SOME TAUBERIAN THEOREMS FOR NÖRLUND SUMMABILITY

by M. R. PARAMESWARAN, *Ramanujan Institute of Mathematics, Madras 5*

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ABSTRACT

Not many Tauberian results are known for the Nörlund methods of summability. In this paper are proved, for Nörlund methods, certain Tauberian theorems which are strict analogues of results for Abel summability contained in a recent paper by Jakimovski and Parameswaran (1958). In fact, it is an essential step in the proof of each of the theorems given below that, under the (Tauberian) conditions of the theorem, summability by the Nörlund method implies summability by the Abel method and the conclusion of the theorem follows from the corresponding theorem for Abel summability. Some results which have been earlier obtained by Agnew (1946) and Jakimovski (1952) are seen to be corollaries of the results proved here.

INTRODUCTION

Let $s = \{s_n\}$, $n = 0, 1, 2, \ldots$ be a real sequence. (In this note, we shall be concerned with real sequences only.) Let $\{p_n\}$ be a sequence of real numbers and let

$$P_n \equiv p_0+p_1+ \ldots +p_n \neq 0 \quad \text{for any } n. \qquad (1)$$

If

$$u_n = \frac{p_n s_0+p_{n-1}s_1+ \ldots +p_0 s_n}{P_n} \qquad (n = 0, 1, 2, \ldots) \qquad (2)$$

and

$$u_n \to l \text{ as } n \to \infty, \qquad (3)$$

then we say that s is summable by the Nörlund method (N, p_n) to the value l. (Throughout this paper l will denote a finite number.) The method (N, p_n) is regular, i.e. $s_n \to l$ implies $u_n \to l$ if and only if

$$p_n = o(P_n) \text{ and } \sum_{i=0}^{n} |p_i| = O(P_n).$$

In particular, it is not necessary that $p_n \geqslant 0$. The infinite system of equations (2) can be written for short as

$$u = Ns \qquad (4)$$

where $u = \{u_n\}$ and $s = \{s_n\}$ are column-vectors and N is the matrix of the Nörlund method (N, p_n), i.e. of the transformation (2).

For every real number α, let the Hölder transform of order α be defined by

$$H^{\alpha}s \equiv t = \{t_n\},$$

where

$$\Delta^n t_0 = (n+1)^{-\alpha}\Delta^n s_0, \quad \Delta^1 s_n = s_n - s_{n+1}, \quad \Delta^0 s_n = s_n.$$

Then the Hölder method H^{α} is regular for $\alpha > 0$.

If $v = \{v_n\}$ is any real sequence, we write also, for the sake of convenience:

(i) $v \to l$ for $\lim_{n\to\infty} v_n = l$,

(ii) $v = O(1)$ for $v_n = O(1)$ as $n \to \infty$,

(iii) $v = O_L(1)[O_R(1)]$ for $v_n = O_L(1)[O_R(1)]$ as $n \to \infty$,

(iv) $v \leqslant K$ for $v_n \leqslant K$ for all large n

and similarly $v \geqslant K$, etc., in other cases.

We first give two known Tauberian results for Nörlund summability and show how they are included in the results of this paper.

THEOREM A. (Agnew, 1946.) *Let (N, p_n) be a regular Nörlund method with $p_n \geqslant 0$. If*

(i) $Ns \to l$,

and (ii) $n(s_n - s_{n-1}) = O(1)$,

then $s_n \to l$.

Agnew gives the condition (ii) in a slightly different form which implies (ii), and he uses (ii) in the proof. Theorem A is included in the following result due to Jakimovski (1952).

