FINAL PERIOD DECAY-LAW IN THE CASE OF LONGITUDINALLY HOMOGENEOUS TURBULENCE WITH SYMMETRY ABOUT A LOCALIZED AXIS

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Abstract

The final period decay-law of a longitudinally homogeneous turbulence with symmetry about a localized axis has been studied and found to be proportional to $(\nu t)^{-5/2}$. The decay-laws so obtained for component directions which are perpendicular to the axis of symmetry are found to be identical.

The relevant energy values at a point at great distance from the axis of symmetry vanish and this may be considered as satisfying the condition at infinity of the type of turbulence considered here.

1. Introduction

From the works of Batchelor and Townsend (1948), and of Batchelor and Stewart (1950), we learn that in the final period of decay of homogeneous and isotropic turbulence the decay-law is given by $\overline{u^2} \propto (\nu t)^{-5/2}$. This has indeed received experimental confirmation. The final period decay-law for homogeneous and axisymmetric turbulence has been worked out by S. Chandrasekhar (1950*b*). Chandrasekhar has shown that the two mean square velocity components ultimately decay as

$$\overline{u_{\parallel}^2} = \frac{\Lambda_1}{24(2\pi)^{\frac{1}{2}}(\nu t)^{5/2}}, \text{ and } \overline{u_{\perp}^2} = \frac{2\Lambda_1+\Lambda_2}{48(2\pi)^{\frac{1}{2}}(\nu t)^{5/2}}$$

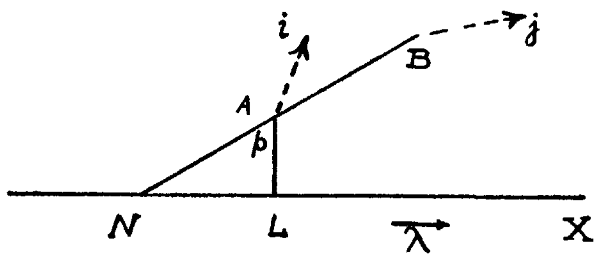
where $\overline{u_{\parallel}^2}$ and $\overline{u_{\perp}^2}$ imply twice the mean square energies associated with the velocities in the preferential and its perpendicular direction respectively, and Λ_1, Λ_2 are two finite integrals of Loitsiansky type. The same author pictured the axisymmetric turbulence as a superposition of two non-interacting fields which are isotropic and axisymmetric respectively. We shall follow Chandrasekhar in our consideration of the final decay motion of localized axisymmetric turbulence in this paper.

In an earlier work, the present author dealt with the kinematics of the longitudinally homogeneous localized axisymmetric turbulence. In continuation of that work, we shall here derive the final period decay-law for turbulence of this type of symmetry. The second-order two-point velocity correlations that will appear in subsequent calculations are directly written from gauge invariance as outlined by S. Chandrasekhar (1950*a*). In working with the Navier-Stokes equations we shall neglect the non-linear part together with the pressure term which are known to have negligible effects on the decay-law at the final stage. The decay-laws associated with the velocity components in directions (i) parallel, and (ii) perpendicular to the localized axis, and also in (iii) a direction perpendicular to the meridian plane containing the point under consideration and the localized axis have been worked out.

One interesting point of our final result is that the effects of the type of turbulence we consider here vanish at large distance from the localized axis. For instance, the turbulence characteristics and the total value of energy at a point which is at an infinitely great distance from the localized axis are found to vanish. This makes our result suitable for application to the case of spreading of turbulence generated along a fixed axis in an infinite medium, as for instance that produced by a jet.

2. Form of R_{ij} Solenoidal in j, and Deduction of the Dynamical Equations in Terms of Defining Scalars

Let the unit vector λ denote the direction of the localized axis, and A an arbitrarily chosen fixed point outside the axis in the infinite turbulent fluid with no mean motion. Let the distance of A from the λ-axis be p. We next choose a second point B in the turbulent fluid such that when A and B are joined, the line



TEXT-FIG. 1

BA (or BA produced as in the figure) cuts the λ-axis at N. Since we are only interested in the energy-value at A and its time dependence at the final stage of decay of turbulence, which will be obtained by a limiting process in which B approaches and coincides with A, the choice of B on the meridian plane through A and the λ-axis will not in any way imply a loss of generality.

