THE EVALUATION OF COVOLUME FUNCTION IN HUNT-HINDS' AND GOLDIE'S METHODS OF INTERNAL BALLISTICS BY THE USE OF RUSSIAN TABLES

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(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received October 19, 1957 ; read June 26, 1958*)

Abstract

In the present paper we have shown how the Russian tables of internal ballistics can be utilized to evaluate the covolume function in the methods of Hunt-Hinds and Goldie.

1. Introduction

The integration of the equations of internal ballistics is quite straightforward except for the evaluation of the covolume function. In almost all the standard methods of internal ballistics (Sugot, 1928; Hunt and Hinds, 1929; Drozdov, 1936; Goldie, 1945; Billiard, 1948), this function can be expressed as a definite integral. It has been shown by Kapur (1957*a*) that this integral can be expressed in terms of incomplete Beta (and in the particular case $\theta' = 0$ in terms of incomplete Gamma) functions and this would have completely solved the problem except for the fact that for the evaluation of the function, new tables of $\beta(p, q)$ for

$$p = 8{\cdot}00,\ 8{\cdot}05,\ 8{\cdot}10,\ 8{\cdot}15,\ 8{\cdot}20,\ 8{\cdot}25,\ 8{\cdot}30,\ 8{\cdot}35,\ 8{\cdot}40,\ 8{\cdot}45,\ 8{\cdot}50$$
$$q = 1{\cdot}00,\ 1{\cdot}05,\ 1{\cdot}10,\ 1{\cdot}15,\ 1{\cdot}20,\ 1{\cdot}25,\ 1{\cdot}30,\ 1{\cdot}35,\ 1{\cdot}40,\ 1{\cdot}45,\ 1{\cdot}50$$

would have to be prepared. In the absence of these tables very wide interpolations have to be done in the existing tables of Pearson (1934).

To avoid this difficulty of numerical evaluation of definite integrals, Goldie (1945) suggested an approximate formula for his covolume function $G(f)$ (Corner, 1950) and stated that the error by the use of his formula would be greatest at all-burnt and that this would not exceed ± 10 per cent. Kapur (1957*b*), however, showed by actually evaluating this function in one particular case that the error could be as high as 30 per cent and that, to a first approximation, the error is not necessarily greatest at all-burnt and can actually be a minimum there in some cases.

To resolve this question of the validity of Goldie's approximation formula more exactly, Dr. Corner kindly offered to the author a few months back to get the function $G(f)$ tabulated on an electronic computor in the U.K. for three values of z, four of θ, six of f and the usual range of M from 0 to 4.*

In the present paper, we have shown however that with the help of the Russian tables of internal ballistics much of the labour can be avoided so that the calculation can be done on an ordinary calculating machine. We have given a simple transformation which expresses $G(f)$ in terms of the function $\int_0^\beta Z^{\frac{B}{B_1}}\, d\beta$ occurring in the

* These tables have since been received by the author and are being published separately, with Dr. Corner's permission.

method of Drozdov (1936) and tabulated in Serebrykov (1949). We have used these tables to deduce certain results regarding the validity of Goldie's approximate formula.

2. Covolume Function in Hunt-Hinds' Methods

The basic equations of internal ballistics (H.M.S.O., 1951, p. 83) are:

$$z = \zeta(\xi - \bar{B}z) + \tfrac{1}{2}(\gamma - 1)\frac{\eta^2}{M} \quad \ldots \quad (1)$$

$$M\zeta = \eta\frac{d\eta}{d\xi} \quad \ldots \quad (2)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (3)$$

$$\zeta = -\eta\frac{df}{d\xi}, \quad \ldots \quad (4)$$

where

$$\xi = 1 + \frac{x}{l}, \quad \ldots \quad (5)$$

$$\eta = \frac{AD}{FC\beta}v, \quad \ldots \quad (6)$$

$$\zeta = \frac{Al}{FC}p, \quad \ldots \quad (7)$$

$$M = \frac{A^2D^2}{FC\beta^2 W_1}, \quad \ldots \quad (8)$$

$$\bar{B} = \left(b - \frac{1}{\delta}\right)\frac{C}{Al}, \quad \ldots \quad (9)$$

$$Al = K_0 - \frac{C}{\delta}. \quad \ldots \quad (10)$$

In Hunt-Hinds' method (H.M.S.O., 1951, pp. 84–86) it is shown that:

$$\xi = \xi'(1 - \bar{B}C) = H^N(1 - \bar{B}C), \quad \ldots \quad (11)$$

where

$$\log H = \int_0^\eta \frac{\eta d\eta}{MNz_0 + \eta N(1 - \theta + 2\theta f_0) - \eta^2} = \int_0^\eta \frac{\eta d\eta}{(a-\eta)(b+\eta)} \quad \ldots \quad (12)$$

$$N = \frac{M}{\theta'}, \quad \theta' = \theta + \tfrac{1}{2}(\bar{\gamma} - 1)M \quad \ldots \quad (13)$$

and

$$C = z_0 - \frac{z}{\xi'} + \int_{z_0}^{z} \frac{1}{\xi'}dz. \quad \ldots \quad (14)$$

