ON THE CESÀRO SUMMABILITY OF THE ULTRASPHERICAL SERIES (2)

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Abstract

In this paper the Cesàro summability of the ultraspherical series has been investigated. The result obtained includes as a particular case that of Du Plessis (1952) for Laplace series.

1. Introduction

In a previous paper published in this journal (Gupta, 1958), I have investigated the Cesàro summability (C, δ) of the ultraspherical series for $\delta > \lambda$. In this paper I propose to extend the investigation to cover the case where $\lambda - 1 < \delta < \lambda$.

The ultraspherical polynomials $P_n^{(\lambda)}(x)$ are defined as follows :—

$$(1.1) \qquad (1-2\rho\mu+\rho^2)^{-\lambda} = \sum_{n=0}^{\infty} \rho^n P_n^{(\lambda)}(\mu), \qquad \lambda > 0.$$

Let $f(\theta, \phi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ on a sphere S. The ultraspherical series associated with this function is

$$(1.2) \qquad f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \iint_S \frac{P_n^{(\lambda)}(\cos\omega) f(\theta', \phi')\, d\sigma'}{[\sin^2\theta' \sin^2(\phi-\phi')]^{\frac{1}{2}-\lambda}}, \qquad \lambda > 0,$$

where $\qquad \cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi').$

The Laplace series is a particular case of the series (1.2) for $\lambda = \frac{1}{2}$, while it reduces to the trigonometric series in the limit as $\lambda \to 0$, because

$$\lim_{\lambda\to 0} \frac{1}{\lambda} P_n^{(\lambda)}(\cos\theta) = \frac{2}{n}\cos n\theta, \qquad n \geqslant 1.$$

Positive order Cesàro summability of the ultraspherical series (1.2) has been discussed by Darboux (1878), Kogbetliantz (1917, 1919, 1924, 1926), Obrechkoff (1936) and Szegö (1939). Kogbetliantz (1923) and Du Plessis (1952) have also studied the (C, k) summability of the Laplace series for $-\frac{1}{2} < k < \frac{1}{2}$. Du Plessis has obtained the following result :—

For $-\frac{1}{2} < k < \frac{1}{2}$, the Laplace series of $f(\theta, \phi)$ on a sphere S is summable (C, k) at the point (θ, ϕ) of the sphere to the sum $F_\rho(\theta')$, provided that

$$F_\rho(\theta') \equiv \int_0^{2\pi} f(\theta', \phi')\, d\phi' \in \text{lip}^* \left(\frac{1}{2} - k\right), \ (\theta, \phi) \text{ being the pole.}$$

The above result corresponds to Hardy's theorem (Hardy and Littlewood, 1928) on the negative order summability of Fourier series, viz., For $0 < k < 1$, the Fourier series of $f(x)$ is everywhere summable $(C, -k)$ to the sum $f(x)$ if

$$f(x) \epsilon \text{ Lip}^* k.$$

We suppose throughout that the function

$$f(\theta', \phi') [\sin^2 \theta' \sin^2 (\phi - \phi')]^{\lambda - \frac{1}{2}} \tag{1.3}$$

is integrable (L) over the whole surface of the unit sphere, and following Kogbetliantz (1924, p. 114) we define the generalised mean value of $f(\theta, \phi)$ as follows :—

$$f(\omega) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)}{\Gamma(\lambda)\, 2\pi\, (\sin \omega)^{2\lambda}} \int_{C_\omega} \frac{f(\theta', \phi')\, ds'}{[\sin^2 \theta' \sin^2 (\phi - \phi')]^{\frac{1}{2}-\lambda}}, \tag{1.4}$$

where the integral is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is ω.
We write

$$F(\omega) \equiv f(\omega)\ (\sin \omega)^{2\lambda - 1}.$$

The following theorem will be proved:

THEOREM: *For* $\lambda - 1 < k < \lambda$ *and* $0 < \lambda \leqslant \frac{1}{2}$, *the ultraspherical series* (1.2) *is* (C, k) *summable at the point* (θ, ϕ) *to the sum* $f(\omega)$, *provided that*

$$F(\omega) \epsilon \text{ lip}^* (\lambda - k). \tag{1.5}$$

The result obtained by Du Plessis will be a particular case of our theorem for $\lambda = \frac{1}{2}$.

2. Cesàro means of ultraspherical series :—It is known that (Szegö, 1939, p. 84)

$$\sum_{k=0}^{m} (k+\lambda)\, P_k^{(\lambda)} (\cos \theta) = \frac{1}{2}\, \frac{(m+2\lambda) P_m^{(\lambda)} (\cos \theta) - (m+1) P_{m+1}^{(\lambda)} (\cos \theta)}{1 - \cos \theta} \tag{2.1}$$

$$= \frac{1}{2} \left[\frac{d}{dx} \left\{ P_m^{(\lambda)} (x) \right\} + \frac{d}{dx} \left\{ P_{m+1}^{(\lambda)} (x) \right\} \right]_{x = \cos \theta}. \tag{2.2}$$

So, the mth partial sum S_m of the series (1.2) is given by

$$S_m = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi f(\omega) \left[\frac{d}{dx} \left\{ P_{m+1}^{(\lambda)} (x) + P_m^{(\lambda)} (x) \right\} \right]_{x = \cos \omega} (\sin \omega)^{2\lambda}\, d\omega.$$

Thus

$$S_m - f(P) = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \cdot \int_0^\pi \{ f(\omega) - f(0) \} \left[\frac{d}{dx} \left\{ P_{m+1}^{(\lambda)} (x) + P_m^{(\lambda)} (x) \right\} \right]_{x = \cos \omega} (\sin \omega)^{2\lambda} d\omega.$$

Without loss of generality we may set $f(P) = f(0) = 0$.
Consequently

$$S_m = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \cdot \int_0^\pi f(\omega) \left[\frac{d}{dx} \left\{ P_{m+1}^{(\lambda)} (x) + P_m^{(\lambda)} (x) \right\} \right]_{x = \cos \omega} (\sin \omega)^{2\lambda}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \int_0^\pi F(\omega) \left[\frac{d}{dx} \left\{ P_{m+1}^{(\lambda)} (x) + P_m^{(\lambda)} (x) \right\} \right]_{x = \cos \omega} \sin \omega\, d\omega. \tag{2.3}$$

Thus the (C, k) mean $\sigma_n^k(P)$ of the series (1.2) at the point P is then given by

$$\sigma_n^k = \int_0^\pi F(\omega) L_n^k(\omega)\, d\omega,$$

where

$$L_n^k(\omega) = \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \left(A_n^k\right)^{-1} \sum_{m=0}^{n} A_{n-m}^{k-1} \left[\frac{d}{dx}\left\{P_{m+1}^{(\lambda)}(x) + P_m^{(\lambda)}(x)\right\}\right]_{x=\cos\omega} \sin\omega$$

$$= \frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \left(A_n^k\right)^{-1} S_n^k(\omega), \text{ say,} \tag{2.4}$$

and

$$A_n^k = \binom{n+k}{k}.$$

It is also quite clear that

$$S_n^k(\omega) = \sum_{m=0}^{n} A_{n-m}^k \,.\, 2(m+\lambda) P_m^{(\lambda)}(\cos\omega) \sin\omega. \tag{2.5}$$

3. For the proof of the theorem, we divide it into two parts, viz., $0 < k < \lambda$ and $\lambda - 1 < k \leqslant 0$. We first take up the case $0 < k < \lambda$, for which we require the following lemmas :—

Lemma 1. *For $0 < k < \lambda$ and $0 < \lambda < 1$,*

$$L_n^k(\omega) = O(n^{2\lambda+1}\omega). \tag{3.1}$$

Proof. From Szegö (1939, p. 166) we have

$$P_n^{(\lambda)}(\cos\theta) = O(n^{2\lambda-1}) \qquad \text{for } 0 \leqslant \theta \leqslant \pi.$$