In this system the turbulent fluid is supposed to extend to infinity in all directions, and further the symmetry about the λ-axis, as also homogeneity in the longitudinal direction are assumed. This means that the rigid configuration of vectors $(\vec{i}, \vec{AB}, \vec{j})$, where i, j are unit vectors at A and B respectively remains invariant for a shift parallel to the λ-axis of the entire configuration, as also for a rotation about the λ-axis, and a reflection in any plane through λ-axis. The equivalent isotropic and homogeneous system, that can be formed to help representation of the type of turbulence that is considered, is given by the rigid configuration $\vec{j}$ at B, $\vec{BN}$, $\vec{i}$ at A, and $\vec{\lambda}$. To consider the velocity correlation formed with u_i at A, and u_j at B represented by $R_{ij} \equiv \overline{u_i u'_j}$ (dashed and undashed components corresponding to those at B and A respectively), we shall proceed to construct the gauge-invariant form of the tensor R_{ij}. Let us denote LA (AL being the perpendicular from A on the λ-axis) by p, NL the distance of the foot of this perpendicular from the point N by l, and the vector NB by $\vec{\eta}$. Further let us take $|AB| = r$, $|\eta| = mr$ and $(\eta.\lambda) = \eta\mu$. If we denote $\vec{NA}$ by $\vec{\bar{\eta}}$, then $\bar{\eta}^2 = l^2+p^2$. In terms of these notations, we write the gauge-invariant form of R_{ij} (where the continuity condition at B is satisfied from the form of the tensor R_{ij} itself), as (cf. S. Chandrasekhar, 1950*a*)

$$\begin{aligned} R_{ij} = {} & [\eta_i\eta_j D_\eta - \delta_{ij}(\eta^2 D_\eta + \eta\mu D_\mu + 2) + \lambda_i\eta_j D_\mu]Q_1 \\ & + [\eta_i\eta_j D_\eta - \delta_{ij}\{\eta^2(1-\mu^2)D_\eta + 1\} + \lambda_i\lambda_j(\eta^2 D_\eta + 1) - (\lambda_i\eta_j + \lambda_j\eta_i)\eta\mu D_\eta]Q_2 \\ & + [-\eta_i\eta_j D_\mu + \delta_{ij}\{\eta^2(1-\mu^2)D_\mu - \eta\mu\} - \lambda_i\lambda_j\eta^2 D_\mu + (\lambda_i\eta_j + \lambda_j\eta_i)\eta\mu D_\mu + \lambda_j\eta_i]Q_3 \end{aligned} \quad (1)$$

where Q_1, Q_2, Q_3 are functions of $(\eta.\eta)$, $(\eta.\lambda)$, l, and p; and D_η and D_μ are operators defined by

$$D_\eta = \frac{1}{\eta}\frac{\partial}{\partial \eta} - \frac{\mu}{\eta^2}\frac{\partial}{\partial \mu}, \quad D_\mu = \frac{1}{\eta}\frac{\partial}{\partial \mu}. \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

From the Navier-Stokes equations of motion we find in the usual way the equation for the time-rate of change of R_{ij} as

$$\frac{\partial}{\partial t} R_{ij} = \nu \nabla_\eta^2 R_{ij} + \nu \left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial p^2}\right) R_{ij} + S_{ij} \quad \ldots \quad \ldots \quad \ldots \quad (3)^*$$

where ν is the kinematic viscosity, and

$$S_{ij} = -\frac{\partial}{\partial \eta_k}\overline{u_i u'_j u'_k} - \frac{\partial}{\partial \eta_j}\frac{\overline{p' u_i}}{\rho} - \left(\lambda_k \frac{\partial}{\partial l} + \psi_k \frac{\partial}{\partial p}\right)\overline{u_i u'_j u_k} - \left(\lambda_i \frac{\partial}{\partial l} + \psi_i \frac{\partial}{\partial p}\right)\frac{\overline{p u'_j}}{\rho}. \quad \ldots \quad (4)$$

From (3) we conclude that whatever invariant properties and forms may be attributed to

$$R_{ij}, \; \nabla_\eta^2 R_{ij}, \text{ and } \left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial p^2}\right) R_{ij},$$

S_{ij} will have exactly the same symmetry properties and forms. R_{ij} as defined in (1) is expressible in terms of three defining scalars Q_1, Q_2, Q_3; it is known that $\nabla_\eta^2 R_{ij}$ can similarly be defined in terms of three defining scalars ΔQ_1, $\Delta Q_2 + 2D_{\mu\mu}Q_1$, and $\Delta Q_3 + 2D_{\eta\mu}Q_1$, where Δ is defined as

$$\Delta \equiv \frac{\partial^2}{\partial \eta^2} + \frac{4}{\eta}\frac{\partial}{\partial \eta} + \frac{1-\mu^2}{\eta^2}\frac{\partial^2}{\partial \mu^2} - \frac{4\mu}{\eta^2}\frac{\partial}{\partial \mu}.$$

In the same way we assume that S_{ij} may be defined in terms of three other defining scalars S_1, S_2, S_3 in a form analogous to (1). Hence the Navier-Stokes equation (3) can be rewritten in terms of these scalars as

$$\left.\begin{aligned} \left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial p^2}\right)\right\} Q_1 &= \nu \Delta Q_1 + S_1 \\ \left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial p^2}\right)\right\} Q_2 &= \nu \Delta Q_2 + 2\nu D_{\mu\mu} Q_1 + S_2 \\ \left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2} + \frac{\partial^2}{\partial p^2}\right)\right\} Q_3 &= \nu \Delta Q_3 + 2\nu D_{\eta\mu} Q_1 + S_3; \end{aligned}\right\} \quad \ldots \quad (5)$$

so we find that a single tensor equation (3) breaks up into three scalar equations, and since R_{ij}, $\nabla_\eta^2 R_{ij}$, and S_{ij} are all derived from consideration of gauge-invariance

