TRANSVERSE WAVES IN PLASMA

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Transverse electromagnetic waves have been studied in collisionless, one-component classical plasma using Vlasov's linearized equation and Maxwell's equations and it is also extended to the case when a constant external magnetic field B_0 permeates plasma. The respective characteristic equation is solved by means of Van Kampen's method of stationary solutions following Case closely. The dispersion equation for the cold plasma is obtained from general dispersion equation.

1. INTRODUCTION

Landau (1946), Van Kampen (1955), Case (1959), Backus (1960), Zelazny (1962*a*) solved the initial value problem for longitudinal waves in plasma extensively. Zelazny (1962*b*) and Felderhof (1963*a*) studied transverse waves in plasma in the absence of an external magnetic field following the method of stationary solutions; Pradhan (1957) and Felderhof (1963*b*) discussed transverse waves in plasma under a constant external magnetic field following the same method. Case's approach (1959) has helped a good deal to understand the problem of longitudinal waves in plasma. In Part I we extend his approach to study transverse waves in hot, classical and collisionless plasma in the absence of any external field and in Part II we further extend it when a constant external magnetic field B_0 permeates the plasma. The positive ions are assumed to be smeared out to form a uniform positive neutralizing background. For simplicity, the reduced distribution functions, which give clear insight into the important physical features of the problem such as the density of the electrons, the electric field and Landau-damping in the plasma, are used. The basic equations of the theory are Vlasov's linearized equation for the single-particle distribution function and Maxwell's field equations.

PART I

(Plasma free from any External Field)

2. *Basic Equations.*—The Vlasov-Boltzmann equation is used for collisionless plasma. For transverse waves, Maxwell's equations are combined with the V-B equation. The electron distribution is given by single particle distribution function $f(r, v, t)$ and it is broken into two parts as

$$f(r, v, t) = n_0 f_0(v) + f_1(r, v, t) \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

where n_0 and f_0 refer to the equilibrium state and $f_1(r, v, t)$ is the perturbation. In the absence of any external electric or magnetic field the V-B equation reduces to the first order in f_1 as

$$\frac{\partial f_1}{\partial t}+V\cdot\frac{\partial f_1}{\partial r}+\frac{n_0 e}{m}\left\{E(r, t)+\frac{V\times B(r, t)}{c}\right\}\cdot\frac{\partial f_0}{\partial V}=0. \quad \ldots \quad \ldots \quad (2.2)$$

Maxwell's equations are

$$\nabla\times B=\frac{4\pi}{c}j+\frac{1}{c}\frac{\partial E}{\partial t}, \quad \nabla\cdot B=0,$$

$$\nabla\times E=-\frac{1}{c}\frac{\partial B}{\partial t}, \quad \nabla\cdot E=4\pi\rho \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

where E and B are the electric and magnetic field strengths. $\rho(r, t)$ and $j(r, t)$ are the charge and current densities respectively and measured as follows:

$$\rho(r, t)=e\int f_1(r, V, t)dV, j(r, t)=e\int Vf_1(r, V, t)dV. \quad \ldots \quad (2.4)$$

The coefficients of linear equations (2.2) and (2.3) are independent of space coordinates; therefore, Fourier transform technique can be used. Fourier components depend on space coordinates according as exp $(ik.r)$, where k is a real vector. Here solutions pertaining to a given k are considered.

Let the k-vector be along the z-axis; then by integrating over the velocity components v_x and v_y we define the following reduced distribution functions:

$$f_x(z, v_z, t)=\iint v_x f_1(z, V, t)dv_x dv_y,$$

$$f_y(z, v_z, t)=\iint v_y f_1(z, V, t)dv_x dv_y,$$

$$f_z(z, v_z, t)=\iint f_1(z, V, t)dv_x dv_y. \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

The linearized V-B equation (2.2) is multiplied by v_x, v_y and 1 respectively and then integrated over v_x and v_y components of velocity. We get the reduced distribution functions (2.5) and on combining these functions with relevant field components from Maxwell's equations, three different sets of equations are obtained.