Consequently, from (2.1)

$$\frac{(m+2\lambda) P_m^{(\lambda)}(\cos\omega) - (m+1) P_{m+1}^{(\lambda)}(\cos\omega)}{1-\cos\omega} = O\left\{\sum_{0}^{m} (k+\lambda) k^{2\lambda-1}\right\}$$
$$= O(m^{2\lambda+1}).$$

It follows that

$$L_n^k(\omega) = O\left\{n^{-k}\omega \sum_{m=0}^{n} A_{n-m}^{k-1} m^{2\lambda+1}\right\}$$
$$= O(n^{2\lambda+1}\omega).$$

Lemma 2. *For $0 < \lambda < 1$ and $\pi - \frac{1}{n} \leqslant \omega \leqslant \pi$,*

$$L_n^k(\omega) = O(n^{2\lambda}\sin\omega). \tag{3.2}$$

Proof. We have

$$\text{(3.2.1)}\quad \left[\frac{d}{dx}\left\{P^{(\lambda)}_{m+1}(x)\right\}+\frac{d}{dx}\left\{P^{(\lambda)}_{m}(x)\right\}\right]_{\substack{x=\cos\omega\\ =-1+\delta}}$$

$$=\sum_{s=0}^{m}\left[\frac{d^{s+1}}{dx^{s+1}}\left\{P^{(\lambda)}_{m+1}(x)\right\}+\frac{d^{s+1}}{dx^{s+1}}\left\{P^{(\lambda)}_{m}(x)\right\}\right]_{x=-1}\cdot\frac{\delta^s}{\lfloor s}.$$

Differentiating the relation (1.1) r times with respect to μ and putting $\mu=-1$ we get (Carlitz, 1954)

$$\left\{\frac{d^r P^{(\lambda)}_n(\mu)}{d\mu^r}\right\}_{\mu=-1}=\frac{(2\lambda)_{n+r}(-1)^{n-r}}{2^r(\lambda+\frac{1}{2})_r\lfloor n-r},$$

where

$$(\alpha)_n=\alpha(\alpha+1)\ldots\ldots(\alpha+n-1).$$

Thus, the right hand side of (3.2.1) becomes

$$\sum_{s=0}^{m}(-1)^{m-s}\frac{\delta^s}{\lfloor s}\left[\frac{2\lambda(2\lambda+1)\ldots\ldots(2\lambda+m+s+1)}{2^{s+1}(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots(\lambda+\frac{1}{2}+s)}\cdot\frac{1}{\lfloor m-s}\right.$$

$$\left.-\frac{2\lambda(2\lambda+1)\ldots\ldots(2\lambda+m+s)}{2^{s+1}(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots\ldots(\lambda+\frac{1}{2}+s)}\cdot\frac{1}{\lfloor m-s-1}\right]$$

$$=O(m^{2\lambda})+(-1)^m\sum_{s=1}^{m}(-1)^s\frac{\delta^s}{2^{s+1}\lfloor s\,\lfloor m-s}\left[\frac{2\lambda(2\lambda+1)\ldots\ldots(2\lambda+m+s)}{(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots\ldots(\lambda+s-\frac{1}{2})}\right].$$

For $0<\delta<\dfrac{1}{2m^2}$, the modular value of each term of the above series is less than

$$\frac{\delta^s}{2^{s+1}}\cdot\frac{1}{\lfloor s}\cdot\frac{m^s}{\lfloor m}\,\frac{2\lambda(2\lambda+1)\ldots(2\lambda+m)(2\lambda+m+1)\ldots(2\lambda+m+s)}{(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots\ldots(\lambda+s-\frac{1}{2})}$$

$$<\frac{1}{2^{2s+1}}\cdot\frac{1}{m^{2s}}\cdot\frac{1}{\lfloor s}\cdot m^s\cdot\left[\frac{2\lambda(2\lambda+1)\ldots(2\lambda+m)}{\lfloor m}\right]\left[\frac{(2m+2\lambda)^s}{(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots(\lambda+s-\frac{1}{2})}\right]$$

$$<\frac{1}{2^{s+1}\lfloor s}\left[\frac{2\lambda(2\lambda+1)\ldots(2\lambda+m)}{\lfloor m}\right]\frac{1}{(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots(\lambda+s-\frac{1}{2})}.$$

Thus the modular value of the whole series is less than

$$\frac{2\lambda(2\lambda+1)\ldots\ldots(2\lambda+m)}{\lfloor m}\sum_{s=1}^{m}\frac{1}{2^{s+1}\lfloor s}\cdot\frac{1}{(\lambda+\frac{1}{2})(\lambda+\frac{3}{2})\ldots(\lambda+s-\frac{1}{2})}$$

$$=O\left[\frac{(m+2\lambda)(m+2\lambda-1)\ldots 2\lambda}{\lfloor m}\right]=Om(^{2\lambda}).$$

It follows that

$$\left[\frac{d}{dx}\left\{P^{(\lambda)}_{m+1}(x)\right\}+\frac{d}{dx}\left\{P^{(\lambda)}_{m}(x)\right\}\right]_{\substack{x=\cos\omega\\=-1+\delta}}.\sin\omega$$

(3.2.2) $$=\frac{d}{dx}\left[P^{(\lambda)}_{m+1}\{-1+2\sin^2\tfrac{1}{2}(\pi-\omega)\}+P^{(\lambda)}_{m}\{-1+2\sin^2\tfrac{1}{2}(\pi-\omega)\}\right]\sin\omega$$

$$=O(m^{2\lambda}\sin\omega).$$

So,

$$L^K_n(\omega)=O\left[n^{-k}\sum_{m=1}^{n-1}(n-m)^{k-1}m^{2\lambda}\sin\omega\right]$$

$$=O(n^{2\lambda}\sin\omega).$$

LEMMA 3. *If* $\alpha_n\leqslant\theta\leqslant\pi-\frac{1}{n}$, $\left(\alpha_n\geqslant\frac{1}{n}\right)$ *and if* E_n *be the real part of*

$$-\frac{2^{2-2\lambda}}{\pi}\lambda\sin\lambda\pi\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}e^{i\pi(\frac{1}{2}-\lambda)}\int_{-\infty}(\theta-t)^{-\lambda-1}\sum_{m=0}^{n}A^{k-1}_{n-m}.$$

$$\{e^{i(m+\lambda+\frac{1}{2})\theta}-e^{i(m+\lambda+\frac{1}{2})t}\}dt,$$

then

(3.3) $$L^k_n(\theta)=\frac{\Gamma(\lambda)}{2\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)}(A^k_n)^{-1}E_n+O\{n^{\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda}$$
$$+n^{-1}(\sin\theta)^{-\lambda-1}+n^{\lambda-1}\theta^{-\lambda}(\sin\theta)^{-1}\}$$

Proof. Stieltje's generalisation of Laplace's first integral gives (Whittakar and Watson, 1927, p. 315)