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* From fig. 1, it is apparent that if the co-ordinates of N, A, B be respectively given by X_i, x_i, x'_i we have

$$\frac{\partial R_{ij}}{\partial x'_i} = \frac{\partial R_{ij}}{\partial \eta_i}, \quad \text{and} \quad \frac{\partial R_{ij}}{\partial x_i} = \lambda_i \frac{\partial R_{ij}}{\partial l} + \psi_i \frac{\partial R_{ij}}{\partial p} \quad \ldots \quad \ldots \quad (3.1)$$

where ψ is the unit vector in the direction perpendicular to λ, viz. in the direction of LA. Here ∇_η^2 should be read as

$$\frac{\partial^2}{\partial \eta_1^2} + \frac{\partial^2}{\partial \eta_2^2} + \frac{\partial^2}{\partial \eta_3^2}.$$

their representations are unique when all the symmetry conditions have been taken into consideration. A new symmetry condition arises when B is made to coincide with A, in which case $(R_{ij})_{AB\to 0} = (R_{ji})_{AB\to 0}$. This will be used in section 4. Equations (5) together with the above-mentioned symmetry condition would give us a correct picture of the decay of turbulence.

It is clear from the works of Batchelor and Townsend (1948), and of S. Chandrasekhar (1950), that in the final period of decay the predominant rôle is played by the viscosity term in the Navier-Stokes equation, and in fact this represents the controlling force. Hence for the study of decay-law in the final stage we may neglect S_1, S_2, S_3 in (5). Equations (5) may be approximated for the final stage of decay by

$$\left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2}+\frac{\partial^2}{\partial p^2}\right)\right\} Q_1 = \nu \Delta Q_1 \quad \ldots \quad (6.1)$$

$$\left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2}+\frac{\partial^2}{\partial p^2}\right)\right\} Q_2 = \nu \Delta Q_2 + 2\nu D_{\mu\mu} Q_1 \quad \ldots \quad (6.2)$$

$$\left\{\frac{\partial}{\partial t} - \nu\left(\frac{\partial^2}{\partial l^2}+\frac{\partial^2}{\partial p^2}\right)\right\} Q_3 = \nu \Delta Q_3 + 2\nu D_{\eta\mu} Q_1. \quad \ldots \quad (6.3)$$

In the next two sections we solve the above equations for which the analysis of Chandrasekhar has been closely followed.

3. The Solution of (6.1) for Q_1

Equation (6.1) is in fact the equation of heat conduction in an axisymmetric five-dimensional space (Δ being considered a five-dimensional operator); we have to solve the equation in terms of the given initial values of the variable Q_1 at $t = 0$. A solution can be written in a five-dimensional space as

$$\frac{1}{(8\pi\nu t)^{5/2}}\left[e^{-\frac{(\eta_3''-\eta_3)^2+(\eta_4''-\eta_4)^2+(\eta_5''-\eta_5)^2}{4\nu t}}\left\{e^{-\frac{(\eta_1''-\eta_1)^2+(\eta_2''-\eta_2)^2}{4\nu t}}+ +e^{-\frac{(\eta_1''-l)^2+(\eta_2''-p)^2}{4\nu t}}\right\}\right]$$

which corresponds to two unit sources at $\eta'' = \eta$, and $\eta'' = \bar{\eta}$ at time $t = 0$, η'' meaning the five-dimensional vector whose components are η_1'', η_2'', η_3'', η_4'', η_5'' in the five-dimensional cartesian frame. The solution of (6.1) satisfying the given initial values, viz.

$$Q_1 = Q_1(\eta'', \mu''; 0)^* \text{ at } t = 0, \text{ is} \quad \ldots \quad (7)$$

$$Q_1(\eta, \mu; l, p; t) = \frac{1}{(8\pi\nu t)^{5/2}}\int\int\int\int\int \{e^{-|\eta-\eta''|^2/(4\nu t)} + e^{-|\bar{\eta}-\eta''|^2/(4\nu t)}\} \times Q_1(\eta'', \mu''; 0)\, d\eta_1'' d\eta_2'' d\eta_3'' d\eta_4'' d\eta_5''. \quad \ldots \quad (8)$$

Here we have for the moment taken the plane of the figure as the plane of η_1, η_2, the co-ordinates of the point A (now denoted by vector $\bar{\eta}$) as $(l, p, 0, 0, 0)$, and the λ- and ψ-directions as the directions of η_1 and η_2 respectively. It is to be noted