I set:

$$\frac{\partial f_x}{\partial t}+v_z\frac{\partial f_x}{\partial z}-\frac{n_0 e}{m}E_x F(v_z)=0,$$

$$\frac{\partial E_x}{\partial t}=-c\frac{\partial B_y}{\partial z}-4\pi e\int f_x(z, v_z, t)dv_z,$$

$$\frac{\partial B_y}{\partial t}=-c\frac{\partial E_x}{\partial z} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

where

$$F(v_z) = \int\int f_0(v) dv_x dv_y. \qquad \ldots \quad \ldots \quad \ldots \quad (2.7)$$

$F(v_z)$ is an even, non-negative function of v_z; it decreases monotonously for $v_z > 0$ and normalized to unity. I set involves only the x-component of the electric field.

II set:

$$\frac{\partial f_y}{\partial t} + v_z \frac{\partial f_y}{\partial z} - \frac{n_0 e}{m} E_y F(v_z) = 0,$$

$$\frac{\partial E_y}{\partial t} = c \frac{\partial B_x}{\partial z} - 4\pi e \int f_y(z, v_z, t)\, dv_z,$$

$$\frac{\partial B_x}{\partial t} = c \frac{\partial E_y}{\partial z}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

It involves only E_y component.

III set:

$$\frac{\partial f_z}{\partial t} + v_z \frac{\partial f_z}{\partial z} + \frac{n_0 e}{m} E_z \frac{dF(v_z)}{dv_z} = 0,$$

$$\frac{\partial E_z}{\partial z} = 4\pi e \int f_z(z, v_z, t)\, dv_z. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

It involves only E_z. From these sets of equations the factor exp (ikz) and an implied index k have been omitted. Now there are three reduced problems. And all these three can be dealt with individually as an initial value problem. The first two sets correspond to transverse electromagnetic waves with different direction of polarization, and the III set to the longitudinal plasma waves. The last set of equations has been studied extensively by Landau (1946), Van Kampen (1955), Case (1959), Zelazny (1962*a*) and Backus (1960). Here we shall discuss only transverse waves.

3. *Stationary Solutions.*—The I and II sets form equivalent problems for transverse electromagnetic waves and here we shall restrict to the I set only. The fundamental equations of this problem are

$$\frac{\partial f_x}{\partial t} = -ikv_z f_x + \frac{n_0 e}{m} E_x F(v_z),$$

$$\frac{\partial E_x}{\partial t} = -ikc\, B_y - 4\pi e \int f_x(v_z, t) dv_z, \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

$$\frac{\partial B_y}{\partial t} = -ikc\, E_x.$$

To solve equations (3.1) let us consider the distribution and field components vary exponentially with time as exp $(-iwt)$. Putting

$$f_x = g(v_z) e^{-iwt}$$

and taking

$$u = \frac{w}{k},$$

we get for equations (3.1)

$$(u-v_z)g^u(v_z) = i\left(\frac{n_0 e}{mk}\right)E^u F(v_z),$$

$$uE^u = cB^u - \left(\frac{4\pi ie}{k}\right)\int g^u(v_z')\,dv_z', \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

$$uB^u = cE^u.$$

Eliminating B^u from the last two equations of (3.2) we get

$$\frac{(u^2-c^2)}{u}E^u = -\left(\frac{4\pi ie}{k}\right)\int g^u(v_z')\,dv_z'. \qquad \ldots \quad \ldots \quad (3.3)$$

Substituting the value of E^u in the first equation of (3.2) from (3.3) we obtain

$$(u-v_z)g^u(v_z) = u_p^2 u\,\frac{F(v_z)}{(u^2-c^2)}\int g^u(v_z')\,dv_z' \qquad \ldots \quad \ldots \quad (3.4)$$

where

$$u_p^2 = \frac{w_p^2}{k^2} \text{ and } w_p^2 = \frac{4\pi n_0 e^2}{m}.$$

Let us put

$$u_p^2 u F(v_z) = \eta(v_z) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

where $\eta(v_z)$ is an arbitrary function. Using equation (3.5), equation (3.4) becomes

$$(u-v_z)g^u(v_z) = \frac{\eta(v_z)}{(u^2-c^2)}\int_{-\infty}^{\infty} g^u(v_z')dv_z'. \qquad \ldots \quad \ldots \quad (3.6)$$

Considering the different values of u and the properties of the function $\eta(v_z)$, the eigen functions are divided into four groups.