(3.3.1) $$\frac{\pi}{2}P^{(\lambda)}_m(\cos\theta)=\sin\lambda\pi\,\mathfrak{R}\left\{\exp i[(m+2\lambda)\theta-\lambda\pi].\right.$$
$$\left.\int_0^1 t^{m+2\lambda-1}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda}dt\right\},\text{ for } 0<\lambda<1,$$

whence

(3.3.2) $$-\frac{\pi}{2}\left[\frac{d}{dx}\left\{P^{(\lambda)}_{m+1}(x)+P^{(\lambda)}_m(x)\right\}\right]_{x=\cos\theta}\sin\theta\,\text{cosec}\,\lambda\pi$$

$$=\mathfrak{R}\left[(m+2\lambda)\exp i\left\{(m+2\lambda)\,\theta+\frac{\pi}{2}-\lambda\pi\right\}\int_0^1 t^{m+2\lambda-1}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda}dt\right.$$

$$+(m+2\lambda+1)\exp i\left\{(m+2\lambda+1)\,\theta+\frac{\pi}{2}-\lambda\pi\right\}\int_0^1 t^{m+2\lambda}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda}dt$$

$$+2\lambda\exp i\left\{(m+2\lambda+2)\,\theta+\frac{\pi}{2}-\lambda\pi\right\}\int_0^1 t^{m+2\lambda}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda-1}dt$$

$$\left.+2\lambda\exp i\left\{(m+2\lambda+3)\,\theta+\frac{\pi}{2}-\lambda\pi\right\}\int_0^1 t^{m+2\lambda+1}(1-t)^{-\lambda}(1-te^{2i\theta})^{-\lambda-1}dt\right].$$

Let

$$g = g(\theta, u) = \{1 - ue^{i\theta}(2i\sin\theta)^{-1}\}^{-\lambda}. \tag{3.3.3}$$

The right hand side of (3.3.2) becomes after some simplification

$$\mathfrak{R}[J_m^{(1)} + J_m^{(2)} + \dots + J_m^{(6)}], \text{ where}$$

$$J_m^{(1)} = 2^{1-2\lambda}(m+\lambda+\tfrac{1}{2}) \exp i\left[(m+\lambda+\tfrac{1}{2})\theta + \frac{\pi}{2} - \frac{\lambda\pi}{2}\right]\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}.$$
$$\int_0^1 (1-u)^{m+2\lambda-1} u^{-\lambda} g(\theta, u)\, du;$$

$$J_m^{(2)} = 2^{1-2\lambda}(\lambda-\tfrac{1}{2}) \exp i\left[(m+\lambda+\tfrac{1}{2})\theta + \frac{\pi}{2} - \frac{\lambda\pi}{2}\right]\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}.$$
$$\int_0^1 (1-u)^{m+2\lambda-1} u^{-\lambda} g(\theta, u)\, du;$$

$$J_m^{(3)} = -(m+2\lambda) \exp i\left[(m+\lambda+1)\,\theta + \frac{\pi}{2} - \frac{\lambda\pi}{2}\right](2\sin\theta)^{-\lambda}.$$
$$\int_0^1 (1-u)^{m+2\lambda-1}\, u^{1-\lambda}\, g\, du;$$

$$J_m^{(4)} = \exp i\left[(m+\lambda+1)\,\theta + \frac{\pi}{2} - \frac{\lambda\pi}{2}\right](2\sin\theta)^{-\lambda}\int_0^1 (1-u)^{m+2\lambda}\, u^{-\lambda} g\, du;$$

$$J_m^{(5)} = \lambda \exp i\left[\left(m+\lambda+\frac{3}{2}\right)\theta + \pi - \frac{\lambda\pi}{2}\right]\left(\sin\frac{\theta}{2}\right)^{-1}(2\sin\theta)^{-\lambda}.$$
$$\int_0^1 (1-u)^{m+2\lambda}\, u^{-\lambda} g^{\frac{\lambda+1}{\lambda}}\, du;$$

$$J_m^{(6)} = -2\lambda \exp i\left[(m+\lambda+2)\,\theta + \pi - \frac{\lambda\pi}{2}\right](2\sin\theta)^{-\lambda-1}.$$
$$\int_0^1 (1-u)^{m+2\lambda}\, u^{1-\lambda} g^{\frac{\lambda+1}{\lambda}}\, du.$$

Consequently

$$S_n^k(\theta) = \mathfrak{R}\left[-\frac{2}{\pi}\sum_{m=0}^{n} A_{n-m}^{k-1}\left\{J_m^{(1)} + J_m^{(2)} + \dots + J_m^{(6)}\right\}\sin\lambda\pi\right] \tag{3.3.4}$$
$$= \mathfrak{R}[S_n^{(1)} + S_n^{(2)} + \dots + S_n^{(6)}].$$

It can be easily seen that for $0 < u < 1$ and $0 < \theta < \pi$,

$$|g| < 2^{\lambda}.$$

So, for $m \geqslant 1$,

$$\left|J_m^{(2)}\right| \leqslant Am^{\lambda-1}(\sin\theta)^{-\lambda}, \qquad \left|J_m^{(3)}\right| \leqslant Am^{\lambda-1}(\sin\theta)^{-\lambda},$$

$$\left|J_m^{(4)}\right| \leqslant Am^{\lambda-1}(\sin\theta)^{-\lambda}, \qquad \left|J_m^{(5)}\right| \leqslant Am^{\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda},$$

$$\left|J_m^{(6)}\right| \leqslant Am^{\lambda-2}(\sin\theta)^{-\lambda-1}.$$

Hence

$$S_n^{(2)}, S_n^{(3)}, S_n^{(4)} = O\left\{n^{k-1} + \sum_{m=1}^{n-1}(n-m)^{k-1}m^{\lambda-1} + n^{\lambda-1}\right\}(\sin\theta)^{-\lambda}$$

$$= O\{n^{k+\lambda-1}(\sin\theta)^{-\lambda}\}.$$

Similarly,

$$S_n^{(5)} = O\{n^{k+\lambda-1}(\sin\theta)^{-\lambda}\theta^{-1}\},$$

and

$$S_n^{(6)} = O\left\{n^{k-1} + \sum_{m=1}^{n-1}(n-m)^{k-1}m^{\lambda-2} + n^{\lambda-2}\right\}(\sin\theta)^{-\lambda-1}$$

$$= O\{n^{k-1}(\sin\theta)^{-\lambda-1}\}.$$

Finally

$$-\frac{\pi}{2}S_n^{(1)} = \sum_{m=0}^{n} A_{n-m}^{k-1}\sin\lambda\pi \,.\, 2^{1-2\lambda}(m+\lambda+\tfrac{1}{2})\exp i\left\{(m+\lambda+\tfrac{1}{2})\theta + \frac{\pi}{2} - \frac{\lambda\pi}{2}\right\}.$$

$$\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}\int_0^1 (1-u)^{m+2\lambda-1}u^{-\lambda}g\,du$$

$$= e^{\frac{i\pi}{2}(1-\lambda)}\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}2^{1-2\lambda}\sin\lambda\pi.$$

$$\sum_{m=0}^{n} A_{n-m}^{k-1}(m+\lambda+\tfrac{1}{2})\, e^{i(m+\lambda+\frac{1}{2})\theta} I_m,$$

where

$$I_m = \int_0^1 u^{-\lambda}(1-u)^{m+2\lambda-1}g\,du$$

$$= \int_0^1 u^{-\lambda}(1-u)^{m+2\lambda-1}\left[1 - \frac{\lambda}{2}ie^{i\theta}(\sin\theta)^{-1}\int_0^u g^{1+\frac{1}{\lambda}}(\theta, v)\,dv\right]du.$$

But, since $\int_0^u g^{1+\frac{1}{\lambda}}(\theta, v)\,dv = O(u)$, we have

$$I_m = \left[\frac{\Gamma(1-\lambda)\Gamma(m+2\lambda)}{\Gamma(m+\lambda+1)} + O\left\{\frac{\Gamma(2-\lambda)\Gamma(m+2\lambda)}{\Gamma(m+\lambda+2)}(\sin\theta)^{-1}\right\}\right]$$

$$= [\Gamma(1-\lambda)(m+2\lambda)^{\lambda-1} + O\{m^{\lambda-2}(\sin\theta)^{-1}\}]$$

$$= [\Gamma(1-\lambda)(m+\lambda+\tfrac{1}{2})^{\lambda-1} + O\{m^{\lambda-2}(\sin\theta)^{-1}\}].$$

