A NOTE ON ROTATING CONFIGURATION ASSOCIATED WITH TOROIDAL MAGNETIC FIELD

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ABSTRACT

In the present note, the equilibrium form of an incompressible fluid mass rotating about an axis in presence of a toroidal magnetic field, whose axis coincides with the axis of rotation, is discussed and the conditions that the equilibrium configuration may be a sphere, a prolate spheroid or an oblate spheroid are derived.

1. E. Tandberg-Hanssen (1953) has studied the effect of magnetic field on the configuration of an incompressible fluid mass having an infinitesimally small velocity of rotation, the magnetic field being parallel to the axis of rotation. John Sykes (1957) has studied the equilibrium configuration of an incompressible fluid mass having a toroidal velocity $\tilde{\omega}\sqrt{\beta^2\tilde{\omega}^2P+V_0^2}$ in an axi-symmetric magnetic field discussed by Chandrasekhar (1956) :

$$\frac{\vec{H}}{\sqrt{4\pi\rho}} = -\frac{\partial}{\partial\mu}[(1-\mu^2)P]I_r - \frac{\sqrt{1-\mu^2}}{r}\frac{\partial}{\partial r}(r^2P)I_\theta + r\sqrt{1-\mu^2\alpha P I_\phi},$$

where ρ is the density of the fluid ; r, θ, ϕ are spherical polar co-ordinates ; I_r, I_θ, I_ϕ are unit vectors in these co-ordinates ; P is a scalar function independent of ϕ and

$$\mu = \cos\theta,\ \tilde{\omega} = r\sqrt{1-\mu^2}.$$

De (1958) has studied the equilibrium form of a fluid mass under the action of a toroidal magnetic field $H_\phi = H_0\,\tilde{\omega}$ rotating rigidly with constant angular velocity αH_0 about the polar axis.

In this note, we consider the equilibrium configuration of an incompressible fluid mass, rotating about an axis, which possesses electric currents giving rise to toroidal magnetic field whose axis coincides with the axis of rotation. It is found that the equilibrium configuration is a prolate spheroid, sphere or an oblate spheroid according as the toroidal fluid velocity at every point of the configuration is greater than, equal to or less than the Alfvén wave velocity corresponding to the toroidal magnetic field at that point.

2. The equations of motion of a fluid mass, rotating in presence of a magnetic field $\vec{H}$, are

$$-\rho w^2\tilde{\omega} = -\text{grad } p - \rho \text{ grad } V + \frac{1}{4\pi}(\text{curl } H)\,\chi H$$

where

W = velocity of rotation,
p = hydrostatic pressure,
V = gravitational potential,
$\vec{j}$ = the volume current density vector.

In the case of only a toroidal magnetic field H_ϕ, these equations reduce to

$$\rho w^2\tilde{\omega} = \frac{\partial}{\partial\tilde{\omega}}\left[p+\rho V+\frac{1}{8\pi}H_\phi^2\right]+\frac{1}{4\pi}\frac{H_\phi^2}{\tilde{\omega}} \quad \ldots \quad (1)$$

$$O = \frac{\partial}{\partial z}\left[p+\rho V+\frac{1}{8\pi}H_\phi^2\right] \quad \ldots \quad (2)$$

we have from (2)

$$p+\rho V+\frac{1}{8\pi}H_\phi^2 = f(\tilde{\omega}) \quad \ldots \quad (3)$$

where $f(\tilde{\omega})$ is some arbitrary function of $\tilde{\omega}$ only, and from (1),

$$\frac{\partial}{\partial\tilde{\omega}}\left(p+\rho V+\frac{1}{8\pi}H_\phi^2\right) = \frac{\rho}{\tilde{\omega}}\left(w^2\tilde{\omega}^2-\frac{H_\phi^2}{4\pi\rho}\right) \quad \ldots \quad (4)$$

or

$$\frac{\tilde{\omega}}{\rho}\frac{\partial}{\partial\tilde{\omega}}\left(p+\rho V+\frac{1}{8\pi}H_\phi^2\right) = V_\phi^2 - V_A^2 \quad \ldots \quad (5)$$

where

$$V_A = \frac{H_\phi}{(4\pi\rho)^{1/2}} \quad \ldots \quad (6)$$

and

$$V_\phi = \tilde{\omega}w \quad \ldots \quad (7)$$

If $V_\phi = V_A$, i.e. if at every point the velocity is equal to Alfvén wave velocity at that point,

$$\frac{\partial}{\partial\tilde{\omega}}\left(p+\rho V+\frac{1}{8\pi}H_\phi^2\right) = 0 \quad \ldots \quad (8)$$

and consequently, from (3), we have

$$p+\rho V+\frac{1}{8\pi}H_\phi^2 = \text{const.} \quad \ldots \quad (9)$$

We know that outside an axisymmetric configuration $[H_\phi]_{\text{external}} = 0$. From the continuity of magnetic field, there will be no surface currents if $[H_\phi]_{\text{internal}}$ vanishes at the boundary, and the boundary will be defined by $p = 0$. If, however, $[H_\phi]_{\text{internal}} \neq 0$ on the boundary, there will be surface currents and on the boundary, total pressure $P = p+\frac{H_\phi^2}{8\pi} = C$. Hence, from (9), we have, in either case,

$V = \text{const.}$ on the boundary.