* Here and in the following lines μ'' represents cosine of an angular variable belonging to the point η''.

that the choice of the line NAB on (η_1, η_2)-plane of the five-dimensional space does not imply any loss of generality in the present problem. We rewrite (8) as

$$Q_1(\eta, \mu; l, p; t) = I_1 + I_2,$$

where

$$I_1 = \frac{1}{(8\pi\nu t)^{5/2}} \int\int\int\int\int e^{-|\eta-\eta''|^2/(4\nu t)} Q_1(\eta'', \mu''; 0)\, d\eta_1'' d\eta_2'' d\eta_3'' d\eta_4'' d\eta_5'' \quad .. \quad (9.1)$$

$$I_2 = \frac{1}{(8\pi\nu t)^{5/2}} \int\int\int\int\int e^{-|\bar{\eta}-\eta''|^2/(4\nu t)} Q_1(\eta'', \mu''; 0)\, d\eta_1'' d\eta_2'' d\eta_3'' d\eta_4'' d\eta_5'' \quad .. \quad (9.2)$$

Next we introduce the polar co-ordinates by

$$\eta_1'' = \eta'' \cos \theta'', \; \eta_2'' = \eta'' \sin \theta'' \cos \phi_1'', \; \eta_3'' = \eta'' \sin \theta'' \sin \phi_1'' \cos \phi_2'',$$

$$\eta_4'' = \eta'' \sin \theta'' \sin \phi_1'' \sin \phi_2'' \cos \phi_3'', \; \eta_5'' = \eta'' \sin \theta'' \sin \phi_1'' \sin \phi_2'' \sin \phi_3'',$$

where by η'' we mean $|\eta''|$; we then proceed to simplify (9.1) thus

$$I_1 = \frac{4\pi}{(8\pi\nu t)^{5/2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-(\eta^2+\eta''^2-2\eta\eta'' \cos \odot)/(4\nu t)} Q_1(\eta'', \mu''; 0)$$
$$\times \eta''^4 \sin^3 \theta'' \sin^2 \phi_1''\, d\eta'' d\theta'' d\phi_1'' \quad .. \quad (10)$$

where

$$\cos \odot = \cos \theta \cos \theta'' + \sin \theta \sin \theta'' \sin \phi_1''.$$

We expand the exponential function in the manner of Chandrasekhar as follows

$$e^{\eta\eta'' \cos \odot/(2\nu t)} = 2^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \sum_{m=0}^{\infty} \left(m+\frac{3}{2}\right) C_m^{\frac{3}{2}}(\cos \odot) \frac{I_{m+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)}{\left(\frac{\eta\eta''}{2\nu t}\right)^{\frac{3}{2}}},$$

where $C_m^{\frac{3}{2}}(\cos \odot)$ are the Gegenbaner polynomials, and $I_{m+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)$ Bessel functions of half-integral order for purely imaginary arguments and in addition using the following relation

$$\int_0^\pi C_m^{\frac{3}{2}}(\cos \odot) \sin^2 \phi_1'' d\phi_1'' = \frac{2^2 m! \left[\Gamma\left(\frac{3}{2}\right)\right]^2}{\Gamma(m+3)} \cdot C_m^{\frac{3}{2}}(\mu)\, C_m^{\frac{3}{2}}(\mu'').$$

We can reduce the form (10) as follows

$$I_1 = \frac{4\pi e^{-\eta^2/(4\nu t)}}{(8\pi\nu t)^{5/2}} \sum \frac{2^{\frac{7}{2}} \left[\Gamma\left(\frac{3}{2}\right)\right]^3 \left(m+\frac{3}{2}\right) C_m^{\frac{3}{2}}(\mu)}{(m+2)(m+1)}$$
$$\times \int_0^\infty \int_{-1}^1 e^{-\eta''^2/(4\nu t)} \eta''^4 (1-\mu''^2)\, Q_1(\eta'', \mu''; 0)\, C_m^{\frac{3}{2}}(\mu'')$$
$$\times \frac{I_{m+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)}{\left(\frac{\eta\eta''}{2\nu t}\right)^{\frac{3}{2}}} d\eta'' d\mu''. \quad .. \quad .. \quad .. \quad (11)$$

Without loss of generality we may write $Q_1(\eta'', \mu''; 0)$ as a series in Gegenbaner polynomials $C_m^{\frac{3}{2}}(\cos \odot)$. Further Q_1 is to be an even function in μ'' as required for the type of symmetry considered for R_{ij}, and consequently the expansion will have only $C_{2m}^{\frac{3}{2}}(\mu'')$ terms. Hence

$$Q_1(\eta'', \mu''; 0) = \sum_{n=0}^{\infty} q_{2n}^{(1)}(\eta'')\, C_{2n}^{\frac{3}{2}}(\mu'') \ldots \qquad \ldots \qquad \ldots \quad (12)$$