Therefore

$$-\frac{\pi}{2}S_n^{(1)} = e^{\frac{i\pi}{2}(1-\lambda)}\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}2^{1-2\lambda}\sin\lambda\pi\,.$$

$$\sum_{m=0}^{n} A_{n-m}^{k-1} e^{i(m+\lambda+\frac{1}{2})\theta}\{\Gamma(1-\lambda)(m+\lambda+\tfrac{1}{2})^{\lambda}+O(m^{\lambda-1})(\sin\theta)^{-1}\}.$$

Again, since

$$\int_{-\infty}^{\theta}(\theta-t)^{-\lambda-1}\{e^{i(m+\lambda+\frac{1}{2})\theta}-e^{i(m+\lambda+\frac{1}{2})t}\}\,dt = \frac{\Gamma(1-\lambda)}{\lambda}(m+\lambda+\tfrac{1}{2})^{\lambda}e^{i(m+\lambda+\frac{1}{2})\theta}e^{\frac{i\pi\lambda}{2}},$$

we may write

$$S_n^{(1)} = -\frac{2^{2-2\lambda}}{\pi}\lambda\sin\pi\lambda\, e^{\frac{i\pi}{2}(1-2\lambda)}\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda} \tag{3.3.5}$$

$$\times\int_{-\infty}^{\theta}(\theta-t)^{-\lambda-1}\sum_{m=0}^{n}A_{n-m}^{k-1}\{e^{i(m+\lambda+\frac{1}{2})\theta}-e^{i(m+\lambda+\frac{1}{2})t}\}\,dt$$

$$+O\left[\theta^{-\lambda}\sum_{m=1}^{n-1}(n-m)^{k-1}m^{\lambda-1}(\sin\theta)^{-1}\right].$$

This completes the proof of the lemma.

LEMMA 4. *If* $\alpha_n \leqslant \theta \leqslant \pi-\frac{1}{n}$ *and* $\mu_n = \frac{\pi}{n+\lambda+\frac{1}{2}}$, *then*

$$E_n = \theta^{-\lambda}\phi(\theta)e^{i(n+\lambda+\frac{1}{2})\theta}$$

where $\phi(\theta)$ *is such that*

$$\phi(\theta) = O(n^{\lambda}\theta^{-k})\,;\quad \phi(\theta+\mu_n)-\phi(\theta) = O(n^{k+\lambda-1}\theta^{-1}\log n). \tag{3.4}$$

Proof. Putting $\theta-t=u$, the integral in (3.3.5) becomes

$$\int_0^{\infty}u^{-\lambda-1}\sum_{m=0}^{n}A_m^{k-1}e^{i(n+\lambda+\frac{1}{2})\theta}\{e^{-im\theta}-e^{-i(n+\lambda+\frac{1}{2})u-im(\theta-u)}\}du$$

$$= e^{i(n+\lambda+\frac{1}{2})\theta}\int_0^{\infty}u^{-\lambda-1}[K_n(\theta)-e^{-i(n+\lambda+\frac{1}{2})u}K_n(\theta-u)]\,du,$$

where

$$K_n(t) = \sum_{m=0}^{n}A_m^{k-1}e^{-imt}\,.$$

Du Plessis (1952, p. 342) has shown that

$$K_n(t) = O(n^k),\qquad K_n'(t) = O(n^{k+1})\,; \tag{3.4.1}$$

$$K_n(t) = O(t^{-k}),\qquad \text{for } \frac{1}{n} < t < \pi\,; \tag{3.4.2}$$

$$K_n'(t) = O(n^k t^{-1}), \qquad \text{for } \frac{1}{n} \leqslant t \leqslant \pi; \tag{3.4.3}$$

$$K_n''(t) = O(n^{k+1} t^{-1}), \quad \text{for } \frac{1}{n} \leqslant t \leqslant \pi. \tag{3.4.4}$$

We now write

$$\phi(\theta) = I_1 + I_2 + \ldots + I_5,$$

where

$$I_1 = \int_0^{\frac{1}{n}} u^{-\lambda-1}\left[K_n(\theta) - e^{-i(n+\lambda+\frac{1}{2})u} K_n(\theta-u)\right] du,$$

$$I_2 = \int_{\frac{1}{n}}^{\infty} u^{-\lambda-1} K_n(\theta)\, du,$$

$$I_3 + I_4 + I_5 = -\left(\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} + \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} + \int_{\theta+\frac{1}{n}}^{\infty}\right) u^{-\lambda-1} e^{-i(n+\lambda+\frac{1}{2})u} K_n(\theta-u)\, du.$$

In this lemma we are concerned with $\theta \geqslant \alpha_n$, and so we may say that

$$\theta \pm \frac{1}{n} = O(\theta) \text{ as } n \to \infty.$$

We now have

$$I_2 = O(n^\lambda \theta^{-k});$$

$$I_3 = O\left\{\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1}(\theta-u)^{-k}\, du\right\} = O\left\{\int_{\frac{1}{n\theta}}^{1-\frac{1}{n\theta}} \theta^{-k-\lambda} v^{-\lambda-1}(1-v)^{-k}\, dv\right\}$$

$$= O(\theta^{-k} n^\lambda);$$

$$I_4 = O\left\{n^k \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1}\, du\right\} = O\left(n^{k-1}\, \theta^{-\lambda-1}\right)$$

$$= O\left(n^\lambda \theta^{-k}\right).$$

$$I_5 = \int_{\theta+\frac{1}{n}}^{\pi} + \sum_{m=1}^{\infty} \int_{(2m-1)\pi}^{(2m+1)\pi}$$

and here the second term may be written as

$$\int_{-\pi}^{\pi} e^{-i(n+\lambda+\frac{1}{2})u} K_n(\theta-u) \sum_{m=1}^{\infty} e^{-i(n+\lambda+\frac{1}{2})2m\pi} (u+2m\pi)^{-\lambda-1}\, du.$$

It follows that

$$I_5 = O\left\{ \int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1}(u-\theta)^{-k}\,du + \int_{-\pi}^{\theta-\frac{1}{n}} (\theta-u)^{-k}\,du + \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} n^k\,du + \int_{\theta+\frac{1}{n}}^{\pi} (u-\theta)^{-k}du \right\}$$

$$= O\{\theta^{-k-\lambda}\} + O(n^{k-\lambda}) = O(n^\lambda \theta^{-k}).$$

Finally,

$$I_1 = O\left[\int_0^{\frac{1}{n}} u^{-\lambda} \left\{ \frac{d}{d\xi} e^{-i(n+\lambda+\frac{1}{2})\xi} K_n(\theta-\xi) \right\} du \right],$$

where $$o < \xi < u,$$

and therefore

$$I_1 = O\left[\int_0^{\frac{1}{n}} u^{-\lambda} \{ n(\theta-\xi)^{-k} + n^k(\theta-\xi)^{-1} \}\, du \right]$$

$$= O\left[n\theta^{-k} \int_0^{\frac{1}{n}} u^{-\lambda}\, du \right]$$

$$= O[n^\lambda \theta^{-k}].$$

Thus the first part of the lemma is proved.
Next,

$$\phi(\theta+\mu_n) - \phi(\theta) = J_1 + J_2 + \cdots + J_5,$$

where

$$J_1 = \int_0^{\frac{1}{n}} u^{-\lambda-1} [\{ K_n(\theta+\mu_n) - e^{-i(n+\lambda+\frac{1}{2})u} K_n(\theta+\mu_n-u) \}$$
$$- \{ K_n(\theta) - e^{-i(n+\lambda+\frac{1}{2})u} K_n(\theta-u) \}]\, du;$$

$$J_2 = \int_{\frac{1}{n}}^{\infty} u^{-\lambda-1} [K_n(\theta+\mu_n) - K_n(\theta)] du;$$

$$J_3 + J_4 + J_5 = -\left(\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} + \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} + \int_{\theta+\frac{1}{n}}^{\infty} \right) u^{-\lambda-1} e^{-i(n+\lambda+\frac{1}{2})u}$$
$$\times \{ K_n(\theta+\mu_n-u) - K_n(\theta-u) \}\, du.$$