Class 1(a):

There are solutions when u is real and $\eta(u)$ does not vanish. These solutions form a set of such points u and we have

$$g^u(v_z) = P\,\frac{\eta(v_z)}{u-v_z} + \lambda(u)\delta(u-v_z). \qquad \ldots \quad \ldots \quad (3.7)$$

Normalizing the solutions such as

$$\int_{-\infty}^{\infty} g^u(v_z')dv_z' = u^2-c^2. \qquad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

Hence

$$\lambda(u) = u^2-c^2+P\int \frac{\eta(v_z)}{v_z-u}\,dv_z. \qquad \ldots \quad \ldots \quad (3.9)$$

Here P indicates the Cauchy principal value while integrating over v_z and δ denotes the Dirac delta function.

Class 1(b):

In these solutions u is real and $\eta(u)$ vanishes. The solution is

$$g^u(v_z) = P\,\frac{\eta(v_z)}{u-v_z}+\lambda(u)\delta(u-v_z) \qquad \ldots \quad \ldots \quad \ldots \quad (3.10)$$

and again

$$\lambda(u) = u^2-c^2+P\int_{-\infty}^{\infty} \frac{\eta(v_z)}{v_z-u}\,dv_z. \qquad \ldots \quad \ldots \quad (3.11)$$

Here in equation (3.10) the principal value sign is not necessary.

Class 1(c):

These solutions cover those cases of class 1(b) where u is real, $\eta(u) = 0$ and $\lambda(u)$ also vanishes. They occur only for a finite number of points u_i where $i = 1, 2 \ldots m$. The solutions are

$$g^{u_i}(v_z) = g_i(v_z) = \frac{\eta(v_z)}{u-v_z} \qquad \ldots \quad \ldots \quad \ldots \quad (3.12)$$

and condition (3.8) is also satisfied.

Class 2:

For these solutions u is complex and the normalization condition (3.8) is also satisfied. Then we have

$$g^u(v_z) = \frac{\eta(v_z)}{u-v_z}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.13)$$

According to the normalization condition there are a finite discrete set of points u_j, where $j = m+1, m+2, \ldots n$.

Thus we have a continuum of solutions for all real u such that not simultaneously

$$\eta(u_i) = 0 = \lambda(u_i) \equiv u^2-c^2+\int_{-\infty}^{\infty} \frac{\eta(v_z)}{v_z-u}\,dv_z. \qquad \ldots \quad \ldots \quad (3.14)$$

There is a discrete set of solutions such that either u_i is complex and

$$\int_{-\infty}^{\infty} \frac{\eta(v_z)}{v_z-u}\,dv_z = u^2-c^2$$

or u_i is real with $\eta(u_i) = \lambda(u_i) = 0$.

These two points should be noted.

(*a*) All solutions have been normalized such that

$$\int_{-\infty}^{\infty} g^u(v_z)dv_z = u^2 - c^2. \quad \ldots \quad \ldots \quad \ldots \quad (3.15)$$

(*b*) For simplicity all roots u_i are assumed to be simple.

4. *The Adjoint Equations.*—Let us consider the adjoint equation of equation (3.6) as

$$(u - v_z)\bar{g}^u(v_z) = \frac{1}{u^2 - c^2}\int \eta(v_z')\bar{g}^u(v_z')dv_z' \quad \ldots \quad \ldots \quad (4.1)$$

and we see that to each eigen value u of equation (3.6) there is an eigen function of the adjoint equation (4.1). Thus for each u of the previous section we consider equation (4.1) and discuss u under similar classes.

