A NOTE ON THE SOLUTION OF THE EQUATIONS OF INTERNAL BALLISTICS FOR THE GENERAL LINEAR LAW OF BURNING

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Abstract

In the present paper, we have given a fifth method of integrating the differential equations of internal ballistics for the general linear law of burning and compared it with the other four methods given earlier.

1. Introduction

In a recent paper, Kapur (1956) has given four methods of solving the equations of internal ballistics when the rate of burning is a linear function of the pressure. In the present paper, we propose a fifth method. From the point of view of numerical integration, the basic differential equation of this method is simpler than the corresponding differential equations of the other four methods, but it suffers from the defect that the study of the effect of the term βp_1 in the general linear law of burning, made possible by the other four methods, is not possible with this method.

In the discussion of the present method, as of others, we use non-dimensional variables and constants, for convenience of application.

2. The Basic Equations

The basic differential equations for the general linear law of burning are: (Kapur, 1956)

$$z = \zeta\xi + \tfrac{1}{2}(\gamma-1)\frac{\eta^2}{M} \quad \ldots \quad (1)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \quad \ldots \quad (2)$$

$$\eta\frac{df}{d\xi} = -(\zeta+\zeta_1) \quad \ldots \quad (3)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (4)$$

where

$$\xi = 1+\frac{x}{l} \quad \ldots \quad (5)$$

$$\eta = \frac{AD}{FC\beta}v \quad \ldots \quad (6)$$

$$\zeta = \frac{Al}{FC}p,\ \zeta_1 = \frac{Al}{FC}p_1 \quad \ldots \quad (7)$$

$$M = \frac{A^2D^2}{FC\beta^2 w_1} \quad \ldots \quad (8)$$

Now we define

$$\tau = \frac{\beta}{D}\frac{FC}{Al}t \qquad \text{(9)}$$

so that τ is the dimensionless variable corresponding to time and

$$\frac{d\xi}{d\tau} = \frac{1}{l}\frac{dx}{\frac{\beta}{D}\frac{FC}{Al}dt} = \frac{AD}{FC\beta}\frac{dx}{dt} = \frac{AD}{FC\beta}v = \eta, \qquad \text{(10)}$$

so that the basic equations can be written as:

$$(1-f)(1+\theta f) = \zeta\xi + \tfrac{1}{2}(\gamma-1)\frac{\eta^2}{M} \qquad \text{(11)}$$

$$\frac{d\eta}{d\tau} = M\zeta \qquad \text{(12)}$$

$$\frac{df}{d\tau} = -(\zeta+\zeta_1) \qquad \text{(13)}$$

Before the shot-starts, the equations are:

$$(1-f)(1+\theta f) = \zeta \qquad \text{(14)}$$

$$\frac{df}{d\tau} = -(\zeta+\zeta_1) \qquad \text{(15)}$$

From (14) and (15)

$$\int_1^{f_0} \frac{df}{(1-f)(1+\theta f)+\zeta_1} = -\int_0^{\tau_0} d\tau, \qquad \text{(16)}$$

where f_0 and τ_0 are the values of f and τ at shot-start.

Also from (14), at shot-start,

$$(1-f_0)(1+\theta f_0) = \zeta_0 = \frac{Al}{FC}p_0 \qquad \text{(17)}$$

From (16) and (17), we determine f_0 and τ_0 in terms of shot-start pressure.

3. Motion after Shot-start

From (12) and (15)

$$\frac{df}{d\tau} = -\left(\frac{1}{M}\frac{d\eta}{d\tau}+\zeta_1\right)$$

Integrating

$$f = f_0 - \frac{\eta}{M} - \zeta_1(\tau-\tau_0) \qquad \text{(18)}$$

Using (11) and simplifying, we get

$$\xi\frac{d^2\xi}{d\tau^2} + \left[\frac{\theta}{M}+\frac{1}{2}(\gamma-1)\right]\left(\frac{d\xi}{d\tau}\right)^2 - \frac{d\xi}{d\tau}\left[1-\theta+2\theta f_0-2\theta\zeta_1(\tau-\tau_0)\right]$$
$$-M[1-f_0+\zeta_1(\tau-\tau_0)][1+\theta f_0-\theta\zeta_1(\tau-\tau_0)] = 0 \qquad \text{(19)}$$

This is the basic differential equation to be integrated numerically subject to the initial condition

$$\tau = \tau_0, \ \xi = 1, \ \frac{d\xi}{d\tau} = 0 \qquad \qquad (20)$$

The numerical integration of (19) will determine ξ and $\frac{d\xi}{d\tau}$ (or η) as functions of time. (18) and (19) will determine then respectively f and ζ as functions of time. We can then easily determine the maximum pressure and the all-burnt position. The motion after all-burnt is discussed as in the other four methods.