THEOREM B. *Same as Theorem A, but with (ii) replaced by*

$$\text{(ii)}' \quad \lim_{\lambda\to 1+0} \overline{\lim_{n\to\infty}} \max_{n\leqslant m\leqslant \lambda n} |s_m - s_n| = 0.$$

Theorem B is itself contained in Theorem 4 of this paper. For, by a result of Szász (1929, Satz a), the condition (ii)′ implies

$$s_n - \frac{s_0+s_1+\dots+s_n}{n+1} = O(1) \qquad (5)$$

i.e.

$$(H^0 - H^1)s = O(1) \qquad (6)$$

so that we may apply Theorem 4 with $\alpha = c = 0$ and get that

$$H^1 s \to l. \qquad (7)$$

It is a classical result that (7) and (ii)′ imply convergence of s_n to l and thus Theorem B is proved.

Both Agnew (1946) and Jakimovski (1952) state their results for Nörlund methods (N, p_n) with $p_n \geqslant 0$, but this restriction is not essential. Agnew states also, without proof, that, if in Theorem A we have merely that

$$n(s_n - s_{n-1}) = O_L(1) \qquad (8)$$

then we could conclude:

$$H^1 s \to l. \qquad (9)$$

In the proof of the results of this paper, we shall need in particular the notion of *quasi-monotone sequences* and Tauberian theorems for Abel summability, contained in a recent paper by Jakimovski and Parameswaran (1958; Lemmas 1, 2) and a result of Jurkat and Peyerimhoff (1955; Lemma 3) extending a well-known result (Hardy, 1949; Theorem 18), besides other classical results.

LEMMA 1. *If the real sequence* $s = \{s_n\}$ *satisfies the condition*

$$(H^{-1}-\overline{c+1}\,H^0)s \leqslant 0 \quad \textit{for some real number } c \qquad \ldots \qquad \ldots \quad (10)$$

then $\{s_n\}$ *is 'quasi-monotone decreasing' in the sense that it will satisfy the condition*

$$s_{n+1}\left(1-\frac{c}{n+1}\right) \leqslant s_n \qquad (n > n_0(c)), \qquad \ldots \qquad \ldots \quad (11)$$

or the equivalent condition that there exists a real number d, of the same sign as c and such that

$$s_{n+1} \leqslant \left(1+\frac{d}{n}\right)s_n \qquad (n \geqslant n_0(d)) \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

which necessarily implies that s_n *is ultimately of the same sign.*

LEMMA 2. *If the real sequence* $s = \{s_n\}$ *is Abel-summable to l and if there exist real numbers* α, *c*, *K such that*

$$t \equiv (H^{\alpha}-\overline{c+1}\,H^{\alpha+1})s \leqslant K \qquad \ldots \qquad \ldots \qquad \ldots \quad (13)$$

then

$$H^{\alpha+1}\,s \to l. \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (14)$$

If, instead of (13), $t = O(1)$ [*or t is convergent*], *then* $H^{\alpha+\epsilon}\,s \to l$ *for every* $\epsilon > 0$ [*or* $H^{\alpha}s \to l$].

LEMMA 3. *If* $s = \{s_n\}$ *is summable to l by the regular* (N, p_n) *method, then the series* $(1-z)\sum_{n=0}^{\infty} s_n z^n$ *has a positive radius of convergence and defines an analytic function* $\sigma(z)$ *which has at most poles in* $|z| < 1$. *Furthermore,* $\sigma(z)$ *is regular for* $1-\epsilon < x < 1$ *with some* $0 < \epsilon \leqslant 1$, *and* $\sigma(z) \to l$ *when* $x \to 1-0$.

THEOREM 1. *Let* $s = \{s_n\}$ *be a real sequence summable by the regular Nörlund method* (N, p_n) *to the value l. Let there exist real numbers* α, *K*, *c*, λ, *L such that*

$$(H^{\alpha}-\overline{c+1}\,H^{\alpha+1})s \leqslant K, \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (15)$$

$$H^{\lambda}s \geqslant L. \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (16)$$

Then $H^{\alpha+1}s \to l$.