Class 1(*a*):

Here u is real and $\eta(u)$ does not vanish. The normalization condition is

$$\int_{-\infty}^{\infty} \eta(v_z')\bar{g}^u(v_z')dv_z' = u^2 - c^2. \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

Then equation (4.1) becomes

$$(u - v_z)\bar{g}^u(v_z) = 1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3)$$

and the solution of equation (4.1) is

$$\bar{g}^u(v_z) = P\frac{1}{u - v_z} + \bar{\lambda}(u)\delta(u - v_z). \quad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

Multiplying equation (4.4) by $\eta(v_z)$ integrating over v_z and using the normalization condition we get

$$\bar{\lambda}(u)\eta(u) = u^2 - c^2 + P\int \frac{\eta(v_z)}{v_z - u}dv_z = \lambda(u).$$

Therefore

$$\bar{\lambda}(u) = \frac{\lambda(u)}{\eta(u)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

Class 1(*b*):

When u is real, $\eta(u) = 0$ and $\lambda(u)$ does not vanish, the solution is

$$\bar{g}^u(v_z) = \delta(u - v_z). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

Substituting this in the right-hand side of equation (4.1) and integrating, we get

$$\int \bar{g}^u(v_z)\eta(v_z)dv_z = \eta(u) = 0.$$

Hence equation (4.6) gives the eigen function solution for the characteristic equation (4.1).

Class 1(c):

When $\eta(u_i) = 0 = \lambda(u_i)$ with u_i real and the normalization condition (4.2) holds good, then the eigen function solutions of equation (4.1) are

$$\bar{g}^{u_i}(v_z) = \bar{g}_i(v_z) = \frac{1}{u_i - v_z}. \quad \cdots \quad (4.7)$$

Substituting this value in the normalization condition and using the identity (3.14), we can define characteristic values u_i.

Class 2:

This class consists of complex values of u and the normalization condition is again (4.2). Then the characteristic solutions are given by

$$\bar{g}_i(v_z) = \frac{1}{u_i - v_z}. \quad \cdots \quad (4.8)$$

In this case also the normalization condition becomes an identity and accordingly the characteristic values u_i are defined.

5. *Solutions of the Initial Value Problem :*

We are interested in solving the problem posed by the set of equations (3.1) with a prescribed initial value $g(v_z, 0)$. Using the completeness result and expanding in terms of eigen functions as has been proposed by Case (1959) for the problem of longitudinal plasma oscillations, we have

$$g(v_z, 0) = \sum_i a_i g_i(v_z) + \int A(u) g^u(v_z)\, du. \quad \cdots \quad (5.1)$$

Following Case (1959) to make use of the orthogonality property with respect to the adjoint function, we get

$$a_i = \frac{1}{c_i} \int \bar{g}_i(v_z) g(v_z, 0)\, dv_z$$

$$= \frac{\displaystyle\int \frac{g(v_z, 0)}{u_i - v_z}\, dv_z}{\displaystyle\int \frac{\eta(v_z)}{(u_i - v_z)^2}\, dv_z}, \quad \cdots \quad (5.2)$$

and

$$A(u) = \frac{1}{c_u} \int \bar{g}^{(u)}(v_z) g(v_z, 0)\, dv_z$$

$$= \left\{ \frac{1}{\lambda^2(u) + \pi^2 \eta^2(u)} \left[\lambda(u) g(u, 0) - \eta(u) P \int \frac{\eta(v_z) g(v_z, 0)}{u - v_z}\, dv_z \right] \right\}. \quad \cdots \quad (5.3)$$

Now the solution of $g(v_z, t)$ is obtained directly from (5.1) by introducing the appropriate time dependence of the eigen functions. Thus

$$g(v_z, t) = \sum_i a_i e^{-iku_i t} g_i(v_z) + \int A(u) e^{-ikut} g^u(v_z)\, du \quad \cdots \quad (5.4)$$

since

$$\int g^u(v_z)\,dv_z = u^2-c^2,$$

and

$$E^u(t) = -\left(\frac{4\pi ie}{k}\right)\frac{u}{u^2-c^2}\int g^u(v_z,t)\,dv_z, \quad \ldots \quad (5.5)$$

we have

$$E^u(t) = -\left(\frac{4\pi ie}{k}\right)\left\{\sum_i a_i u_i e^{-iku_it}+\int A(u)ue^{-ikut}\,du\right\}. \quad \ldots \quad (5.6)$$

6. *Dispersion Equation.*—For real u we have seen in section 3 that

$$\lambda(u) = u^2-c^2+P\int\frac{\eta(v_z)}{v_z-u}\,dv_z \quad \ldots \quad (6.1)$$

or

$$\lambda(u) = u^2-c^2+u_p^2uP\int\frac{F(v_z)}{v_z-u}\,dv_z. \quad \ldots \quad (6.2)$$

It is obvious that for high velocities the equilibrium distribution function $f_0(v)$ is small and hence $F(v_z)$ becomes very small.