Now,

$$J_1 = \mu_n \int_0^{\frac{1}{n}} u^{-\lambda-1}\left[K_n'(t) - e^{-i(n+\lambda+\frac{1}{2})u} K_n'(t-u)\right] du, \qquad \theta < t < \theta + \mu_n,$$

$$= \mu_n \int_0^{\frac{1}{n}} u^{-\lambda} \left[\frac{d}{d\xi}\left\{e^{-i(n+\lambda+\frac{1}{2})\xi} K_n'(t-\xi)\right\}\right] du, \qquad 0 < \xi < u,$$

$$= O\left[\frac{1}{n}\int_0^{\frac{1}{n}} u^{-\lambda} \,.\, n^{k+1}\theta^{-1}\, du\right] = O\left(n^{k+\lambda-1}\theta^{-1}\right).$$

$$J_2 = O\left[\mu_n K_n'(\theta)\int_{\frac{1}{n}}^{\infty} u^{-\lambda-1}\, du\right]$$

$$= O\left[\frac{1}{n}\,.\, n^k\theta^{-1} n^\lambda\right] = O\left(n^{k+\lambda-1}\theta^{-1}\right).$$

$$J_3 = O\left[\mu_n \int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1} K_n'(\theta-u)\, du\right] = O\left[n^{k-1}\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1}(\theta-u)^{-1} du\right]$$

$$= O\left[n^{k-1}\theta^{-\lambda-1}\int_{\frac{1}{n\theta}}^{1-\frac{1}{n\theta}} u^{-\lambda-1}(1-u)^{-1} du\right]$$

$$= O\left[n^{k-1}\theta^{-\lambda-1} n^\lambda \theta^\lambda \int_{\frac{1}{n\theta}}^{1-\frac{1}{n\theta}} \frac{1}{u(1-u)}\, du\right] = O(n^{k+\lambda-1}\theta^{-1}\log n).$$

$$J_4 = O\left[\mu_n \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1} K_n'(\theta-u)\, du\right]$$

$$= O\left[\mu_n \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1} n^{k+1}\, du\right] = O\left(n^{k+\lambda-1}\theta^{-1}\right).$$

$$J_5 = \int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1} e^{-i(n+\lambda+\frac{1}{2})u}[K_n(\theta+\mu_n-u) - K_n(\theta-u)]\, du$$

$$+ \int_{-\pi}^{\pi} e^{-i(n+\lambda+\frac{1}{2})u}[K_n(\theta+\mu_n-u) - K_n(\theta-u)]$$

$$\times \sum_{m=1}^{\infty} e^{-i(n+\lambda+\frac{1}{2})2m\pi}(u+2m\pi)^{-\lambda-1} du$$

$= J_{5,1} + J_{5,2}$, say.

Here

$$J_{5,1}=O\left[\mu_n\int_{\theta+\frac{1}{n}}^{\pi}u^{-\lambda-1}K_n'(\theta-u)du\right]$$

$$=O\left[n^{-1}\int_{\theta+\frac{1}{n}}^{\pi}u^{-\lambda-1}n^k(u-\theta)^{-1}du\right].$$

The integral in the above expression is less than

$$n^k\theta^{-\lambda-1}\int_{1+\frac{1}{n\theta}}^{\infty}v^{-\lambda-1}(v-1)^{-1}dv=O(n^k\theta^{-\lambda-1}\log n)\ .$$

Therefore

$$J_{5,1}=O(n^{k-1}\theta^{-\lambda-1}\log n)=O(n^{k+\lambda-1}\theta^{-1}\log n).$$

$$J_{5,2}=\mu_n\,O\left[\int_{-\pi}^{\theta-\frac{1}{n}}\left|K_n'(\theta-u)\right|du+\int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}}\left|K_n'(\theta-u)\right|du+\int_{\theta+\frac{1}{n}}^{\pi}\left|K_n'(\theta-u)\right|du\right]$$

$$=O\left[n^{k-1}\int_{-\pi}^{\theta-\frac{1}{n}}(\theta-u)^{-1}du+n^{-1}\,.\,n^{k+1}\,.\,n^{-1}+n^{k-1}\int_{\theta+\frac{1}{n}}^{\pi}(u-\theta)^{-1}du\right]$$

$$=O(n^{k-1}\log n).$$

Consequently

$$J_5=O(n^{k+\lambda-1}\theta^{-1}\log n).$$

This completes the proof of the lemma.
Summing up lemmas 3 and 4, we state the following :—

LEMMA 5. *For* $\alpha_n\leqslant\theta\leqslant\pi-\dfrac{1}{n}$,

$$L_n^k(\theta)=\mathfrak{J}\left[\psi(\theta)\theta^{-\lambda}e^{i(n+\lambda+\frac{1}{2})\theta}\right]+O\left\{n\lambda^{-1}\theta^{-1}(\sin\theta)^{-\lambda}\right.$$
$$\left.+n^{-1}(\sin\theta)^{-\lambda-1}+n\lambda^{-1}\theta^{-\lambda}(\sin\theta)^{-1}\right\},$$

where $\psi(\theta)$ *is such that for* $\mu_n=\dfrac{\pi}{n+\lambda+\frac{1}{2}}$,

(3.5) $$\psi(\theta)=O(n^{\lambda-k}\theta^{-k})\ ;\ \psi(\theta+\mu_n)-\psi(\theta)=O(n^{\lambda-1}\theta^{-1}\log n).$$

4. Proof of the theorem for $0<k<\lambda$:—In order to prove the theorem, we have to show that

$$\sigma_n^k(0)\to 0\quad\text{as } n\to\infty.$$

Now,

$$\sigma_n^k(0)=\left(\int_0^{\alpha_n}+\int_{\alpha_n}^{\pi-\frac{1}{n}}+\int_{\pi-\frac{1}{n}}^{\pi}\right)F(\theta)L_n^k(\theta)d\theta$$

$$\equiv I_1+I_2+I_3,\ \text{say}.$$

First, using lemma 1 and the fact that $F(\theta) = o(\theta^{\lambda-k})$,

$$I_1 = o\left\{ n^{2\lambda+1} \int_0^{\alpha_n} \theta \,.\, \theta^{\lambda-k} d\theta \right\}$$

$$= o\left(n^{2\lambda+1} \alpha_n^{2+\lambda-k}\right) = o(1),$$

provided that we take $\alpha_n = n^{-(2\lambda+1)(2+\lambda-k)^{-1}}$, which is greater than $\dfrac{1}{n}$ if $0 < \lambda \leqslant \frac{1}{2}$.

