GRAVITATIONAL INSTABILITY OF AN INFINITE HOMOGENEOUS AND STATIONARY TURBULENT MEDIUM

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Abstract

By considering an infinite turbulent medium to be homogeneous in space and stationary in time, Jeans' criterion of stability has been discussed. Using the equations of motion and of continuity for a compressible fluid, the equation in density fluctuations $\bar{\omega}(r, t)$ $[= \overline{\delta\rho'\delta\rho''}]$ has been deduced. Then, on the basis of the assumptions that the fourth order correlation is related to the second order correlations in the same manner as in a joint-Gaussian distribution, and that the variations in density and pressure are adiabatic, the condition derived for gravitationally unstable turbulent medium is $k^2 < \dfrac{4\pi G\bar{\rho}}{C^2 + \frac{1}{3}\overline{u^2}}$ which differs from Jeans' condition in having $C^2 + \frac{1}{3}\overline{u^2}$ in place of C^2.

1. Introduction

Jeans (1929) has considered the problem of the gravitational stability of an infinite homogeneous medium. His analysis can be summarized as follows:—

Using the equation of continuity and the equations of motion for a compressible fluid, we have

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho u_i u_j) + \nabla^2 p - \frac{\partial}{\partial x_i}\left(\rho \frac{\partial V}{\partial x_i}\right) \quad \ldots \quad (1)$$

where ρ, p and V denote the density, the pressure and the gravitational potential respectively.

We introduce the perturbation given by

$$\rho = \bar{\rho} + \delta\rho,\ p = \bar{p} + \delta p \text{ and } V = \bar{V} + \delta V \quad \ldots \quad (2)$$

where $\bar{\rho}$, $\bar{p}$ and $\bar{V}$ are certain constants.

With the help of the adiabatic relation for the changes in the pressure and the density, that is

$$\frac{\delta p}{\bar{p}} = \gamma \frac{\delta \rho}{\rho} \quad \ldots \quad (3)$$

we obtain

$$\frac{\partial^2(\delta\rho)}{\partial t^2} = C^2 \nabla^2 \delta\rho - \bar{\rho}\nabla^2 \delta V \quad \ldots \quad (4)$$

In the last two equations, γ denotes the ratio of the specific heats and $C\left[= \left(\gamma \frac{\bar{p}}{\bar{\rho}}\right)^{\frac{1}{2}}\right]$ denotes the velocity of sound. In deducing equation (4), the inertial term $\dfrac{\partial^2(\rho u_i u_j)}{\partial x_i \partial x_j}$ has been neglected as a quantity of higher order of smallness as compared to those terms which have been retained. Then, using a variation of Poisson's equation (cf. eq. 13, section 2), equation (4) takes the final form

$$\frac{\partial^2(\delta\rho)}{\partial t^2} = C^2 \nabla^2\delta\rho + 4\pi G\bar{\rho}\delta\rho. \qquad (5)$$

Considering a solution of equation (5) of the form of a plane wave, viz.

$$\delta\rho = A(t)e^{i\underline{k}\cdot\underline{\gamma}} \qquad (6)$$

where $A(t)$ denotes the amplitude of the wave, we get

$$\frac{d^2A}{dt^2} = -(k^2C^2 - 4\pi G\bar{\rho})A \qquad (6a)$$

Hence the amplitude A can increase indefinitely with time provided

$$k^2 < \frac{4\pi G\bar{\rho}}{C^2} \qquad (7)$$

Therefore, the medium is gravitationally unstable if the wave length λ exceeds the value $\lambda_C \left[= \frac{2\pi C}{\sqrt{4\pi G\bar{\rho}}} \right]$.

The conclusion drawn from it is that an infinite homogeneous medium will break up into 'condensations' having linear dimensions of the order of $\frac{1}{2}\lambda_C$.

S. Chandrasekhar (1951*b*) has discussed the difficulties which one may encounter in the astronomical applications of Jeans' criterion. He has come to the conclusion that for such applications the Reynolds number of the hydro-dynamical motion will be large enough for the medium to be considered highly turbulent. In the light of this result, Chandrasekhar has considered the problem of gravitational stability in a homogeneous turbulent medium. He has taken the viscous medium and has made use of the adiabatic relation in addition to the other approximations. But the assumption of the adiabatic relation is inconsistent with the retention of the viscous terms in the equations of motion. Therefore, in order to avoid this inconsistency, we have considered the problem of gravitational instability in a turbulent medium by neglecting the viscous terms in the equations of motion in the very beginning. It means that we are discussing the instability problem in that range of the turbulent medium in which the inertial forces are dominant and the viscous forces are negligible. Further, we have introduced correlations at two points ρ' and ρ'' and at two different times t' and t''. We have assumed that these correlations depend on the vector $\underline{\xi}$ $[= \underline{\gamma}'' - \underline{\gamma}']$ and the difference in times $t = |t' - t''|$ (Chandrasekhar, 1955). With the help of this method, we have solved the problem by making use of the quasi-normality hypothesis as has been done in the paper 'Density fluctuations in a compressible inviscid turbulent fluid' by the present writer. The criterion for stability, so obtained, is exactly the same as obtained by Chandrasekhar (1951*b*); the condition being $k^2 < \frac{4\pi G\bar{\rho}}{C^2 + \frac{1}{3}\overline{u^2}}$ which differs from Jeans' criterion in having $\frac{1}{3}\overline{u^2}$ as the additional term in the denominator. Thus we have shown that Chandrasekhar's criterion holds even under his new theory of turbulence (Chandrasekhar, 1955).