Therefore

$$\lambda(u) = u^2-c^2+u_p^2uP\int\frac{F(v_z)}{v_z-u}\,dv_z = 0. \quad \ldots \quad (6.3)$$

For finite values of these quantities the denominator in (6.3) is expanded and one finds on integration

$$u^2 = c^2+u_p^2\left(1+\left\langle v_z^2\right\rangle\Big/u^2+\ldots\right) \quad \ldots \quad (6.4)$$

where

$$\left\langle v_z^2\right\rangle = \int v_z^2F(v_z)\,dv_z.$$

If $F(v_z)$ approximates to a delta function then (6.4) reduces to the familiar expression for the cold-plasma, *i.e.*

$$w^2 = k^2c^2+w_p^2. \quad \ldots \quad (6.5)$$

In the asymptotic sense discussed above it is useful to define a refractive index n as follows :

$$n^2 = \frac{c^2k^2}{w^2} = 1-\frac{w_p^2}{w^2}P\int\frac{F(v_z)}{w-kv_z}\,dv_z. \quad \ldots \quad (6.6)$$

Here n can be obtained as a function of w by solving equation (6.6) for k in terms of w. In the limit of small k, *i.e.* for a cold plasma, equation (6.6) reduces to the familiar expression

$$n^2 = 1-\frac{w_p^2}{w^2}. \quad \ldots \quad (6.7)$$

For real values of k the frequency is always larger than the plasma frequency. For this range of frequencies the refractive index is real.

PART II

(Plasma under Constant External Magnetic Field B_0)

7. *Fundamental Equation.*—For collisionless plasma we combine the Vlasov-Boltzmann equation with Maxwell's field equations. We use the single-particle distribution function $f(r, v, t)$ which separates into two parts.

$$f(r, v, t) = n_0 f_0(v) + f_1(r, v, t) \qquad \ldots \quad \ldots \quad \ldots \quad (7.1)$$

where n_0 and $f_0(v)$ correspond to equilibrium state and in the presence of constant external magnetic field B_0, $f_0(v)$ may be anisotropic in the velocity space. The V-B equation for this case is

$$\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial r} + \frac{e}{m} \frac{V \times B_0}{c} \cdot \frac{\partial f_1}{\partial v} + \frac{n_0 e}{m} \left\{ E(r, t) + \frac{V \times B(r, t)}{c} \right\} \cdot \frac{\partial f_0}{\partial V} \equiv 0. \quad \ldots \quad (7.2)$$

The equilibrium distribution f_0 is normalized to unity and

$$(V \times B_0) \cdot \frac{\partial f_0}{\partial V} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.3)$$

must hold true to have a time-independent solution of equation (7.2). Let us take B_0 in the direction of z-axis, then the general solution of equation (7.3) is

$$f_0(V) = f_0(v_x^2 + v_y^2, v_z). \qquad \ldots \quad \ldots \quad \ldots \quad (7.4)$$

We are interested in a particular case of wave propagation along the direction of the external magnetic field B_0 and further for the sake of simplicity we take B_0 to be in the direction of z-axis. For convenience sake the reduced distribution functions (2.5) are used.

Multiplying the linearized V-B equation (7.2) by v_x, v_y and 1 respectively and integrating over v_x and v_y we obtain

$$\frac{\partial f_x}{\partial t} + v_z \frac{\partial f_x}{\partial z} - \frac{eB_0}{mc} f_y + \frac{n_0 e}{m} \left[-E_x F(v_z) + \frac{B_y}{c} \{ v_z F(v_z) - G(v_z) \} \right] = 0,$$

$$\frac{\partial f_y}{\partial t} + v_z \frac{\partial f_y}{\partial z} + \frac{eB_0}{mc} f_x + \frac{n_0 e}{m} \left[-E_y F(v_z) - \frac{B_x}{c} \{ v_z F(v_z) - G(v_z) \} \right] = 0. \quad \ldots \quad (7.5)$$

$$\frac{\partial f_z}{\partial t} + v_z \frac{\partial f_z}{\partial z} + \frac{n_0 e}{m} E_z \frac{dF(v_z)}{dv_z} = 0$$

where

$$F(v_z) = \iint f_0(V) dv_x dv_y, \; G(v_z) = -\tfrac{1}{2} \frac{d}{dv_z} \iint (v_x^2 + v_y^2) f_0 dv_x dv_y. \quad \ldots \quad (7.6)$$