Next, using lemma 2 and the boundedness of $F(\theta)$,

$$I_3 = O\left[\int_{\pi-\frac{1}{n}}^{\pi} n^{2\lambda} \sin\theta \, d\theta \right] = o(1).$$

Finally, by lemma 5 we have

$$I_2 = \mathfrak{R} \left\{ \int_{\alpha_n}^{\pi-\frac{1}{n}} F(\theta)\psi(\theta)\theta^{-\lambda} e^{i(n+\lambda+\frac{1}{2})\theta} d\theta \right\} + O\left\{ \int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{\lambda-1}\theta^{-1} (\sin\theta)^{-\lambda} d\theta \right\}$$

$$+ O\left\{ \int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{-1} (\sin\theta)^{-\lambda-1} d\theta \right\} + O\left\{ \int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{\lambda-1}\theta^{-\lambda} (\sin\theta)^{-1} d\theta \right\}.$$

The error term is

$$O\left\{ n^{\lambda-1} \left(\int_{\alpha_n}^{\frac{\pi}{2}} \theta^{\lambda-k}\theta^{-\lambda-1} d\theta + \int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}} (\sin\theta)^{-\lambda} d\theta \right) \right\}$$

$$+ O\left\{ n^{-1} \left(\int_{\alpha_n}^{\frac{\pi}{2}} \theta^{\lambda-k}\theta^{-\lambda-1} d\theta + \int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}} (\sin\theta)^{-\lambda-1} d\theta \right) \right\}$$

$$+ O\left\{ n^{\lambda-1} \left(\int_{\alpha_n}^{\frac{\pi}{2}} \theta^{\lambda-k}\theta^{-\lambda-1} d\theta + \int_{\frac{\pi}{2}}^{\pi-\frac{1}{n}} (\sin\theta)^{-1} d\theta \right) \right\}$$

$$= o(1) \qquad \text{if } 0 < \lambda \leqslant \tfrac{1}{2}.$$

The integral may be rewritten in the form

$$\tfrac{1}{2} \left\{ \int_{\alpha_n}^{\pi-\frac{1}{n}} F(\theta)\psi(\theta)\theta^{-\lambda} e^{i(n+\lambda+\frac{1}{2})\theta} d\theta - \int_{\alpha_n-\mu_n}^{\pi-\frac{1}{n}-\mu_n} F(\theta+\mu_n)\psi(\theta+\mu_n)(\theta+\mu_n)^{-\lambda} e^{i(n+\lambda+\frac{1}{2})\theta} d\theta \right\}$$

and is consequently less than

$$\tfrac{1}{2}\{J_1+J_2+\dots+J_5\},$$

where

$$J_1=\int_{\alpha_n-\mu_n}^{\alpha_n}|F(\theta+\mu_n)\psi(\theta+\mu_n)(\theta+\mu_n)^{-\lambda}|\,d\theta,$$

$$J_2=\int_{\pi-\frac{1}{n}-\mu_n}^{\pi-\frac{1}{n}}|F(\theta)\psi(\theta)\theta^{-\lambda}|\,d\theta,$$

$$J_3=\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n}|F(\theta+\mu_n)-F(\theta)|\,|\psi(\theta+\mu_n)|\,(\theta+\mu_n)^{-\lambda}\,d\theta,$$

$$J_4=\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n}|\psi(\theta+\mu_n)-\psi(\theta)|\,|F(\theta)|\,(\theta+\mu_n)^{-\lambda}\,d\theta,$$

and

$$J_5=\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n}|(\theta+\mu_n)^{-\lambda}-\theta^{-\lambda}|\,|F(\theta)|\,|\psi(\theta)|\,d\theta.$$

Here

$$J_1=O\left[\int_{\alpha_n-\mu_n}^{\alpha_n}n^{\lambda-k}(\theta+\mu_n)^{-k}(\theta+\mu_n)^{-\lambda}(\theta+\mu_n)^{\lambda-k}d\theta\right]$$

$$=O[n^{\lambda-k}\,\alpha_n^{-k-\lambda}(\alpha_n+\mu_n)^{\lambda-k}\mu_n]$$

$$=O[n^{\lambda-k-1}\alpha_n^{-2k}]=o(1),\quad \text{as } 0<\lambda<\tfrac{1}{2}.$$

$$J_2=O\left[\int_{\pi-\frac{1}{n}-\mu_n}^{\pi-\frac{1}{n}}n^{\lambda-k}\theta^{-k-\lambda}d\theta\right]=O(n^{\lambda-k}\mu_n)=o(1).$$

$$J_3=o\left[\mu_n^{\lambda-k}\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n}n^{\lambda-k}(\theta+\mu_n)^{-k-\lambda}d\theta\right]$$

$$=o\left[\mu_n^{\lambda-k}n^{\lambda-k}\int_{\alpha_n}^{\pi}\theta^{-k-\lambda}d\theta\right]$$

$$=o(1),\quad \text{for}\quad 0<\lambda<\tfrac{1}{2}.$$

$$J_4 = O\left[n^{\lambda-1}\log n\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n}\theta^{-1}\theta^{\lambda-k}\theta^{-\lambda}d\theta\right]$$

$$= O\left[n^{\lambda-1}\log n\int_{\alpha_n}^{\pi}\theta^{-k-1}d\theta\right]$$

$$= O[n^{\lambda-1}\log n]+O[n^{\lambda-1}\log n.\, n^{k}] = o(1), \quad \text{for } 0<\lambda<\tfrac{1}{2}.$$

$$J_5 = O\left[\mu_n\int_{\alpha_n}^{\pi}\theta^{-\lambda-1}\theta^{\lambda-k}n^{\lambda-k}\theta^{-k}d\theta\right]$$

$$= O\left[\mu_n\int_{\alpha_n}^{\pi}n^{\lambda-k}\theta^{-2k-1}d\theta\right]$$

$$= O[n^{\lambda-k-1}n^{2k}] = o(1), \qquad \text{if } 0<\lambda\leqslant\tfrac{1}{2}.$$

This proves the theorem for $0<k<\lambda$ and $0<\lambda\leqslant\frac{1}{2}$.

5. We now proceed to prove the theorem for the case $\lambda-1<k\leqslant 0$. We write $k=-p$ and work with the parameter p satisfying $0\leqslant p<1-\lambda$. For the proof we shall require the lemmas in a suitably modified form.

LEMMA 6. *For* $0<p<1-\lambda$, *we have*

$$L_n^{-p}(\theta) = O(n^{2\lambda+1}\theta). \tag{5.6}$$

Proof. Since $P_n^{(\lambda)}(\cos\theta) = O(n^{2\lambda-1})$, we have from (2.5)

$$L_n^{-p}(\theta) = O\left[n^{p}\sum_{1}^{n-1}(n-m)^{-p}m^{2\lambda-1}m\sin\theta\right]$$

$$= O(n^{2\lambda+1}\theta).$$

LEMMA 7. *For* $o<p<1-\lambda$ *and* $\pi-\dfrac{1}{n}\leqslant\theta\leqslant\pi$,

$$L_n^{-p}(\theta) = O(n^{2\lambda+p}\sin\theta). \tag{5.7}$$

Proof. As in lemma 2, we have

$$L_n^{-p}(\theta) = O\left[n^{p}\sum_{m=1}^{n-1}(n-m)^{-p-1}m^{2\lambda}\sin\theta\right]$$

$$= O[n^{2\lambda+p}\sin\theta].$$

When $p=0$, the result is evident from (2.3) and (3.2.2).

