A NOTE ON THE UNIFICATION OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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ABSTRACT

In this note, a condition is deduced under which a polynomial $p_n(x)$ having the Rodrigues formula $p_n(x) = \frac{1}{k_n w(x)} \left(\frac{d}{dx}\right)^n \left\{ w(x) \times \left[X(x) \right]^n \right\}$ satisfies the second order differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

When $X(x)$ is a quadratic expression in x, all the classical polynomials are included in $p_n(x)$.

1. Tricomi * has defined the orthogonal polynomials $p_n(x)$, which satisfy a differential equation of the form

$$A(x)y'' + B(x)y' + \lambda_n y = 0 \quad \ldots \quad (1)$$

where $A(x)$, $B(x)$ are independent of n, and λ_n is independent of x, having the Rodrigues formula

$$p_n(x) = \frac{1}{k_n \omega(x)} \left(\frac{d}{dx}\right)^n \left[\omega(x) \{X(x)\}^n\right] \quad \ldots \quad (2)$$

where K_n is a constant and X is a polynomial in x of degree k whose coefficients are independent of n.

In the present note we have derived a condition under which the polynomials having the Rodrigues formula (2) should satisfy a differential equation (1), where now λ_n is also a function of x; unlike in the work of Tricomi, $A(x)$, $B(x)$ may contain n.

We have discussed in full the case when X is quadratic, which includes the Bessel polynomial in addition to the polynomials of Legendre, Gegenbauer, Tchebycheff, Jacobi, Hermite and Laguerre; and at the end we mention the case when X is a cubic polynomial.

2. We shall prove the following theorems :

Theorem (*a*).† The condition that the polynomials defined by

$$p_m(n, x) = \frac{1}{k_m w(x)} \left(\frac{d}{dx}\right)^m (w(x) X^n) \quad \ldots \quad (3)$$

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* I am grateful to Prof. Erdélyi for bringing Tricomi's (1955) work to my notice.

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[(m, n) are positive integers] may have a second order differential equation is

$$X^{(iv)}+\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(iii)}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

The numerals appearing here (and henceforward) as superscripts denote the orders of differentiation ; X is a polynomial of degree k in x ($k \leqslant m$) and

$$\log \omega = -\int^{x} \frac{h(t)}{X(t)}\, dt \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where $h(t)$ is a polynomial of degree $(k-1)$.

PROOF. Let $y = w \,.\, X^{n}$;

then $$Xy^{(i)} = \left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)y.$$

Differentiating this $(m+1)$ times and equating the coefficients of $y^{(m-1)}, \ldots, y$ to zero, in order to get a second order differential equation in $y^{(m)} = \left(\frac{d}{dx}\right)^{m} [w(x)X^{n}]$, we get the set of conditions :

$$(m+2-r)X^{(r)}-r\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(r-1)}=0 \qquad \ldots \quad \ldots \quad (6)$$

for $r = 3, 4, \ldots\ldots, (m+1)$.
Changing r into $r-1$ we obtain

$$(m+3-r)\, X^{(r-1)}-(r-1)\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(n-2)}=0 \qquad \ldots \quad \ldots \quad (6')$$

Differentiating (6′) and subtracting from (6) we get

$$X^{(n)}+\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(n-1)}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

This procedure can be repeated for every adjacent pair of (6), and (7) holds for $r = 4, 5, \ldots\ldots, (m+1)$.

From (7) we have for $r = (m+1)$

$$X^{(m+1)}+\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(m)}=0.$$

Putting this in $$X^{(m+1)}-(m+1)\left(nX^{(i)}+\frac{\omega^{(i)}}{\omega}X\right)^{(m)}=0$$

obtained from (6) with $r = (m+1)$, we find

$$X^{(m+1)} = 0\,, \text{ and } \left(X\frac{\omega^{(i)}}{\omega}\right)^{(m)}=0$$

implying thereby that X can *at most* be a polynomial of degree m ; and $X\dfrac{w^{(i)}}{w} = -h$, a polynomial of degree one less than that of g.

Hence $$\log \omega = -\int^x \frac{h(t)}{X(t)}\,dt.$$

If X is of degree k, then $r = k+1$ in (7), and proceeding as above, we can show that h is of degree $(k-1)$.
Hence the theorem.

Cor. 1: If X is of degree k, $p_m(n, x)$ is of degree $(nk-m)$.

Cor. 2: The differential equation which (3) satisfies under these conditions is

$$Xv^{(ii)} + \left[(m-n+1)\,X^{(i)} - h\right] v^{(i)} + \left[(m+1)\,(\tfrac{1}{2}m-n)\,X^{(ii)} + nh^{(i)} - (m-n)(\log \omega)^{(ii)}X\right] v = 0 \qquad .. \quad (8)$$

From now on, we consider the case $m = n$.

Theorem (*b*). The least degree of X which makes $\{p_n(x)\}$ an orthogonal set is 2.

