $\lambda_n = (\log n)^{-1-\beta-\frac{1}{2}-\epsilon}$.

For the case $\beta = 0$, he showed later that $\{\lambda_n\}$ may be any one of the sequences given in Theorem C (Cheng, 1948*a*). He also pointed out the truth of a similar result for the conjugate series. Pati (1954) has extended the part concerning Fourier series of Theorem C to the case in which the multiplying factor is λ_n, where $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent. In Theorem 2 of this paper the corresponding result for the conjugate series is given. Further the proof given suggests an alternative proof for the case of the Fourier series as well.

Quite recently Prasad and Bhatt (1957) have shown that in Theorem D, we can, instead, take λ_n such that the sequence $\{[\log (n+1)]^{\frac{1}{2}+\beta}\lambda_n\}$ is convex and the series $\Sigma n^{-1} \{\log (n+1)\}^{\frac{1}{2}+\beta}\lambda_n$ is convergent. Theorem 3 extends the result of Theorem D to the case $\beta \geqslant -\frac{1}{2}$, and for $\beta > -\frac{1}{2}$ with the generalization due to Prasad and Bhatt, and also gives the corresponding result for the conjugate series.

Theorems 2 and 3 of this paper and the other results, quoted here, start with order conditions on

$$\int_0^t |\phi (u)|\, du,$$

or on

$$\int_0^t |\psi(u)|\, du$$

according as we consider the Fourier series or the series conjugate to it, and obtain summability $|C, \alpha|$, $\alpha \geqslant 1$, of the factored series. In Theorems 4 and 5 we establish similar results, for positive orders in general. In fact we start in these with order conditions on

$$\int_0^t |\phi_\alpha(u)|\, du$$

and

$$\int_0^t |\psi_\alpha(u)|\, du,$$

$\alpha > 0$, and arrive at summability $|C, \delta|$, $\delta \geqslant \alpha+1$, of the factored Fourier series and the conjugate series respectively.

2. We require the following lemmas for the proof of the subsequent theorems.

2.1. LEMMA 1 (see Bosanquet and Chow, 1957; Theorems 2 and Y). *If the sequence* $\{na_n\}$ *is bounded* $(C, \alpha+1)$, *then the series* $\Sigma a_n\lambda_n$ *is summable* $|C, \alpha+1|$, *where* $|\Delta\lambda_n|$ *is non-increasing, the series* $\Sigma n^{-1} |\lambda_n|$ *and*

$$\Sigma n^{[\alpha]+1} |\Delta^{[\alpha]+2}\lambda_n|$$

are convergent.

This lemma is a slightly different version of a theorem of Borwein (1954) for integrals. The corresponding v ersion of Borwein's theorem can also be obtained by the method of Theorem 1 of Paper 1.

2.2. LEMMA 2 (Chow, 1941; Lemmas III and IV). *If* $\{\lambda_n\}$ *is a convex sequence such that the series* $\Sigma n^{-1} \lambda_n$ *is convergent, then*

$$n\Delta\lambda_n \to 0,$$

and

$$\Sigma(n+1)\Delta^2\lambda_n$$

is convergent.

2.3. LEMMA 3 (Bosanquet, 1936*b*). *If* $g^{\alpha}(n, t)$ *be the n-th Cesàro mean of the sequence* $\{2\pi^{-1}\sin nt\}$, *then*

$$\left(\frac{d}{dt}\right)^{\lambda} g^{\alpha}(nt) = O(n^{\lambda}) \min\{1, (nt)^{-\alpha}\}, \ \alpha < \lambda+1,$$
$$= O(n^{-1}t^{-(\lambda+1)}), \ 0 \leqslant \lambda \leqslant \alpha-1.$$

2.4. LEMMA 4. *If, for* $\alpha \geqslant 0$, $\Psi_{\alpha}(+0) = 0$ *and*

$$\int_0^{\pi} t^{-\alpha} |d\Psi_{\alpha}(t)| < \infty,$$

then the sequence $\{nB_n(t)\}$, *at* $t = x$, *is bounded* (C, α).

Proof. We have

$$nB_n(x) = \frac{2}{\pi}\int_0^{\pi} \psi(t)\frac{d}{dt}\sin nt.$$

If we denote by τ_n^{α} the (C, α) mean of the sequence $\{nB_n(x)\}$, then

$$\tau_n^{\alpha} = \int_0^{\pi} \psi(t)\frac{d}{dt}g^{\alpha}(n, t)\, dt.$$

On integration by parts, we have

$$\int_0^{\pi} \psi(t)\frac{d}{dt}g^{\alpha}(n, t)\, dt$$
$$= \left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Psi_{\rho}(t)\left(\frac{d}{dt}\right)^{\rho} g^{\alpha}(n, t)\right]_{t=0}^{\pi} - (-1)^{h}\int_0^{\pi}\Psi_h(t)\left(\frac{d}{dt}\right)^{h+1} g^{\alpha}(n, t)\, dt,$$