LEMMA 8. *If* $\alpha_n \leqslant \theta \leqslant \pi - \frac{1}{n}$ *and* E_n *be the real part of*

$$-\frac{2^{2-2\lambda}}{\pi}\lambda \sin \lambda\pi e^{i\pi(\frac{1}{2}-\lambda)}\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}.$$

$$\int_{-\infty}^{\theta}(\theta-t)^{-\lambda-1}\sum_{m=0}^{n}A_{n-m}^{-p-1}\{e^{i(m+\lambda+\frac{1}{2})\theta}-e^{i(m+\lambda+\frac{1}{2})t}\}\,dt,$$

then

$$(5.8)\quad L_n^{-p}(\theta) = (A_n^{-p})^{-1}E_n + O[n^{p+\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda} + n^{-1}(\sin\theta)^{-\lambda-1} + n^{p+\lambda-1}\theta^{-\lambda}(\sin\theta)^{-1}].$$

Proof. $J_m^{(2)}, J_m^{(3)}, \ldots\ldots J_m^{(6)}$ are estimated as in the proof of lemma 3 and then for $p \neq 0$

$$S_n^{(2)},\ S_n^{(3)},\ S_n^{(4)} = O\left[\sum_{1}^{n-1}(n-m)^{-p-1}m^{\lambda-1}\right](\sin\theta)^{-\lambda}$$

$$= O[n^{\lambda-1}(\sin\theta)^{-\lambda}];$$

$$S_n^{(5)} = O[n^{\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda}];$$

$$S_n^{(6)} = O\left[\left\{n^{-p-1} + \sum_{1}^{n-1}(n-m)^{-p-1}m^{\lambda-2} + n^{\lambda-2}\right\}(\sin\theta)^{-\lambda-1}\right]$$

$$= O[n^{-p-1}(\sin\theta)^{-\lambda-1}].$$

When $p = 0$; $S_n^{(2)}, S_n^{(3)}, \ldots\ldots S_n^{(6)}$ simply become $J_m^{(1)}, J_m^{(2)}, \ldots\ldots J_m^{(6)}$ and therefore the above results still hold. Beyond this, the proof in lemma 3 survives unchanged.

LEMMA 9. *If* $\alpha_n \leqslant \theta \leqslant \pi - \frac{1}{n}$ *and* $\mu_n = \frac{\pi}{n+\lambda+\frac{1}{2}}$, *then*

$$E_n = \Re\{\theta^{-\lambda}\phi(\theta)e^{i(n+\lambda+\frac{1}{2})\theta}\}$$

where $\phi(\theta)$ *is such that*

$$\phi(\theta) = O(n^{\lambda}\theta^{p}) + O(1);\ \phi(\theta+\mu_n) - \phi(\theta) = O(n^{\lambda-1}\theta^{p-1}).$$

Proof. Following Du Plessis again (1952, p. 348) we write

$$K_n(t) = \sum_{m=0}^{n}A_m^{-p-1}e^{-imt}.$$

Here

$$K_n(t) = \begin{cases} O(n^{-p}),\ 0 \leqslant t \leqslant \frac{1}{n}, \\ O(t^{p}),\ \frac{1}{n} \leqslant t \leqslant \lambda. \end{cases}$$

$$K_n'(t) = \begin{cases} O(n^{1-p}), \\ O(t^{p-1}), \ \frac{1}{n} \leqslant t \leqslant \pi. \end{cases}$$

$$K_n''(t) = \begin{cases} O(n^{2-p}), \\ O(n^{1-p}t^{-1}), \frac{1}{n} \leqslant t \leqslant \pi. \end{cases}$$

Again, we write

$$\phi(\theta) = I_1 + I_2 + \dots + I_5 \text{ as in lemma 4.}$$

So,

$$I_2 = O\left(\int_{\frac{1}{n}}^{\infty} u^{-\lambda-1}\theta^p \, du\right) = O(n^{\lambda}\theta^p);$$

$$I_3 = O\left\{\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1}(\theta-u)^p \, du\right\} = O(n^{\lambda}\theta^p);$$

$$I_4 = O\left\{n^{-p}\int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1} \, du\right\} = O(n^{-p-1}\theta^{-\lambda-1})$$

$$= O(n^{\lambda}\theta^p);$$

and

$$I_5 = \int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1}e^{-i(n+\lambda+\frac{1}{2})u}K_n(\theta-u) \, du$$

$$+\int_{-\pi}^{\pi} e^{-i(n+\lambda+\frac{1}{2})u}K_n(\theta-u) \sum_{m=1}^{\infty} e^{-i(n+\lambda+\frac{1}{2})2m\pi}(u+2m\pi)^{-\lambda-1} \, du.$$

Also

$$\int_{\theta+\frac{1}{n}}^{\pi} = O\left\{\int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1}(\theta-u)^p \, du\right\} = O(\theta^{p-\lambda}) + O(1)$$

$$= O(n^{\lambda}\theta^p) + O(1),$$

and

$$\int_{-\pi}^{\pi} = \int_{-\pi}^{\theta-\frac{1}{n}} [\sin(\theta-u)]^p \, du + \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} n^{-p} \, du + \int_{\theta+\frac{1}{n}}^{\pi} [\sin(u-\theta)]^p \, du$$

$$= O(1).$$

Finally

$$I_1 = O\left\{\int_0^{\frac{1}{n}} u^{-\lambda}\left[\frac{d}{d\xi}\, e^{-i(n+\lambda+\frac{1}{2})\xi}\, K_n(\theta-\xi)\right] du\right\}; \quad o < \xi < u,$$

$$= O\left[\int_0^{\frac{1}{n}} u^{-\lambda}.\, n\theta^p\, du\right]$$

$$= O(n^{\lambda}\theta^p).$$

As in lemma 3, we divide $\phi(\theta+\mu_n)-\phi(\theta)$ into

$$J_1+J_2+\,\ldots\ldots\,+J_5,$$

where

$$J_1 = \mu_n \int_0^{\frac{1}{n}} u^{-\lambda}\frac{d}{d\xi}\left[e^{-i(n+\lambda+\frac{1}{2})\xi}\, K_n'(t-\xi)\right] du;$$

$$\theta < t < \theta+\mu_n \quad \text{and} \quad o < \xi < u,$$

$$= O\left[\frac{1}{n}\int_0^{\frac{1}{n}} u^{-\lambda}\{n^{1-p}\theta^{-1}+n\theta^{p-1}\}\, du\right]$$

$$= O(\theta^{p-1}\, n^{\lambda-1});$$

$$J_2 = O\left[\mu_n\, K_n'(\theta)\int_{\frac{1}{n}}^{\infty} u^{-\lambda-1}\, du\right] = O\left[\frac{1}{n}\,\theta^{p-1}n^{\lambda}\right]$$

$$= O[n^{\lambda-1}\,\theta^{p-1}];$$

$$J_3 = O\left[\mu_n\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1}\, K_n'(\theta-u)\, du\right]$$

$$= O\left[\frac{1}{n}\int_{\frac{1}{n}}^{\theta-\frac{1}{n}} u^{-\lambda-1}\,(\theta-u)^{p-1}\, du\right]$$

$$= O[n^{-1}.\,\theta^{p-1-\lambda}n^{\lambda}\,\theta^{\lambda}] = O(n^{\lambda-1}\,\theta^{p-1})\cdot$$

$$J_4 = O\left[\mu_n\int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1}\, K_n'(\theta-u)\, du\right]$$

$$= O\left[\frac{1}{n}\int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} u^{-\lambda-1}\, n^{1-p}\, du\right]$$

$$= O[n^{\lambda-1}\,\theta^{p-1}].$$

If $p \neq 0$,

$$J_5 = O\left[\mu_n \int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1} K_n'(\theta-u)\,du + \mu_n\left(\int_{-\pi}^{\theta-\frac{1}{n}} + \int_{\theta-\frac{1}{n}}^{\theta+\frac{1}{n}} + \int_{\theta+\frac{1}{n}}^{\pi}\right) K_n'(\theta-u)\,du\right].$$

The first term is

$$O\left[n^{-1}\int_{\theta+\frac{1}{n}}^{\pi} u^{-\lambda-1}(\theta-u)^{p-1}\,du\right]$$

$$= O\left[n^{-1}\theta^{p-\lambda-1}\int_{1+\frac{1}{n\theta}}^{\infty} v^{-\lambda-1}(1-v)^{p-1}\,dv\right]$$

$$= O[n^{\lambda-1}\theta^{p-1}].$$

The second term is easily seen to be $O\left(\frac{1}{n}\right)$.