$\log H$ has been tabulated (H.M.S.O., 1951, p. 227) for values of $\frac{b}{a}$ going from $-0{\cdot}20$ to $+0{\cdot}20$ and for $\frac{\eta}{a}$ going from $-0{\cdot}60$ to $0{\cdot}60$, but the definite integral on the R.H.S.

of (14) has not been tabulated; only some methods for evaluating it approximately when $\bar{B}$ is small, are given.

3. Covolume Function in Drozdov's Method

The following two functions occur in this method:

$$\log Z = \int_0^X \frac{XdX}{X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}} = \int_0^\beta \frac{\beta d\beta}{\beta^2 - \beta - \gamma} \quad \text{.. .. (15a)}$$

and

$$I\left(\beta, \gamma, \frac{B}{B_1}\right) = \int_0^\beta Z^{\frac{B}{B_1}} d\beta = \frac{B_1}{k_1}\int_0^X Z^{\frac{B}{B_1}} dX, \quad \text{.. .. (15b)}$$

where

$$B = M,\ \psi_0 = z_0,\ k_1 = 1 - \theta + 2\theta f_0 \quad \text{.. .. .. (16)}$$

$$B_1 = \theta + \tfrac{1}{2}B(\bar{\gamma} - 1) = \theta + \tfrac{1}{2}M(\bar{\gamma} - 1) = \theta' \quad \text{.. .. .. (17)}$$

$$\beta = \frac{B_1}{k_1}X,\ \gamma = \frac{B_1\psi_0}{k_1^2} \quad \text{.. .. .. (18)}$$

and

$$X = f_0 - f \quad \text{.. .. .. (19)}$$

From (13), (16) and (17)

$$\frac{B}{B_1} = \frac{M}{\theta'} = N \quad \text{.. .. .. .. (20)}$$

From (2), (4) and (19)

$$\eta = M(f_0 - f) = MX \quad \text{.. .. .. .. (21)}$$

4. Relation between the Two Sets of Functions

From (12), (15a), (15b), (17) and (21)

$$\log Z = -\int_0^\eta \frac{\eta d\eta}{MNz_0 + N\eta(1 - \theta + 2\theta f_0) - \eta^2} = -\log H$$

$$\therefore\ Z = \frac{1}{H} \quad \text{.. .. .. (22)}$$

Also

$$\int_{z_0}^z \frac{1}{\xi'}dz = \int_{z_0}^z H^{-N}dz = \int_{z_0}^z Z^{\frac{B}{B_1}}dz$$

$$= \int_0^\eta Z^{\frac{B}{B_1}}\left\{(1 - \theta + 2\theta f_0)\frac{1}{M} - \frac{2\theta\eta}{M^2}\right\}d\eta$$

$$= \int_0^X Z^{\frac{B}{B_1}}\{(1 - \theta + 2\theta f_0) - 2\theta X\}dX \quad \text{.. .. .. (23)}$$

Now let

$$J \equiv \int_0^X Z^{\frac{B}{B_1}} X dX, \; K \equiv \int_0^X Z^{\frac{B}{B_1}} dX \quad .. \quad .. \quad .. \quad (24)$$

but from (15)

$$\frac{dZ}{Z} = \frac{XdX}{X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}}$$

$$\therefore \quad J = \int_0^X Z^{\frac{B}{B_1}-1}\left(X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}\right) dZ$$

$$= \frac{B_1}{B} Z^{\frac{B}{B_1}}\left(X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}\right) - \frac{B_1}{B}\int_0^X Z^{\frac{B}{B_1}}\left(2X - \frac{k_1}{B_1}\right) dX$$

$$\therefore \quad J\left(1 + \frac{2B_1}{B}\right) = \frac{B_1}{B} Z^{\frac{B}{B_1}}\left(X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}\right) + \frac{k_1}{B}\int_0^X Z^{\frac{B}{B_1}} dX$$

or

$$J(B+2B_1) = B_1 Z^{\frac{B}{B_1}}\left(X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}\right) + k_1 K \quad .. \quad .. \quad (25)$$

(23) then gives

$$\int_{z_0}^{z} \frac{1}{\xi'} dz = k_1 K - 2\theta J$$

$$= k_1 K\left(1 - \frac{2\theta}{B+2B_1}\right) - \frac{2\theta}{B+2B_1} B Z^{\frac{B}{B_1}}\left(X^2 - \frac{k_1}{B_1}X - \frac{\psi_0}{B_1}\right)$$