Inserting (12) in (11) and using the orthogonality property

$$\int_{-1}^{1} C_m^{\frac{3}{2}}(\mu'') C_n^{\frac{3}{2}}(\mu'')(1-\mu''^2)d\mu'' = \delta_{mn} \frac{\pi(m+2)(m+1)}{2^2\left(m+\frac{3}{2}\right)\left[\Gamma\left(\frac{3}{2}\right)\right]^2} \qquad \ldots \quad (13)$$

we finally get

$$I_1 = \frac{e^{-\eta^2/(4\nu t)}}{32(\nu t)^{5/2}} \sum_{n=0}^{\infty} C_{2n}^{\frac{3}{2}}(\mu) \int_0^{\infty} e^{-\eta''^2/(4\nu t)} \eta''^4 q_{2n}^{(1)}(\eta'') \frac{I_{2n+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)}{\left(\frac{\eta\eta''}{2\nu t}\right)^{\frac{3}{2}}} d\eta''. \qquad \ldots \quad (14)$$

By a similar procedure we get

$$I_2 = \frac{e^{-\bar{\eta}^2/(4\nu t)}}{32(\nu t)^{5/2}} \sum_{n=0}^{\infty} C_{2n}^{\frac{3}{2}}(\mu) \int_0^{\infty} e^{-\eta''^2/(4\nu t)} \eta''^4 q_{2n}^{(1)}(\eta'') \frac{I_{2n+\frac{3}{2}}\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)}{\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)^{\frac{3}{2}}} d\eta'' \qquad \ldots \quad (15)$$

so we get the solution of (6.1) as
$Q_1(\eta, \mu; l, p; t) = I_1 + I_2$, where I_1 and I_2 are represented as in (14) and (15) respectively. .. .. (16)

4. The Solution for Q_2 and Q_3

To solve for Q_2 and Q_3 from (6.2) and (6.3) respectively we require substitution for Q_1 from (16) in (6.1) and (6.2). Thus by the method of sources we can easily write down, as

$$\begin{aligned} Q_2(\eta, \mu; l, p; t) &= I_1' + I_2' \\ &+ 2\nu \int_0^t \frac{dt'}{[8\pi\nu(t-t')]^{5/2}} \int\!\!\int\!\!\int\!\!\int\!\!\int \left\{ \frac{1}{\eta^2} \frac{\partial^2 (I_1+I_2)}{\partial \mu^2} \right\}_{\eta'', \mu'', t'} \\ &\times \left\{ e^{-|\eta-\eta''|^2/[4\nu(t-t')]} + e^{-|\bar{\eta}-\eta''|^2/[4\nu(t-t')]} \right\} \times d\eta_1'' d\eta_2'' d\eta_3'' d\eta_4'' d\eta_5'' \ldots \qquad \ldots \quad (17) \end{aligned}$$

where

$$I_1' = \frac{e^{-\eta^2/(4\nu t)}}{32(\nu t)^{5/2}} \sum_{n=0}^{\infty} C_{2n}^{3/2}(\mu) \int_0^{\infty} e^{-\eta''^2/(4\nu t)} \eta''^4 q_{2n}^{(2)}(\eta'') \frac{I_{2n+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)}{\left(\frac{\eta\eta''}{2\nu t}\right)^{\frac{3}{2}}} d\eta''$$

and

$$I_2' = \frac{e^{-\bar{\eta}^2/(4\nu t)}}{32(\nu t)^{5/2}} \sum_{n=0}^{\infty} C_{2n}^{3/2}(\mu) \int_0^{\infty} e^{-\eta''^2/(4\nu t)} \eta''^4 q_{2n}^{(2)}(\eta'') \frac{I_{2n+\frac{3}{2}}\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)}{\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)^{\frac{3}{2}}} d\eta''$$

$q_{2n}^{(2)}(\eta'')$ being the coefficients in the expansion of $Q_2(\eta'', \mu''; 0)$.

Similarly

$$Q_3(\eta,\mu\,;l,p\,;t) = I_1''+I_2''$$

$$+2\nu\int_0^t \frac{dt'}{[8\pi\nu(t-t')]^{5/2}}\cdot\iiiint\!\!\int\left[\left\{\frac{1}{\eta^2}\frac{\partial^2}{\partial\eta\partial\mu}-\frac{\mu}{\eta^3}\frac{\partial^2}{\partial\mu^2}-\frac{1}{\eta^3}\frac{\partial}{\partial\mu}\right\}(I_1+I_2)\right]_{\eta'',\mu'',t'}$$

$$\times\left[e^{-|\eta-\eta''|^2/[4\nu(t-t')]}+e^{-|\bar{\eta}-\eta''|^2/[4\nu(t-t')]}\right]d\eta_1''d\eta_2''d\eta_3''d\eta_4''d\eta_5'' \quad .. \quad (18)$$