When $p = 0$,

$$\phi(\theta) = \theta^{\lambda}\left(\cot\frac{\theta}{2}\right)^{1-\lambda}\left(\sin\frac{\theta}{2}\right)^{1-2\lambda}\left(n+\lambda+\frac{1}{2}\right)^{\lambda} e^{i\pi\lambda/2}.$$

Here again the required orders stand true.
Combining the lemmas above we may state

LEMMA 10. *For* $\alpha_n \leqslant \theta \leqslant \pi - \frac{1}{n}$,

$$L_n^{-p}(\theta) = \mathfrak{R}[\psi(\theta)\theta^{-\lambda}e^{i(n+\lambda+\frac{1}{2})\theta}] + O[n^{p+\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda} + n^{-1}(\sin\theta)^{-\lambda-1} + n^{p+\lambda-1}\theta^{-\lambda}(\sin\theta)^{-1}];$$

where $\psi(\theta)$ *is such that for* $\mu_n = \frac{\pi}{n+\lambda+\frac{1}{2}}$,

$$\psi(\theta) = O(\theta^{p}n^{p+\lambda}) + O(n^{p});\quad \psi(\theta+\mu_n) - \psi(\theta) = O(n^{p+\lambda-1}\theta^{p-1}).$$

6. Proof of the theorem for $o \leqslant p < 1-\lambda$ and $o < \lambda \leqslant \frac{1}{2}$:—

$$\sigma_n^{-p} = \left\{\int_0^{\alpha_n} + \int_{\alpha_n}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right\} F(\theta)L_n^{-p}(\theta)\,d\theta$$

$$= I_1 + I_2 + I_3, \text{ say.}$$

Now,

$$|I_1| = \left|\int_0^{\alpha_n} F(\theta)L_n^{-p}(\theta)\,d\theta\right|$$

$$= o\left\{n^{2\lambda+1}\int_0^{\alpha_n} \theta\,.\,\theta^{\lambda+p}\,d\theta\right\}$$

$$= o\left(n^{2\lambda+1}\alpha_n^{\lambda+p+2}\right) = o(1),$$

when

$$\alpha_n = n^{-(2\lambda+1)(\lambda+p+2)^{-1}}.$$

Also

$$|I_3| = O\left(n^{2\lambda+p}\int_{\pi-\frac{1}{n}}^{\pi} \sin\theta\, d\theta\right) = o(1).$$

And

$$I_2 = \int_{\alpha_n}^{\pi-\frac{1}{n}} F(\theta)\psi(\theta)\theta^{-\lambda}e^{i(n+\lambda+\frac{1}{2})\theta}d\theta$$

$$+O\left[\int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{p+\lambda-1}\theta^{-1}(\sin\theta)^{-\lambda}d\theta + \int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{-1}(\sin\theta)^{-\lambda-1}d\theta\right.$$

$$\left.+\int_{\alpha_n}^{\pi-\frac{1}{n}} |F(\theta)|\, n^{p+\lambda-1}\theta^{-\lambda}(\sin\theta)^{-1}\, d\theta\right].$$

The error term is

$$O\left[n^{p+\lambda-1}\left(\int_{\alpha_n}^{\pi/2} \theta^{\lambda+p}\theta^{-\lambda-1}\, d\theta + \int_{\pi/2}^{\pi-\frac{1}{n}} (\sin\theta)^{-\lambda}\, d\theta\right)\right]$$

$$+O\left[n^{-1}\left(\int_{\alpha_n}^{\pi/2} \theta^{\lambda+p}\theta^{-\lambda-1}\, d\theta + \int_{\pi/2}^{\pi-\frac{1}{n}} (\sin\theta)^{-\lambda-1}\, d\theta\right)\right]$$

$$+O\left[n^{p+\lambda-1}\left(\int_{\alpha_n}^{\pi/2} \theta^{\lambda+p}\theta^{-\lambda-1}\, d\theta + \int_{\pi/2}^{\pi-\frac{1}{n}} (\sin\theta)^{-1}\, d\theta\right)\right]$$

$$= o(1).$$

We put

$$I_2 = J_1 + J_2 + \ldots\ldots + J_5 \text{ as before, where}$$

$$J_1 = \int_{\alpha_n-\mu_n}^{\alpha_n} |F(\theta+\mu_n)\psi(\theta+\mu_n)(\theta+\mu_n)^{-\lambda}|\, d\theta$$

$$= O\left(\int_{\alpha_n-\mu_n}^{\alpha_n} (\theta+\mu_n)^{p+\lambda}n^{p+\lambda}(\theta+\mu_n)^{p-\lambda}\, d\theta\right) + O\left(\int_{\alpha_n-\mu_n}^{\alpha_n} n^{p}(\theta+\mu_n)^{p+\lambda}(\theta+\mu_n)^{-\lambda}\, d\theta\right)$$

$$= O(n^{p+\lambda}\alpha_n^{2p}\,\mu_n) + O(\alpha_n^{p}\,\mu_n\,.\,n^{p})$$

$$= o(1);$$

$$J_2 = \int_{\pi-\frac{1}{n}-\mu_n}^{\pi-\frac{1}{n}} |F(\theta)\psi(\theta)\theta^{-\lambda}|\, d\theta$$

$$= O\left[\int_{\pi-\frac{1}{n}-\mu_n}^{\pi-\frac{1}{n}} n^{\lambda+p}\theta^{p-\lambda}\, d\theta\right] + O\left[\int_{\pi-\frac{1}{n}-\mu_n}^{\pi-\frac{1}{n}} n^{p}\theta^{-\lambda}\, d\theta\right]$$

$$= O[n^{\lambda+p}\mu_n] + O[n^p\mu_n] = o(1);$$

$$J_3 = \int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n} |F(\theta+\mu_n)-F(\theta)|\,|\psi(\theta+\mu_n)|\,(\theta+\mu_n)^{-\lambda}\, d\theta$$

$$= o\left\{\mu_n^{p+\lambda}n^{p+\lambda}\int_{\alpha_n}^{\pi}(\theta+\mu_n)^{p-\lambda}\, d\theta\right\} + o\left\{\mu_n^{p+\lambda}\int_{\alpha_n}^{\pi} n^p(\theta+\mu_n)^{-\lambda}\, d\theta\right\}$$

$$= o(1) + o\left(\alpha_n^{p-\lambda+1}\right) + o(1) = o(1);$$

$$J_4 = \int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n} |\psi(\theta+\mu_n)-\psi(\theta)|\,|F(\theta)|\,(\theta+\mu_n)^{-\lambda}\, d\theta$$

$$= O\left[\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n} n^{p+\lambda-1}\theta^{p-1}\theta^{\lambda+p}(\theta+\mu_n)^{-\lambda}\, d\theta\right]$$

$$= O\left[\int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n} n^{p+\lambda-1}\theta^{2p-1}\, d\theta\right]$$

$$= O\left[n^{p+\lambda-1}\left\{O(1)+\alpha_n^{2p}\right\}\right] = o(1);$$

$$J_5 = \int_{\alpha_n}^{\pi-\frac{1}{n}-\mu_n} |(\theta+\mu_n)^{-\lambda}-\theta^{-\lambda}|\,|F(\theta)|\,|\psi(\theta)|\, d\theta$$

$$= O\left[\mu_n\int_{\alpha_n}^{\pi}\theta^{-\lambda-1}n^{p+\lambda}\theta^p\theta^{\lambda+p}\, d\theta\right] + O\left[\mu_n\int_{\alpha_n}^{\pi}\theta^{-\lambda-1}n^p\theta^{\lambda+p}\, d\theta\right]$$

$$= O\left[n^{p+\lambda-1}\left\{O(1)+\alpha_n^{2p}\right\}\right] + O\left[n^{p-1}\left\{O(1)+o(1)\right\}\right]$$

$$= o(1).$$

Hence $\sigma_n^{-p}(o) \to o$ as $n \to \infty$.

This completes the proof of the theorem.

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