But from (15), (18) and (24)

$$K = \frac{k_1}{B_1} I$$

$$\therefore \quad \int_{z_0}^{z} \frac{1}{\xi'} dz = \frac{(1-\theta+2\theta f_0)^2}{\theta'}\left(1 - \frac{2\theta}{M+2\theta'}\right)\int_0^\beta Z^{\frac{B}{B'}} d\beta$$

$$- \frac{2\theta\theta'}{M+2\theta} Z^{\frac{B}{B_1}}\left\{(f_0-f)^2 - \frac{1-\theta+2\theta f_0}{\theta'}(f_0-f) - \frac{(1-f_0)(1+\theta f_0)}{\theta'}\right\} \quad .. \quad (26)$$

Now

$$Z^{\frac{B}{B_1}} = H^{-\frac{M}{\theta'}} = H^{-N} \quad .. \quad .. \quad .. \quad .. \quad (27)$$

and can be evaluated from the tables of log H (H.M.S.O., 1951) or from those of log Z^{-1} tabulated on page 290 of Serebrykov (1949) for range of β from 0 to 0·6 and of γ from 0 to 0·2.

The integral I has been tabulated on pages 660–665 of Serebrykov (1949) for the above ranges of β and γ and for $\frac{B}{B_1} = 5, 6, 7, 8, 9, 10.$

For any given values of f_0, θ, M, $\bar{\gamma}$ and f we can find β, γ and $\frac{B}{B_1}$ and then evaluate the integral on the L.H.S. of (26).

The formula (26) takes a particular simple form for tabular charges ($\theta = 0$) for which it becomes

$$\int_{z_0}^{z} \frac{1}{\xi'} dz = \frac{1}{\frac{1}{2}(\bar{\gamma}-1)M} \int_0^{\beta} Z^{\frac{B}{B_1}} d\beta. \qquad (28)$$

5. Covolume Function in Goldie's Method

In Goldie's method, we use the dependent variable $\xi - \bar{B}z$ instead of ξ used in Hunt-Hinds' method.

From (11)

$$\xi - \bar{B}z = \bar{\xi}(1-\bar{B}C) - \bar{B}z = \xi'\left(1 - \bar{B}C - \frac{\bar{B}z}{\xi'}\right) = \xi'(1-\bar{B}C') \qquad (29)$$

where from (14)

$$C' = C + \frac{z}{\xi'} = z_0 + \int_{z_0}^{z} \frac{1}{\xi'} dz. \qquad (30)$$

By comparison with Goldie's discussion (Corner, 1950, p. 194) we find that C' is the same as his covolume function. Thus Goldie's covolume function can be easily evaluated with the help of (26).

For a tubular charge

$$G(f) = z_0 + \frac{2}{(\bar{\gamma}-1)M} \int_0^{\beta} Z^{\frac{B}{B_1}} d\beta. \qquad (31)$$

6. Goldie's Approximation Formula

Goldie's approximate formula (Corner, 1950, p. 196) is

$$\bar{G}(f) = z_0 + \frac{1+\theta}{(M+\theta')(M+2\theta')} \times \left\{ \bar{\gamma}M(1+\theta) - \left(\frac{1-\frac{1}{2}(\bar{\gamma}-1)M+\theta' f}{1+\theta}\right)^{1+\frac{M}{\theta'}} \times \{M\gamma(1+\theta) - 2\theta(1-f)(M+\theta'\} \right\} \qquad (32)$$

For $z_0 = 0{\cdot}1, f_0 = 0{\cdot}9$, $\theta = 0$, $\bar{\gamma} = 1{\cdot}2$ we get

Table I [$M = 0{\cdot}5$, $\theta = 0$]

f	$G(f)$	$\bar{G}(f)$
0·9	0·100	0·199
0·6	0·400	0·453
0·3	0·680	0·698
0	0·860	0·884

Table II [$M = 1$, $\theta = 0$]

f	$G(f)$	$\bar{G}(f)$
0·9	0·100	0·196
0·6	0·375	0·429
0·3	0·610	0·600
0	0·780	0·729

Table III [$M = 2$, $\theta = 0$]

f	$G(f)$	$\bar{G}(f)$
0·9	0·100	0·152
0·6	0·365	0·266
0·3	0·535	0·371
0	0·680	0·435

Tables I and II show that the error is not necessarily greatest at burnt and all the three tables show that the error can be substantially outside the limits $\pm$ 10 per cent given by Goldie.

Acknowledgements

The author is grateful to Dr. J. Corner, Ph.D., and Professor P. L. Bhatnagar, F.A.Sc., F.N.I., for their active interest and encouragement and to Professor D. S. Kothari, F.N.I., and Dr. R. S. Varma, F.N.I., for making Serebykov's book available to him.

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