where

$$I_1'' = \frac{e^{-\eta^2/(4\nu t)}}{32(\nu t)^{5/2}}\sum_{n=0}^{\infty}C_{2n}^{3/2}(\mu)\int_0^\infty e^{-\eta''^2/(4\nu t)}\eta''^4 q_{2n}^{(3)}(\eta'')\frac{I_{2n+\frac{3}{2}}\left(\frac{\eta\eta''}{2\nu t}\right)}{\left(\frac{\eta\eta''}{2\nu t}\right)^{\frac{3}{2}}}d\eta''$$

$$I_2'' = \frac{e^{-\bar{\eta}^2/(4\nu t)}}{32(\nu t)^{5/2}}\sum_{n=0}^{\infty}C_{2n}^{3/2}(\mu)\int_0^\infty e^{-\eta''^2/(4\nu t)}\eta''^4 q_{2n}^{(3)}(\eta'')\frac{I_{2n+\frac{3}{2}}\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)}{\left(\frac{\bar{\eta}\eta''}{2\nu t}\right)^{\frac{3}{2}}}d\eta'',$$

$q_{2n}^{(3)}(\eta'')$ being the coefficients in the expansion of $Q_3(\eta'',\mu'';0)$.

5. Mean-squares of the Velocities in Terms of the Q's and the Symmetry Condition when the Point B Approaches A

The representation of R_{ij} from the point of view of gauge-invariance is given by (1). But an alternative approach for deducing the form of R_{ij} from the basic invariants, the scalar product of two vectors and the determinant of three vectors give us

$$R_{ij} = A\eta_i\eta_j+B\delta_{ij}+C\lambda_i\lambda_j+D\lambda_i\eta_j+E\lambda_j\eta_i \quad .. \quad .. \quad (19)$$

where A, B, C, D, E are functions of $(\eta.\eta)$, $(\eta.\lambda)$, l, and p. On comparing (19) and (1) we find A, B, C, D, E in terms of the Q's as follows

$$\begin{aligned} A &= D_\eta Q_1 - D_\mu Q_3 + D_\eta Q_2. \\ B &= [-(\eta^2 D_\eta+\eta\mu D_\mu+2)Q_1+\{\eta^2(1-\mu^2)D_\mu-\eta\mu\}Q_3- \\ &\qquad \{\eta^2(1-\mu^2)D_\eta+1\}Q_2] \\ C &= -\eta^2 D_\mu Q_3+(\eta^2 D_\eta+1)Q_2 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (20) \\ D &= D_\mu Q_1-\eta\mu D_\eta Q_2+\eta\mu D_\mu Q_3 \\ E &= -\eta\mu D_\eta Q_2+\eta\mu D_\mu Q_3+Q_3. \end{aligned}$$

It is quite clear from (19) that

$\overline{u_{\parallel}^2}$ = mean-square of the velocity component parallel to $\vec{\lambda}$ at the point A

$$\begin{aligned} &= (\lambda_i\lambda_j R_{ij})_{\text{pt. }B\to\text{pt. }A} \\ &= [A\eta^2\mu^2+B+C+(D+E)\eta\mu] \\ &= [-\{\eta^2(1-\mu^2)D_\eta+2\}Q_1]_{\eta=\bar{\eta}} \quad .. \quad .. \quad .. \quad (21.1) \end{aligned}$$

$\overline{u_\perp^2}$ = mean-square of the velocity component perpendicular to $\vec{\lambda}$ at the point A

$$= (\Psi_i\Psi_jR_{ij})_{\text{pt. }B\rightarrow\text{pt. }A}$$
$$= [A\eta^2(1-\mu^2)+B]_{\eta=\bar{\eta}}$$
$$= [-(\eta^2\mu^2D_\eta+\eta\mu D_\mu+2)Q_1-Q_2-\eta\mu Q_3]_{\eta=\bar{\eta}} \quad \ldots \quad \ldots \quad (21.2)$$

where ψ is the unit vector at the point A perpendicular to λ-axis, and $\overline{u^2_{\text{azimuth}}}$ = mean-square of the velocity component at the point perpendicular to the meridian plane through A

$$= (\chi_i\chi_jR_{ij})_{\text{pt. }B\rightarrow\text{pt. }A}$$
$$= [B]_{\eta=\bar{\eta}}$$
$$= \big[-(\eta^2D_\eta+\eta\mu D_\mu+2)Q_1-\{\eta^2(1-\mu^2)D_\eta+1\}Q_2+ \{\eta^2(1-\mu^2)D_\mu-\eta\mu\}Q_3\big]_{\eta=\bar{\eta}} \quad \ldots \quad \ldots \quad (21.3)$$

where $\vec{\chi}$ is the unit vector perpendicular to the meridian plane at A.