PROOF. Let X be of degree k and let all the derivatives up to and including $(n-1)$th of (wX^n) vanish at (α, β). Consider the scalar product

$$\begin{aligned} < p_n, p_m > &= \int_\alpha^\beta \omega(x)p_n(x)p_m(x)\,dx \\ &= \frac{(-1)^m}{k_m}\int_\alpha^\beta \omega X^m \left(\frac{d}{dx}\right)^m [p_n(x)]\,dx \\ &= 0 \text{ if } m > n\,.\,(k-1), \end{aligned}$$

for by Cor. 1, Theorem (*a*), $p_n(x)$ is of degree $n\,.\,(k-1)$.
Similarly $< p_n, p_m = 0$ if $n > m\,.\,(k-1)$.

Therefore the simplest case in which the sequence $\{p_n(x)\}$ forms an orthogonal set is when $k = 2$.

3. *The classification of the classical orthogonal polynomials*

From theorem (*b*) we see that the simplest of all forms of X is a quadratic, and therefore let

$$X = ax^2+bx+c$$

$$\log w = -\int^x \frac{pt+q}{at^2+bt+c}\,dt.$$

Hence $$p_n(x) = \frac{1}{K_n}\, e^{\int^x \frac{pt+q}{at^2+bt+c}dt} \left(\frac{d}{dx}\right)^n \left[e^{-\int^x \frac{pt+q}{at^2+bt+c}dt} (ax^2+bx+c)^n \right] \qquad .. \quad (9)$$

This, from Theorem (*a*), Cor. 2, satisfies the differential equation

$$(ax^2+bx+c)y^{(ii)} + [(2a-p)x+b-q]y^{(i)} + n[p-(n+1)a]y = 0 \qquad .. \quad (10)$$

By Theorem (*a*), Cor. 1, $p_n(x)$ is of exact degree n and by Theorem (*b*) $\{p_n(x)\}$ forms an orthogonal set.

It is curious to note that Gauss was the first to study a differential equation of the form (10) and its connections with the hypergeometric equation; the polynomial solution of (10), namely (9), may justly be christened '*Gauss Polynomial*'.

We give below the connection of (9) with all the classical orthogonal polynomials; the notations employed here are as in Erdelyi (1953). This suggests $p_n(x)$ to be termed '*Hold-all Polynomial*'.

Polynomial	a	b	c	p	q	$K_n p_n(x)$
Jacobi	1	-1	0	$-\lambda-\mu$	λ	$n!\, p_n^{(\lambda,\mu)}(2x-1)$
Legendre	1	0	-1	0	0	$2^n n!\, p_n(x)$
Hermite	0	0	-1	-2	0	$H_n(x)$
Associated Laguerre	0	1	0	1	$-\alpha$	$n!\, L_n^{\alpha}(x)$
Associated Bessel (Krall and Frink (1949))	1	0	0	$2-\lambda$	$-\mu$	$\mu^n y_n(x, \lambda, \mu)$.

In this manner one arrives at an extension of Jacobi's polynomial. We may note that the above list exhausts all the orthogonal polynomials which can be obtained by taking $k = 2$, i.e., X, a quadratic.

4. *Discussion of the case when* $X = ax^3+bx^2+cx+d$

We give below two non-orthogonal polynomials, which may be of interest, derived by taking a cubic polynomial for X.

If we take $X = ax^3+bx^2+cx+d$,

then
$$\log w = -\int^x \frac{pt^2+qt+r}{at^3+bt^2+ct+d}\,dt \quad \ldots \quad (11)$$

The condition (6), with $r = 3$, shows that

$$p = (2n+1)a \quad \ldots \quad (12)$$

The polynomials thus defined are of degree $2n$, and satisfy the differential equation

$$(ax^3+bx^2+cx+d)y^{(ii)}+[-2(n-1)ax^2+(2b-q)x+(c-r)]y^{(i)} +n[(n-1)ax+q-(n+1)b]y = 0 \quad \ldots \quad (13)$$

(*a*) Let us choose $q = 0 = b = c = d$, $a = 1$, $r = -2$ so that we have

$$A_n(x) = e^{\frac{1}{x^2}}\, x^{2n-1} \left(\frac{d}{dx}\right)^n \left(e^{-\frac{1}{x^2}}\, x^{n-1}\right) \quad \ldots \quad (14)$$

satisfying the differential equation

$$x^3v^{(ii)}-2\,[(n+1)x^2-1]\,v^{(i)}+n(n-1)xv = 0 \quad \ldots \quad (15)$$

(*b*) Let $a = -b = r = 1$; $c = d = q = 0$. Then

$$S_n(x) = e^{\frac{1}{x}}\, x^{2n}\,(x-1) \left(\frac{d}{dx}\right)^n \left(e^{-\frac{1}{x}}\,(x-1)^{n-1}\right) \quad \ldots \quad (16)$$

satisfying the differential equation

$$x^2(x-1)y^{(ii)}-[2(n-1)x^2+2n+1]y^{(i)}+xn(n-1)y = 0 \quad \ldots \quad (17)$$

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References

Erdélyi, A. *et al.* (1953). Higher Transcendental Functions. Bateman Manuscript Project, McGraw Hill.

Krall, H. L. and Frink, O. (1949). A new class of orthogonal polynomials : The Bessel polynomials. *Trans. Amer. Math. Soc.*, **65**, 110.

Tricomi, F. G. (1955). Vorlesungen Über Orthogonalreihen. Springer-Verlag, Berlin.