Proof. Writing $\tau = H^{\alpha+1}s$, we have from (15):

$$(H^{-1}-\overline{c+1}\,H^0)\tau \leqslant K. \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \quad (17)$$

Case (*a*): $c = 0$. Now (17) means that

$$(n+1)(\tau_{n+1}-\tau_n)-c\tau_{n+1} \leqslant K, \qquad \ldots \qquad \ldots \qquad \ldots \quad (18)$$

so that when $c = 0$ we have, writing $\tau_n = t_0+t_1+\ldots+t_n$,

$$(n+1)t_n \leqslant K, \text{ or } t_n \leqslant K/(n+1),$$

and hence

$$\tau_n = t_0+t_1+\ldots+t_n = O_R(\log n). \quad \ldots \qquad \ldots \qquad \ldots \quad (19)$$

If now β is any positive integer such that

$$\beta \geqslant \max(\lambda, \alpha+1), \quad \ldots \quad (20)$$

then (16) and (19) give $L < H^{\beta}s = O_R(\log n)$,

i.e.
$$H^{\beta}s = O(\log n) = o(n), \quad \ldots \quad (21)$$

so that, by a familiar argument (cf. Hardy, 1949 ; Theorem 39) we have

$$s = o(n^{\beta+1}). \quad \ldots \quad (22)$$

It follows from (22) that the series $\Sigma s_n z^n$ converges for $|z| < 1$ and hence Abel's method is applicable to the sequence s. Let $f(z) = \Sigma s_n z^n$ for $|z| < 1$. The application of Lemma 3 shows that $\lim\limits_{x \to 1-0} (1-z)f(z)$ exists and equals l ; that is, the sequence s is Abel-summable to l. It now follows from Lemma 2 that $H^{\alpha+1}s \to l$.

Case (b) : $c \neq 0$. Let the number β be defined as in (20) ; we have then, from (17) and (18) :

$$(H^{-1} - \overline{c+1}\, H^0)\left(H^{\beta}s + \frac{K}{c}\right) \leqslant 0. \quad \ldots \quad (23)$$

Let us write $\sigma = H^{\beta}s + \dfrac{K}{c}$. Then (23) and Lemma 1 give

$$\sigma_{n+1}\left(1 - \frac{c}{n+1}\right) \leqslant \sigma_n,\ \sigma_{n+1} \leqslant \left(1 + \frac{d}{n}\right)\sigma_n, \quad \ldots \quad (24)$$

so that ultimately (i.e. for all large n)

$$\text{either } \sigma \geqslant 0 \text{ or } \sigma \leqslant 0. \quad \ldots \quad (25)$$

There are three possibilities : (i) $c > 0$, $\sigma \geqslant 0$, (ii) $c > 0$, $\sigma < 0$, and (iii) $c < 0$; we shall deal with these cases separately.

(i) *Let* $c > 0$, $\sigma \geqslant 0$. By a result of Szász (1944*b*) $\Gamma(n)\sigma_n/\Gamma(n+c)$ is decreasing, so that $\sigma = O(n^c)$, or $s = O(n^{c+\beta})$, which, together with the Nörlund summability of s, gives, in view of Lemma 3, the Abel-summability of s. The required conclusion follows immediately from Lemma 2.

(ii) *Let* $c > 0$, $\sigma < 0$. It follows from (24) that σ_n is monotone decreasing. Since $H^{\beta}s \geqslant L$ and hence $\sigma_n \geqslant L + K/c$ this means that σ (and therefore $H^{\beta}s$) is convergent or that s is Abel-summable ; Lemma 2 applied to the sequence s yields the required result.

(iii) *Let* $c < 0$. We consider again various possibilities separately : $c = -1$, $c < -1$ and $-1 < c < 0$.

If $c = -1$, then the conditions (15) and (16) give

$$H^{\beta}s = O(1) \text{ or } s = O(n^{\beta}).$$

The required conclusion follows now as in Case (i) above.

If $c < -1$, we can write (15) and (16) as

$$(H^{\beta} - \overline{c+1}\, H^{\beta+1})s \leqslant K,$$

$$(c+1)\, H^{\beta+1}s \leqslant (c+1)L,$$

so that, adding, we get $L < H^{\beta}s < K + (c+1)L$. We have then $s = O(n^{\beta})$ and the proof is again completed as in Case (i).