Now we shall derive a further relation between Q_1 and Q_3 while they assume their asymptotic values as the point B approaches and coincides with A. It is obvious from the physical picture that R_{ij} should then be symmetric in i and j. From (20) one sees clearly that the equality $R_{ij}=R_{ji}$ (in the limiting case) implies

$$(Q_3)_{\eta=\bar{\eta}} = \left(\frac{1}{\eta}\frac{\partial Q_1}{\partial\mu}\right)_{\eta=\bar{\eta}} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

6. The Asymptotic Values of $\overline{u_\parallel^2}$, $\overline{u_\perp^2}$, and $\overline{u_{az}^2}$ for $t\rightarrow\infty$

As $$\frac{I_{2n+\frac{3}{2}}(x)}{x^{\frac{3}{2}}} \longrightarrow \frac{(\frac{1}{2})^{2n+\frac{3}{2}}}{\Gamma(2n+\frac{5}{2})}\cdot x^{2n}, \quad \text{when} \quad t\rightarrow\infty, \text{ i.e. } x\rightarrow 0,$$

$\left[\text{since } x=\dfrac{\eta\eta''}{2\nu t}\right]$, it is clear from (14) that the most significant term of (14) for large t is given by the first term of the series sum, i.e. the term given for $n=0$ as

$$I_1 \longrightarrow \frac{e^{-\frac{\eta^2}{4\nu t}}}{48(2\pi)^{1/2}(\nu t)^{5/2}}\int_0^\infty e^{-\frac{\eta''^2}{4\nu t}}\,\eta''^4 q_0^{(1)}(\eta'')d\eta''$$

or further expanding $e^{-\eta''^2/(4\nu t)}$ in the integrand, and observing that the first term of the expansion contributes the most significant term, we write

$$I_1 \simeq \frac{e^{-\eta^2/(4\nu t)}}{48(2\pi)^{1/2}(\nu t)^{5/2}}\cdot A_1 \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where the integral

$$A_1 = -\int_0^\infty q_0^{(1)}(\eta'')\eta''^4 d\eta'' \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

is assumed to exist.

Similarly

$$I_2 = \frac{e^{-\bar{\eta}^2/(4\nu t)}}{48(2\pi)^{1/2}(\nu t)^{5/2}} \cdot \Lambda_1. \quad \ldots \quad (25)$$

Thus the asymptotic solution for Q_1 as the time $t \to \infty$ is given by

$$Q_1(\eta, \mu; l, p; t) = \frac{e^{-\frac{\eta^2}{4\nu t}} + e^{-\frac{\bar{\eta}^2}{4\nu t}}}{48(2\pi)^{1/2}(\nu t)^{5/2}} \cdot \Lambda_1 . \quad \ldots \quad (26)$$

Proceeding in the same way and retaining only the most significant term for large t we get from (17) and (18) respectively

$$Q_2(\eta, \mu; l, p; t) = \frac{e^{-\frac{\eta^2}{4\nu t}} + e^{-\frac{\bar{\eta}^2}{4\nu t}}}{48(2\pi)^{1/2}(\nu t)^{5/2}} \cdot \Lambda_2 \quad \ldots \quad (27)$$

and

$$Q_3(\eta, \mu; l, p; t) = \frac{e^{-\frac{\eta^2}{4\nu t}} + e^{-\frac{\bar{\eta}^2}{4\nu t}}}{48(2\pi)^{1/2}(\nu t)^{5/2}} \Lambda_3 + \text{higher order terms} \quad \ldots \quad (28)$$

where

$$\Lambda_2 = -\int_0^\infty q_0^{(2)}(\eta'')\eta''^4 d\eta'', \text{ and } \Lambda_3 = -\int_0^\infty q_0^{(3)}(\eta'')\eta''^4 d\eta'' \quad \ldots \quad (29)$$

are assumed to exist, and further in (28) higher order terms begin with $(\nu t)^{-9/2}$.

On the other hand equation (22) shows $(Q_3)_{\eta=\bar{\eta}}$ begins with a term of the order $(\nu t)^{-\frac{9}{2}}$, hence Λ_3 must be zero. Further it may be noted that in (21.1), (21.2), (21.3), the terms under the operation D_η, D_μ will all lead to higher order terms because of the forms of (16), (17), (18). In consideration of these results the significant term for $\overline{u_{11}^2}$, $\overline{u_\perp^2}$, $\overline{u_{az}^2}$ in (21.1), (21.2) and (21.3) are found to be as follows

$$\overline{u_\parallel^2} \simeq [-2Q_1]_{\eta=\bar{\eta}}, \ \overline{u_\perp^2} \simeq [-2Q_1 - Q_2]_{\eta=\bar{\eta}}, \text{ and } \overline{u_{az}^2} \simeq [-2Q_1 - Q_2]_{\eta=\bar{\eta}}. \quad (30)$$