where h is the greatest integer less than α. Also

$$\int_0^{\pi}\Psi_h(t)\left(\frac{d}{dt}\right)^{h+1} g^{\alpha}(n, t)\, dt$$
$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^{\pi}\left(\frac{d}{dt}\right)^{h+1} g^{\alpha}(n, t)\, dt \int_0^{t}(t-u)^{h-\alpha}\, d\Psi_{\alpha}(u)$$
$$= \frac{1}{\Gamma(1+h-\alpha)}\int_0^{\pi} d\Psi_{\alpha}(u)\int_u^{\pi}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1} g^{\alpha}(n, t)$$
$$= \int_0^{\pi} J(n, u)\, d\Psi_{\alpha}(u),$$

say. Now, applying Lemma 3, we obtain

$$\left[\sum_{\rho=1}^{h} \Psi_\rho(t)(-1)^{\rho-1}\left(\frac{d}{dt}\right)^\rho g^\alpha(n,t)\right]_{t=0}^{\pi}$$

$$\leqslant \sum_{\rho=1}^{h}\left|\Psi_\rho(t)\left(\frac{d}{dt}\right)^\rho g^\alpha(n,t)\right|_{t=\pi}$$

$$\leqslant \Psi_h(\pi)\,Kn^h(1+n\pi)^{-\alpha}+\sum_{\rho=1}^{h-1} O(n^{-1})$$

$$=O(1).$$

We are left with

$$\int_0^\pi J(n,u)\,d\Psi_\alpha(u).$$

Now

$$J(n,u)=K\left\{\int_u^{u+n^{-1}}+\int_{u+n^{-1}}^{\pi}\right\}(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\,dt$$

$$=\int_u^{u+n^{-1}} O(n^{h+1})(1+nt)^{-\alpha}(t-u)^{h-\alpha}\,dt$$

$$+\int_{u+n^{-1}}^{\pi} K(t-u)^{h-\alpha}\left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\,dt$$

$$=O\{n^{h+1}(1+nu)^{-\alpha}\,n^{-(h-\alpha+1)}\}$$

$$+n^{-(h-\alpha)}K\int_\xi^\pi\left(\frac{d}{dt}\right)^{h+1} g^\alpha(n,t)\,dt,\ u+n^{-1}<\xi<\pi,$$

$$=O\{(n+u)^{-\alpha}\}+n^{-(h-\alpha)}\,O\{n^h(1+nt)^{-\alpha}\}_{t=\xi}^{\pi}$$

$$=O(u^{-\alpha}).$$

Therefore, by the hypothesis,

$$\int_0^\pi J(n,u)\,d\Psi_\alpha(u)=O\int_0^\pi u^{-\alpha}\,|\,d\Psi_\alpha(u)\,|<\infty.$$

This completes the proof of the Lemma.

2.5. LEMMA 5 (Srivastava*; Theorem 3). *If*

$$\int_0^t |\phi_\alpha(u)|\,du=O\{t(\log 1/t)^\beta\},$$

as $t\to 0$, *then for* $\beta>-\frac{1}{2}$

$$\Sigma A_n(t)=O\{(\log n)^{2\beta+1}\}\ [R,n,\alpha+1,2]$$

*Srivastava, P. Strong summability of Fourier series and the series conjugate to it. (In course of publication in the *Proc. Nat. Acad. Sc. India.*)

and for $\beta = -\frac{1}{2}$

$$\Sigma A_n(t) = O(\log\log n)\ [R, n, \alpha+1, 2].$$

2.6. LEMMA 6 (Srivastava*; Theorem 6). *If*

$$\int_0^t |\psi_\alpha(u)|\, du = O\{t(\log 1/t)^\beta\},$$

as $t \to 0$, *then for* $\beta > -\frac{1}{2}$

$$\{nB_n(t)\} = O\{(\log n)^{2\beta+1}\}\ [R, n, \alpha+2, 2]$$

and for $\beta = -\frac{1}{2}$

$$\{nB_n(t)\} = O(\log\log n)\ [R, n, \alpha+2, 2].$$

2.7. LEMMA 7 (Bosanquet, 1934; Theorem 5). *If* $\alpha \geqslant 0$, *and*

$$\int_0^t |\phi_\alpha(u)|\, du = O(t),$$

as $t \to 0$, *then*

$$\{nA_n(t)\} = O(1)(C, \alpha+1+\delta),\ \delta > 0,$$

at $t = x$.

2.8. LEMMA 8 (Bosanquet, 1934; Theorem 5*a*). *If* $\alpha \geqslant 0$, *and*

$$\int_0^t |\psi_\alpha(u)|\, du = O(t),$$

as $t \to 0$, *then*

$$\{nB_n(t)\} = O(1)(C, \alpha+1+\delta),\ \delta > 0,$$

at $t = x$.