4. Comparison with Other Methods

As compared with the other four methods, the present method has the following *advantages :*

(i) From the point of view of numerical integration, the differential equation (19) is simpler than the corresponding differential equation of the other four methods, and *therefore if it is a question of just considering the numerical integration for the general linear law, the method of this paper is definitely superior to other methods.*

(ii) The present method can be easily extended to the cubic form-function.

$$z = (1-f)(1+\theta f+\psi f^2),$$

for which the basic differential equation is

$$\begin{aligned}\left[1-f_0+\frac{1}{M}\frac{d\xi}{d\tau}+\zeta_1(\tau-\tau_0)\right] &\\ \times\left[1+\theta f_0-\frac{\theta}{M}\frac{d\xi}{d\tau}-\theta\zeta_1(\tau-\tau_0)\right.&\\ \left.+\psi\left(f_0-\frac{1}{M}\frac{d\xi}{d\tau}-\zeta_1(\tau-\tau_0)\right)^2\right] & \qquad (21)\\ =\xi\frac{1}{M}\frac{d^2\xi}{d\tau^2}+\tfrac{1}{2}\frac{\gamma-1}{M}\left(\frac{d\xi}{d\tau}\right)^2 &\end{aligned}$$

and even to the general form-function

$$z = \phi(f)$$

when the corresponding differential equation is

$$M\phi\left[f_0-\frac{1}{M}\frac{d\xi}{d\tau}-\zeta_1(\tau-\tau_0)\right] = \xi\frac{d^2\xi}{d\tau^2}+\tfrac{1}{2}(\gamma-1)\left(\frac{d\xi}{d\tau}\right)^2 \qquad (22)$$

On the other hand, the present method suffers from the following *disadvantages :*

(i) The other four methods could be extended as to enable a discussion of

(*a*) the ballistic effects of bore resistance (Kapur, 1957*a*),

(*b*) the internal ballistics of a tapered-bore gun (Kapur, 1957*b*),

(*c*) the solution of the equations of internal ballistics for the pressure-index law of burning (Kapur, 1958).

The present method cannot be used in these cases. It fails in cases (*a*) and (*b*), since bore resistance and area of cross of the bore of a tapered-bore gun are likely to be given as function of ζ or ξ but never directly as functions of time. For case (*c*), the difficulty is that the equation corresponding to (13) would be

$$\frac{df}{d\tau} = -\zeta^{\alpha} \quad \ldots \quad (23)$$

and this cannot be integrated to give an equation similar to (18).

(ii) For the isothermal model for the specific case of a tubular charge, (19) becomes

$$\xi \frac{d^2\xi}{d\tau^2} - \frac{d\xi}{d\tau} - \overline{M}[(1-f_0)+\zeta_1(\tau-\tau_0)] = 0 \quad \ldots \quad (24)$$

In the particular case when $\zeta_1 = 0$ first integration of (24) can be easily obtained as

$$\xi = e^{\eta}\left[1+\frac{\eta}{\overline{M}(1-f_0)}\right]^{-\overline{M}(1-f_0)} \quad \ldots \quad (25)$$

This is a first order differential equation since $\eta = \dfrac{d\xi}{d\tau}$ but this cannot be further integrated in finite terms and therefore comparison of the general linear law

$$D\frac{df}{dt} = -\beta(p+p_1) \quad \ldots \quad (26)$$

with the usual linear law of burning

$$D\frac{df}{dt} = -\beta p \quad \ldots \quad (27)$$

which was possible with our other methods due to the complete integration of the differential equations in the case $\zeta_1 = 0$ is not possible in this case. As a matter of fact that is the reason why even in the simpler case of law (27), time is never used as an independent variable in the many standard methods available for this case.

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