To consider Q_1 and Q_2 from (26) and (27) respectively for the limiting case $\eta = \bar{\eta}$ we shall break up the exponential multiplier $e^{-\bar{\eta}^2/(4\nu t)}$ into product of two terms $e^{-p^2/(4\nu t)}$, and $e^{-l^2/(4\nu t)}$, where l and p have the same significance as in section 2. For the analysis we have pursued so far, we need not take l very large, as for the limiting process, the point B may be suitably chosen so that l remains finite. Thus in the expansion of $e^{-l^2/(4\nu t)}$ the first term of the expansion, i.e. 1 will give us the most significant term. But in the multiplier $e^{-p^2/(4\nu t)}$ we shall have to take into consideration the order of p^2 which at times has to be considered to be of order higher than, or equal to that of (νt), for the choice of the point under consideration may be such that it is far from the axis of symmetry of turbulence. So after necessary simplification we obtain from (26), (27) and (30) the following asymptotic forms for $\overline{u_\parallel^2}$, $\overline{u_\perp^2}$, and $\overline{u_{az}^2}$ valid for large values of t,

$$\tfrac{1}{2}\overline{u_\parallel^2} \simeq \frac{e^{-\frac{p^2}{4\nu t}} \cdot \Lambda_1}{24(2\pi)^{1/2}(\nu t)^{5/2}}; \ \tfrac{1}{2}\overline{u_\perp^2} \simeq \frac{e^{-\frac{p^2}{4\nu t}} \cdot (2\Lambda_1 + \Lambda_2)}{48(2\pi)^{1/2}(\nu t)^{5/2}}, \text{ and } \tfrac{1}{2}\overline{u_{az}^2} \simeq \frac{e^{-\frac{p^2}{4\nu t}} \cdot (2\Lambda_1 + \Lambda_2)}{48(2\pi)^{1/2}(\nu t)^{5/2}}. \quad (31)$$

For further analysis of (31) we consider three possibilities

(i) $O(p^2) \ll O(\nu t)$, (ii) $O(p^2) \simeq O(\nu t)$, (iii) $O(p^2) \gg O(\nu t)$.

In case of (i), (31) can be rewritten as

$$\frac{1}{2}\overline{u_{\parallel}^2} \simeq \frac{\Lambda_1}{24(2\pi)^{(1/2)}(\nu t)^{(5/2)}}, \quad \frac{1}{2}\overline{u_{\perp}^2} \simeq \frac{2\Lambda_1+\Lambda_2}{48(2\pi)^{1/2}(\nu t)^{5/2}}, \text{ and } \frac{1}{2}\overline{u_{az}^2} \simeq \frac{2\Lambda_1+\Lambda_2}{48(2\pi)^{1/2}(\nu t)^{5/2}} \quad .. \quad (32)$$

For the alternative (ii), the results (32) are to be multiplied by a factor $\frac{1}{\sqrt[4]{e}}$; and for (iii), $\overline{u_{\parallel}^2}$, $\overline{u_{\perp}^2}$, and $\overline{u_{az}^2}$ all vanish.

Remarks

(1) In the derivation of (30) and (31), Λ_3 does not appear, as its value is zero, meaning thereby that Q_3 makes zero contribution in the final decay-law. Further, in cases (i) and (ii) considered above the decay-laws associated with the perpendicular and azimuthal directions are identical for large t, as is shown by the second and third equations of (32). The effect of this in the case $p \to 0$ is the obvious result that the perpendicular and azimuthal directions are now equivalent.

(2) The vanishing of turbulent energy at infinite distance from the localized axis is a very satisfactory feature of the present theory which was missed in Bass's theory of transversely homogeneous turbulence. It makes the present theory suitable for application to an infinite region, e.g. decay of turbulence produced by a straight jet in a medium with otherwise zero mean velocity.

This also supports an assumption made by us in a previous paper (1957) where at the end of §6 we made the suggestions $k_1+k_{11} = 0$, and C_0 not to contain a non-zero constant term in order to be able to extend such type of turbulence to infinity in the transverse direction.

(3) The energy decay-law in the final period for the cases (i) and (ii) above is given by

$$\frac{1}{2}\left(\overline{u_{\parallel}^2}+\overline{u_{\perp}^2}+\overline{u_{az}^2}\right) \propto (\nu t)^{-5/2}.$$

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References

Bass, J. (1954). Space and time correlations in a turbulent fluid. Part I, Univ. California Pub. in Statistics, **2**, No. 3, 55–84.

Batchelor, G. K., and Stewart, R. (1950). Anisotropy of the spectrum of turbulence at small wave-numbers. *Quart. J. Mech. App. Math.*, **3**, 1.

Batchelor, G. K., and Townsend, A. A. (1948). Decay of turbulence in the final period. *Proc. Roy. Soc.*, A, **194**, 527.

Chandrasekhar, S. (1950*a*). The theory of axisymmetric turbulence. *Phil. Trans.*, A, **242**, 557.

——— (1950*b*). The decay of axisymmetric turbulence. *Proc. Roy. Soc.*, A, **203**, 358.

Ghosh, K. M. (1957). On localized axisymmetric turbulence. *Proc. Nat. Inst. Sci. India*, **23**, A, 341.