Here the function $F(v_z)$ is non-negative and normalized to unity; the function $G(v_z)$ is in general positive when $v_z > 0$ and negative for $v_z < 0$, but this is not necessarily the case.

We introduce the quantities

$$f_{(1)(2)} = f_x \pm i f_y, \; E_{(1)(2)} = E_x \pm i E_y, \; B_{(1)(2)} = B_x \pm i B_y. \qquad \ldots \quad (7.7)$$

Now on combining equations (7.5) with Maxwell's equations we get the following three sets of equations:

I Set

$$\left.\begin{aligned}\frac{\partial f_{(1)}}{\partial t}+v_z\frac{\partial f_{(1)}}{\partial z}+iw_cf_{(1)}-\frac{n_0e}{m}\left[E_{(1)}F(v_z)+\frac{iB_{(1)}}{c}H(v_z)\right]=0,\\ \frac{\partial E_{(1)}}{\partial t}=ic\frac{\partial B_{(1)}}{\partial z}-4\pi e\int f_{(1)}(z,\, v_z,\, t)dv_z,\\ \frac{\partial B_{(1)}}{\partial t}=-ic\frac{\partial E_{(1)}}{\partial z}\end{aligned}\right\} \quad .. \quad (7.8)$$

where the cyclotron frequency $w_c=\frac{eB_0}{mc}$ and

$$H(v_z)=v_zF(v_z)-G(v_z). \quad .. \quad .. \quad .. \quad (7.9)$$

The function $H(v_z)$ is a measure of the anisotropy of f_0; it vanishes identically if f_0 is isotropic.

II Set

$$\left.\begin{aligned}\frac{\partial f_{(2)}}{\partial t}+v_z\frac{\partial f_{(2)}}{\partial z}-iw_cf_{(2)}-\frac{n_0e}{m}\left[E_{(2)}F(v_z)-\frac{iB_{(2)}}{c}H(v_z)\right]=0,\\ \frac{\partial E_{(2)}}{\partial t}=-ic\frac{\partial B_{(2)}}{\partial z}-4\pi e\int f_{(2)}(z,\, v_z,\, t)dv_z,\\ \frac{\partial B_{(2)}}{\partial t}=ic\frac{\partial E_{(2)}}{\partial z}.\end{aligned}\right\} \quad .. \quad (7.10)$$

The III set is identical to that of Part I and each of these sets can be solved separately for initial-value problem. Here we are interested in the I and II sets of equations which correspond to transverse waves. For positive phase velocity, the periodic plane wave solution of I set gives right-hand circularly polarized wave and for negative phase velocity, left-hand circularly polarized wave, vice versa is true for solutions of II set.

8. *Stationary Solutions.*—I and II sets of equations form the equivalent problems for transverse waves except the change in sign due to the magnetic field terms. Here we shall restrict only to I set of equations. The coefficients of the equations are independent of space coordinates and Fourier decomposition is applied, the factor exp (ikz) and implied index k are omitted. The fundamental equations become

$$\left.\begin{aligned}\frac{\partial f_{(1)}}{\partial t}+ikv_zf_{(1)}+iw_cf_{(1)}-\frac{n_0e}{m}\left[E_{(1)}F(v_z)+\frac{iB_{(1)}}{c}H(v_z)\right]=0,\\ \frac{\partial E_{(1)}}{\partial t}=-ckB_{(1)}-4\pi e\int f_{(1)}(z,\, v_z,\, t)dv_z,\\ \frac{\partial B_{(1)}}{\partial t}=ckE_{(1)}.\end{aligned}\right\} \quad .. \quad (8.1)$$