Finally, if $-1 < c < 0$ so that $c+1 > 0$ and $|c+1| < 1$, we have from an earlier result of the author (Parameswaran, 1952) that

$T = (H^0 - \overline{c+1}\, H^1)$ has a positive regular inverse T^{-1}, or $TT^{-1} = T^{-1}T = H^0$. Now since $T(H^\beta s) \leqslant K$ and T^{-1} is regular and positive we have:

$$T^{-1} . TH^\beta s = H^\beta s \leqslant K$$

(multiplication being associative, since we are concerned with only lower-semi-matrices). From (16) and the last step we get $H^\beta s = O(1)$, or $s = O(n^\beta)$, when the required conclusion follows again as in Case (i).

This completes the proof of Theorem 1.

The special case of Theorem 1, when $\alpha = -1$, $c > 0$, $\lambda = K = L = 0$ and $p_n \equiv 1$ has been proved by Szász (1944*b*). (See also remark after corollary to Theorem 3.)

THEOREM 2. *Let s be a real sequence summable to l by the regular Nörlund method (N, p_n) and let*

(i) $(H^\circ - \overline{c+1}\, H^1)\, s \leqslant K$ *for some real number* $c > 0$.

(ii) $p_n \geqslant 0$.

Then $H^1 s \to l$.

Proof. As in the proof of Theorem 1, we have that the sequence $\sigma = H^1 s + \dfrac{K}{c}$ is quasi-monotone decreasing and satisfies (24) and (25). If the first alternative in (25), viz. $\sigma \geqslant 0$ is true, then Theorem 1 is applicable and we get that $H^1 s \to l$.

Let now the second alternative in (25) hold, i.e. $\sigma \leqslant 0$. Since $c > 0$, and hence $d > 0$ in (24), we see that σ_n is monotone decreasing. Therefore, either $\sigma_n \to l_1$ (finite), or $\sigma_n \to -\infty$ as $n \to \infty$. If $\sigma_n \to l_1$ then the conclusion of the theorem follows immediately. We shall now show that we can rule out the possibility that $\sigma_n \to -\infty$ since it leads to a contradiction.

For, if $\sigma_n \to -\infty$ then

$$s_n = (n+1)(\sigma_n - \sigma_{n-1}) + \sigma_{n-1} \to -\infty$$

since σ_n is monotone decreasing. But the Nörlund method N is positive and regular and hence by a well-known theorem (Hardy, 1949; Theorem 9) is 'totally regular', i.e. preserves infinite limits also. Hence $\sigma_n \to -\infty$ and $Ns \to l \neq -\infty$ are incompatible.

This completes the proof of the theorem.

THEOREM 3. *Let s be a real sequence summable to l by the regular Nörlund method (N, p_n), and let*

(i) $(H^0 - \overline{c+1} H^1)s \leqslant 0$ *for some real number* $c \geqslant 0$,

(ii) $p_n \geqslant 0$.

Then $H^1 s \to l$.

PROOF. The case $c > 0$ being included in Theorem 2 proved above, we may assume now that $c = 0$. Writing $t_0 + t_1 + \ldots + t_n = \sigma_n$, $\sigma = H^1 s$ we have:

$$(H^{-1} - H^0)\sigma \leqslant 0, \text{ i.e. } (n+1)t_n \leqslant 0$$

and hence $t_n \leqslant 0$. The sequence σ_n is therefore monotone decreasing and either tends to a finite limit or $\sigma_n \to -\infty$ as $n \to \infty$. The second possibility is ruled out as in the proof of Theorem 2, and the theorem stands proved.

COROLLARY. *Let s be a quasi-monotone decreasing sequence and hence satisfying the condition* (11) *; let also $c \geqslant 0$ in* (11). *Then the existence of a regular Nörlund method (N, p_n) with $p_n \geqslant 0$ which sums s implies that $s \to l$.*

The above corollary is a generalization of a result due to Szász (1944*a*; Lemma 5) where he takes, in effect, the Nörlund method to be the H^1-method.