2. Criterion for stability

The equation of continuity for compressible fluid is

$$\frac{\partial\rho'}{\partial t'} + \frac{\partial}{\partial x_i'}(\rho' u_i') = 0 \qquad (8)$$

and the equation of motion for inviscid fluid is

$$\frac{\partial}{\partial t'}(\rho' u_i') + \frac{\partial}{\partial x_j'}(\rho' u_i' u_j') = -\frac{\partial p'}{\partial x_i'} + \rho' X_i' \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where X_i' denotes the component of the external force. If the external force is derived from a gravitational potential V',

$$X_i' = \frac{\partial V'}{\partial x_i'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Differentiating equations (8) and (9) and eliminating the term containing $\rho' u_i'$, we obtain

$$\frac{\partial^2 \rho'}{\partial t'^2} = \frac{\partial^2}{\partial x_i' \partial x_j'}(\rho' u_i' u_j') + \nabla^2 p' - \frac{\partial}{\partial x_i'}\left(\rho' \frac{\partial V'}{\partial x_i'}\right) \quad \ldots \quad \ldots \quad (11)$$

In deriving (11) from (8) and (9) we have used (10) to replace X_i' in terms of V'.

Now, to discuss the gravitational stability of a turbulent medium, we take

$$p = \bar{p} + \delta p, \quad \rho = \bar{\rho} + \delta\rho \text{ and } V = \bar{V} + \delta V \ldots \quad \ldots \quad \ldots \quad (12)$$

where $\bar{p}$, $\bar{\rho}$ and $\bar{V}$ are certain constants.

Therefore, at the point $p'(=x_i')$ under consideration, equation (11) can be reduced as follows :—

Variation of Poisson's equation gives

$$\nabla^2 \delta V = -4\pi G \delta\rho \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Using equations (12) and (13), we get

$$\frac{\partial}{\partial x_i'}\left\{\rho' \frac{\partial V'}{\partial x_i'}\right\} = \frac{\partial}{\partial x_i'}\left\{(\bar{\rho} + \delta\rho') \frac{\partial}{\partial x_i'} \delta V'\right\} \simeq \bar{\rho} \nabla^2 \delta V' = -4\pi G \bar{\rho} \delta\rho' \quad \ldots \quad (14)$$

Putting the value from (14), we obtain the reduced form of (11) as

$$\frac{\partial^2 \rho'}{\partial t'^2} = \frac{\partial^2}{\partial x_i' \partial x_j'}(\rho' u_i' u_j') + \nabla^2 p' + 4\pi G \bar{\rho} \delta\rho' \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

We multiply this equation by ρ''—the value of the density at the point x_j'' and at time t''—and take the averages to obtain

$$\frac{\partial^2 \tilde{\omega}}{\partial t^2} = \frac{\partial^2}{\partial \xi_i \partial \xi_j}\overline{(\rho'' \rho' u_i' u_j')} + \nabla_\xi^2 (\overline{\rho'' p'}) + 4\pi G \bar{\rho} \tilde{\omega} \quad \ldots \quad \ldots \quad (16)$$

where

$$\tilde{\omega}(r, t) = \overline{\delta\rho' \delta\rho''} = \overline{\rho' \rho''} - \bar{\rho}^2. \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

At this stage we introduce the following two assumptions [cf. eqs. 15 and 19, Jain (1958)].

1. The fourth order correlation is related to the second order correlations in the same manner as in a joint-Gaussian distribution.
2. Variations in pressure and in density are governed by the adiabatic law.

Using these two assumptions, equation (16) can be put in the form

$$\frac{\partial^2 \tilde{\omega}}{\partial t^2} = (C^2 + \tfrac{1}{3}\overline{u^2}) \nabla_\xi^2 \tilde{\omega} + 4\pi G \bar{\rho} \tilde{\omega}. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

For a comparison, the equation obtained by Chandrasekhar (1951*a*) corresponding to our equation (18) is

$$\frac{\partial^2 \bar{\omega}}{\partial t^2} = 2(C^2+\tfrac{1}{3}\overline{u^2})\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \bar{\omega}}{\partial r}\right)+8\pi G\bar{\rho}\bar{\omega}-2\frac{\partial^2}{\partial \xi_i \partial \xi_j}\left[\overline{\rho\rho'}\ \overline{u_i u_j'}-\overline{\rho u_i'}\ \overline{\rho u_j'}\right] \quad (19)$$

To obtain the condition of stability we follow the method given by Chandrasekhar (1951*b*).

Equation (18) admits the solution of the form

$$\bar{\omega}(r, t) = \frac{A(t)}{r}e^{ikr}. \quad \ldots \quad (20)$$

Therefore, the expectation is that a superposition of these solutions will represent the asymptotic behaviour of the solution of the equation (18).

Putting $\bar{\omega}(r, t)$ in equation (18), we get

$$\frac{d^2 A}{dt^2} = -\left[(C^2+\tfrac{1}{3}\overline{u^2})k^2-4\pi G\bar{\rho}\right]A \quad \ldots \quad (21)$$

which gives the amplitudes of the spherical waves given by solution (20).

From equation (21)

if

$$k^2 < \frac{4\pi G\bar{\rho}}{C^2+\tfrac{1}{3}\overline{u^2}} \quad \ldots \quad (22)$$

the amplitude of the corresponding spherical wave in the superposition will increase exponentially with t. The physical meaning of this is that in the density fluctuations the eddies, which are greater than a critical size, will be amplified resulting in disintegration and formation of condensations.

The condition for stability obtained above differs from that obtained by Jeans in having $C^2+\frac{1}{3}\overline{u^2}$ in place of C^2. Since we have not restricted our analysis to the case when $\overline{u^2} < C^2$, $\frac{1}{3}\overline{u^2}$ may not be negligible in comparison with C^2. Moreover, the analysis given above is free from the conceptual difficulties mentioned by Chandrasekhar (1951*b*).

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