To solve equations (8.1) let us consider the distribution and field components vary exponentially with time as exp $(-iwt)$ and $f(1) = g(v_z)$ exp $(-iwt)$. Taking the variable $u = w/k$ and omitting the index (1) we have

$$\left.\begin{aligned} (u-u_c-v_z)g^u(v_z) &= \frac{in_0e}{mk}\left[E^uF(v_z)+\frac{iB^u}{c}H(v_z)\right], \\ uE^u &= -icB^u-\frac{4\pi ie}{k}\int g^u(v_z)dv_z, \\ uB^u &= icE^u \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8.2)$$

where

$$u_c = \frac{w_c}{k}.$$

Eliminating B^u from last two equations of (8.2) we have

$$E^u = -\left(\frac{4\pi ie}{k}\right)\left(\frac{u}{u^2-c^2}\right)\int g^u(v_z)\,dv_z. \quad \ldots \quad \ldots \quad \ldots \quad (8.3)$$

Using equation (8.3) and the last equation of (8.2), the first equation of (8.2) becomes

$$(u-u_c-v_z)g^u(v_z) = \frac{u_p^2}{(u^2-c^2)}\left[uF(v_z)-H(v_z)\right]\int g^u(v_z)\,dv_z \quad \ldots \quad (8.4)$$

where

$$u_p^2 = \frac{w_p^2}{k^2} = \frac{4\pi n_0e^2}{mk^2}.$$

Let us put

$$u_p^2[uF(v_z)-H(v_z)] = \eta(v_z)$$

where $\eta(v_z)$ is an arbitrary function.

Therefore

$$(u-u_c-v_z)g^u(v_z) = \frac{\eta(v_z)}{(u^2-c^2)}\int g^u(v_z)\,dv_z. \quad \ldots \quad \ldots \quad (8.5)$$

Equation (8.5) is quite a parallel case to that of equation (3.6) with a difference that here the phase velocity is decreased by a constant u_c due to the presence of static magnetic field B_0 and the function $\eta(v_z)$ is defined differently. Similar types of the eigen function groups depending on different values of the phase velocity and the properties of $\eta(v_z)$ can be obtained as the solutions of equation (8.5) and its corresponding adjoint equation following the same line of arguments as of Part I.

We are interested in solving the problem posed by equations (8.1) with a prescribed initial value $g(v_z, 0)$. Using the completeness result and expanding in terms of eigen functions we obtain the solution of $g(v_z, t)$ by introducing the appropriate time dependence of the eigen functions quite identical to that given by equation (5.4) and hence the expression for electric field can be obtained. Making use of the orthogonality property with respect to the corresponding adjoint function we can calculate the coefficients a_i and $A(u)$ in the usual way following Case (1959).

9. *Dispersion Equation.*—The general dispersion equation for transverse waves is

$$u^2-c^2+P\int\frac{\eta(v_z)}{v_z-u+u_c}dv_z=0 \quad \dots \quad \dots \quad \dots \quad (9.1)$$

or

$$u^2-c^2+u_p^2P\int\frac{\{uF(v_z)-H(v_z)\}}{v_z-u+u_c}dv_z=0. \quad \dots \quad \dots \quad (9.2)$$

For cold isotropic plasma $F(v_z)$ approximates a delta function and $H(v_z)$ vanishes identically. Then the dispersion equation reduces to the form

$$c^2=u^2-\frac{u_p^2u}{u-u_c}. \quad \dots \quad \dots \quad \dots \quad (9.3)$$

The general expression for both the sets I and II combined is

$$c^2=u^2-\frac{u_p^2u}{(u\mp u_c)} \quad \dots \quad \dots \quad \dots \quad \dots \quad (9.4)$$

where the upper sign is for the waves of type (1), the lower sign for waves of type (2).

The refractive index is defined as $n=\frac{c}{u}$.

Therefore

$$n^2=\frac{c^2}{u^2}=1-\frac{u_p^2}{(u^2\mp uu_c)} \quad \dots \quad \dots \quad \dots \quad (9.5)$$

or

$$n^2=1-\frac{w_p^2}{(w^2\mp ww_c)}. \quad \dots \quad \dots \quad \dots \quad (9.6)$$

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