This is possible only if the equilibrium configuration is a sphere. Hence, if the toroidal velocity at every point is equal to the Alfvén wave velocity at that point, the equilibrium configuration is a sphere.

3. In view of (3), we conclude from (5) that $V_\phi^2 - V_A^2$ is a function of $\tilde{\omega}$ only. Regarding $V_\phi^2 - V_A^2$ as a regular function throughout the configuration, we can expand it in the form

$$V_\phi^2 - V_A^2 = \sum_{n=1}^{\infty} a_n\tilde{\omega}^n .$$

We shall, however, consider the special case

$$V_{\phi}^2 - V_A^2 = a_2\tilde{\omega}^2 + a_4\tilde{\omega}^4 \quad \ldots \quad (10)$$

for sake of simplicity.

From (5), we have

$$p + \rho V + \frac{H_{\phi}^2}{8\pi} = \rho\left[\frac{a_2}{2}\tilde{\omega}^2 + \frac{a_4}{4}\tilde{\omega}^4\right] + \text{constant}. \quad \ldots \quad (11)$$

Let the surface of the equilibrium configuration be

$$r_s = R\left[1 + \sum_{n=1}^{\infty} \epsilon_n P_n(\mu)\right] \quad \ldots \quad (12)$$

Then, the gravitational potential of this configuration at an internal point is

$$V = 2\pi G\rho(R^2 - \tfrac{1}{3}r^2) + 4\pi G\rho R^2 \sum_{n=1}^{\infty} \frac{\epsilon_n}{2n+1}\left(\frac{r}{R}\right)^n P_n(\mu) \quad \ldots \quad (13)$$

Neglecting the squares and products of the constants ϵ_n, equation (9) reduces to the form:

$$V = \frac{4\pi}{3} G\rho R^2 - \frac{8\pi}{3} G\rho R^2 \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \epsilon_n P_n(\mu) \quad \ldots \quad (14)$$

on the surface of the configuration. Using (14) in (11), we have, on the boundary of the configuration,

$$\frac{4\pi}{3} G\rho R^2 - \frac{8\pi}{3} G\rho R^2 \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \epsilon_n P_n(\mu)$$

$$= \text{const.} + \tfrac{1}{2}a_2 r_s^2(1-\mu^2) + \frac{a_4}{4} r_s^4(1-\mu^2)^2 \quad \ldots \quad (15)$$

From (15) it is clear that ϵ's are of the same order as a_2 and a_4 and hence on the right hand side of (15) we shall replace r_s by R consistent with our approximation. Expressing $(1-\mu^2)$ and $(1-\mu^2)^2$ in terms of Legendre polynomials and equating the coefficients of various $P_n(\mu)$ on both sides of (15), we have

$$\left.\begin{aligned} \epsilon_2 &= \frac{5}{8\pi G\rho}(a_2 + \tfrac{4}{7}a_4 R^2) \\ \epsilon_4 &= -\frac{9R^2}{140\pi G\rho} a_4 \\ \epsilon_n &= 0,\ n \neq 2, 4 \end{aligned}\right\} \quad \ldots \quad (16)$$

In the special case when $a_4 = 0$, these become

$$\left.\begin{aligned} &\epsilon_n = 0,\ n \neq 2 \\ \text{and} \quad &\epsilon_2 = \frac{5}{8\pi G\rho} a_2 \end{aligned}\right\} \quad \ldots \quad (17)$$

so that

$$r_s = R\left[1 + \frac{5}{8\pi G\rho} a_2 P_2(\mu)\right] \quad \ldots \quad (18)$$

and hence the equilibrium configuration is a prolate or an oblate spheroid according as a_2 is positive or negative, i.e. according as $V_\phi \gtrless V_A$. We shall now discuss the case when $a_4 \neq 0$.

In this case

$$r_s = R\left[1+\frac{5}{8\pi G\rho}(a_2+\tfrac{4}{7}a_4R^2)P_2(\mu)-\frac{9R^2}{140\pi G\rho}a_4P_4(\mu)\right] \quad .. \quad (19)$$

when $\theta = 0$

$$r_s = R\left[1+\frac{5a_2}{8\pi G\rho}+\frac{41}{140}\frac{R^2a_4}{\pi G\rho}\right] \quad .. \quad .. \quad .. \quad (20)$$

and when $\theta = \frac{\pi}{2}$

$$r_s = R\left[1-\frac{5a_2}{16\pi G\rho}-\frac{227}{1120}\frac{R^2a_4}{\pi G\rho}\right] \quad .. \quad .. \quad .. \quad (21)$$

Hence when a_2 and a_4 are positive, i.e. $V_\phi > V_A$, the equilibrium configuration (19) is more elongated than the equilibrium configuration (18) in the direction of the polar axis; on the other hand if a_2 and a_4 are both negative, then (19) is more flattened than (18). Thus increasing the difference between V_ϕ and V_A increases the prolateness or oblateness.

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