3.1. It is easy to see that Lemmas 1 and 4 taken together yield the following theorem.

THEOREM 1. *For* $\alpha > 0$, *if* $\Psi_\alpha(+0) = 0$,

$$\int_0^\pi t^{-\alpha} |d\Psi_\alpha(t)| < \infty,$$

then $\Sigma B_n(x)\lambda_n$ *is summable* $|C, \alpha|$, *where the sequence* $\{\lambda_n\}$ *satisfies the conditions of Lemma 1, with* α *in place of* $\alpha+1$.

In particular $\{\lambda_n\}$ may be any one of the sequences (i) of Theorem C; and, in case $\alpha \leqslant 2$, $\{\lambda_n\}$ may be taken to be a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent, as is evident by Lemma 2. Since, for $\alpha > 1$, the bounded variation of $\phi_\alpha(t)$ implies summability $(C, \alpha-1)$ of the Fourier series $\Sigma A_n(t)$ at $t = x$ (Bosanquet, 1934; footnote to Theorem 4), it follows that the result of Sunouchi (1954) also admits

* Srivastava, P. Strong summability of Fourier series and the series conjugate to it. (In course of publication in the *Proc. Nat. Acad. Sc. India.*)

similar generalization for the multiplying factor $\{\lambda_n\}$. That for the Fourier series, also when $0 \leqslant \alpha < 1$, the same is true has been shown by Prasad and Bhatt (1957, Theorem 5).

3.2. Applying Lemma 1 to Lemma 8, with $\alpha = 0$, and then making use of Lemma 2, we obtain

THEOREM 2. *If $\{\lambda_n\}$ is a convex sequence such that the series $\Sigma n^{-1}\lambda_n$ is convergent, then the series $\Sigma B_n(t)\lambda_n$, at $t = x$, is summable $|C, \alpha|$ for every $\alpha > 1$, provided that*

$$\int_0^t |\psi(u)|\, du = O(t).$$

This also suggests an alternative proof of the corresponding result for the Fourier series, namely Theorem A.

3.3. An application of Lemmas 5 and 6, for $\alpha = 0$, to Theorem 4 of Paper 1 considered along with Lemma 2 gives

THEOREM 3. *If*

$$\{\log (n+1)^{\frac{1}{2}+\beta}\lambda_n\}$$

is a convex sequence such that the series

$$\Sigma n^{-1} \{\log (n+1)\}^{\frac{1}{2}+\beta}\lambda_n$$

is convergent, then the Fourier series $\Sigma A_n(t)$ and the conjugate series $\Sigma B_n(t)$, multiplied by the factor λ_n, are summable $|C, 1|$, at the point $t = x$, provided

$$\int_0^t |\phi(u)|\, du = O\{t\, (\log 1/t)^{\beta}\},$$

$$\int_0^t |\psi(u)|\, du = O\{t\, (\log 1/t)^{\beta}\},$$

hold respectively, for $\beta > -\frac{1}{2}$. For $\beta = -\frac{1}{2}$, $\{\lambda_n\}$ may be any one of the sequences (i) *of Theorem B.*

3.4. THEOREM 4. *If*

$$\int_0^t |\phi_\alpha(u)|\, du = O\{t\, (\log 1/t)^{\beta}\},$$

then the series $\Sigma A_n(x)\lambda_n$ is summable $|C, \alpha+1|$ for $\beta \geqslant -\frac{1}{2}$, where $\lambda_n = k(n)\{\log\ (n+1)\}^{-\frac{1}{2}-\beta}$, $\{k(n)\}$ being any one of the sequences (i) *of Theorem B. For $\beta > -\frac{1}{2}$, $\{\lambda_n\}$ may be such that*

$$\Sigma n^{-1} (\log n)^{\frac{1}{2}+\beta} |\lambda_n|$$

is convergent,

$$|\Delta\lambda_n|$$

is non-increasing and

$$\Sigma n^{[\alpha]+2} (\log n)^{\frac{1}{2}+\beta} |\Delta^{[\alpha]+3}\lambda_n| < \infty.$$

The corresponding results hold for the conjugate series.

The above theorem holds by virtue of Lemmas 5 and 6 and Theorem 4 of Paper 1.

3.5. Finally, by Lemmas 1, 7 and 8, we obtain

THEOREM 5. *If*

$$\int_0^t |\phi_\alpha(u)|\,du = O(t),$$

$\alpha \geqslant 0$, *then the series* $\Sigma A_n(x)\lambda_n$ *is summable* $|C, \delta|$ *for* $\delta > \alpha+1$, *where* $\{\lambda_n\}$ *is such that the series*

$$\Sigma n^{-1}\,|\lambda_n|$$

is convergent,

$$\Sigma\, n^{[\delta]}\,|\,\Delta^{[\delta]+1}\lambda_n| < \infty$$

and

$$|\,\Delta\lambda_n|$$

is non-increasing. A similar result holds for the conjugate series $\Sigma B_n(x)$.

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