THEOREM 4. *Let $N \equiv (N, p_n)$ be any regular Nörlund method and s any real sequence summable by N to l. If there exist real numbers α, c such that*

$$(H^{\alpha}-\overline{c+1}\ H^{\alpha+1})s = O(1) \qquad \ldots \quad \ldots \quad \ldots \quad (26)$$

then $H^{\alpha+\epsilon}s \to l$ for every $\epsilon > 0$.

PROOF. *Case* (*a*) : Let $c < 0$. Then, by a result already used in the proof of Theorem 1, we see that $(H^0-\overline{c+1}\ H^1)$ has an inverse T^{-1} which is a regular method, so that

$$H^{\alpha}s = T^{-1}(H^0-\overline{c+1}\ H^1)H^{\alpha}s = O(1),$$

and hence there follow successively: $s = O(n^{\alpha})$, $\Sigma s_n z^n$ converges for $|z| < 1$, Abel's method is applicable to s and (from Lemma 1) s is Abel-summable. Hence the second part of Lemma 2 (for Abel-summability) proved by Jakimovski and Parameswaran (1958) gives $H^{\alpha+\epsilon}s \to l$.

Case (*b*) : Let $c = 0$. Writing $\tau = H^{\alpha+1}s$, (26) becomes

$$(H^{-1}-\epsilon H^{0})\tau = O(1), \text{ i.e. } (n+1)(\tau_{n+1}-\tau_n) = O(1).$$

Hence $\tau_n = O(\log n)$ and therefore $s = o(n^{\alpha+2})$. This leads as in Case (*a*) again to the Abel-summability of s and the result follows as before from the classical Tauberian theorem for Abel-summability.

Case (*c*) : Let $c > 0$. It follows from (26) that there exists a number $K > 0$ such that

$$-K \leqslant (H^{\alpha}-\overline{c+1}\ H^{\alpha+1})s \leqslant K.$$

We note that if we write $\{-s_n\}$ for $\{s_n\}$, the above inequalities are not changed. From the inequalities we get that both

$$H^{\alpha+1}s+\frac{K}{c} \text{ and } -H^{\alpha+1}s+\frac{K}{c}$$

are quasi-monotone decreasing. Of the two, at least one must be ultimately positive; for we have $K/c > 0$ and if

$$H^{\alpha+1}s+K/c \leqslant 0 \text{ then } -H^{\alpha+1}s+K/c > 0.$$

The required result follows immediately from Theorem 1. This completes the proof of Theorem 4.

COROLLARIES: 1. *If we had $o(1)$ instead of $O(1)$ in* (26), *then the conclusion becomes $H^{\alpha}s \to l$.*

2. *If s is a bounded sequence summable by a regular Nörlund method to the value l, then $H^{\epsilon}s$ converges to l for every $\epsilon > 0$.*

The proof of the first statement is obvious (e.g. from Theorem 4 and Lemma 2) while the second is got as a special case of Theorem 4 by taking $\alpha = 0$, $c = -1$.

REMARKS: 1. It may be noted that the conditions (15) and (16) of Theorem 1 do not imply the condition (26) of Theorem 4. To see this, take $s_n = n$, $c > 1$. Then $s \geqslant 0$ and $(H^{\alpha} - \overline{c+1}\ H^{\alpha+1})s \to -\infty$ for every $\alpha > 0$ and further $\underline{\lim}\,(H^{\alpha} - \overline{c+1}\ H^{\alpha+1})s = -\infty$ for every α; so that again, $(H^{\alpha} - \overline{c+1}\ H^{\alpha+1})s \not\geqslant L$ for any α, L.

2. In Theorems 2 and 3, the condition $p_n \geqslant 0$ cannot be omitted from the hypothesis. For, taking $s = \{-2^n\}$, $p_0 = -\frac{1}{3}$, $p_1 = \frac{2}{3}$, $p_n = 0 (n > 1)$, we have $Ns = 0$ but $H^1 s \to -\infty$ and the theorem fails.

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