THE OSCILLATING RIGID BODY PROBLEM IN MAGNETOHYDRODYNAMICS *

by KASTURI L. ARORA,† *University of Colorado, Boulder, Colorado, U.S.A.*

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The flow field set up by the oscillatory motion of an arbitrary rigid body in a viscous, incompressible and electrically conducting fluid with a transverse magnetic field is studied. It is assumed that the kinematic viscosity and the electrical conductivity are constant throughout the flow field, and that the perturbation in the basic magnetic field is small. The excess charge density and the imposed electric field are assumed to be zero. The expressions for the velocity profile, the frictional force and the energy dissipation are derived. The motion of the fluid is shown to consist of a heavily damped plane sheer wave, the coupling between the different layers being due to viscous friction and magnetic field. The analysis is carried out in two stages: first for an infinite plane and second for a body of arbitrary shape.

I. INTRODUCTION

When a solid body immersed in a homogeneous, viscous and electrically conducting fluid oscillates, the flow thereby set up has a number of characteristic properties. To study these it is convenient to begin with a simple but typical example. A similar example has also been studied by Hide and Roberts (1962).

Suppose that the fluid is bounded by an infinite plane surface which executes a coplanar simple harmonic motion, relative to the body of the fluid, with frequency ω. We solve for the resulting motion of the fluid. We take the solid surface as the xy-plane and the fluid region as $z > 0$; the x-axis is taken in the direction of oscillations. Assume that ν is constant, and that no slip occurs between the fluid and the vibrating surface. Then it is known (Rayleigh 1945) for the simple hydrodynamic case that if the velocity of the vibrating plane is $(u_0 \cos \omega t, 0, 0)$, where u_0 and ω, which are assumed to be constant, are respectively the velocity amplitude and angular frequency of the vibration, the velocity at any point in the fluid, $v = (v_x, v_y, v_z)$ is given by

$$v_x = u_0 \exp\left[-z\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right] \cos\left[\omega t - z\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right], \quad v_y = 0, \; v_z = 0. \quad \ldots \quad (1)$$

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† *Present address:* Physics Department, Panjab University, Chandigarh 3.

To facilitate our discussion for the problem, we will assume that the velocity of the oscillating plane is $A \cos(\omega t+\alpha)$. It is convenient to write this as the real part of a complex quantity:

$$u = \mathcal{R}(u_0 e^{-i\omega t}), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where the constant $u_0 = Ae^{-i\alpha}$ is in general complex, but can always be made real by a proper choice of the origin of time.

So long as the calculations involve only linear operations on the velocity u, we may omit the sign $\mathcal{R}$ and proceed as if u were complex, taking the real part of the final result. Thus we may write

$$v_x = u = u_0 e^{-i\omega t}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

II. Fundamental Equations

1. *The Equations and Their Validity*

We consider a viscous, incompressible and conducting fluid in which magnetic field may be present. The equations governing such a fluid may be taken as

$$\frac{\partial \vec{v}}{\partial t}+\left(\vec{v}\,.\,\nabla\right)\vec{v} = -\frac{1}{\rho}\nabla p+\frac{\sigma}{\rho}\left(\vec{v}\times\vec{B}\right)\times\vec{B}+\nu\nabla^2\vec{v} \quad \ldots \quad \ldots \quad (4)$$

$$\text{div}\,\vec{v} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\text{curl}\,\vec{B} = \vec{j} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$\text{div}\,\vec{B} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

$$\vec{j} = \sigma\left(\vec{v}\times\vec{B}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

We have used Gaussian units with $c = 1$. $\vec{B}$ is the magnetic field, $\vec{j}$ the electric current density, ρ the matter density, $\vec{v}$ the fluid velocity, ν the kinematic viscosity and σ the electrical conductivity. ν and σ are constant throughout the flow field. The excess charge density and the imposed electric field are assumed to be zero. $\vec{v}$, $\vec{B}$ and p are continuous and possess continuous partial derivatives with respect to x, y, z and t up to and including the order appearing in the governing equations.

2. *The Boundary Conditions*

The system is described by eleven scalar equations in eleven unknowns. The region $z \leq 0$ is filled by an insulator. The fluid is at rest at large distances from the surface and does not slip relative to the surface at the surface itself, *i.e.* $\vec{v}(z \to \infty) = 0$ and $\vec{v}(z = 0) = \vec{u} : v_y = v_z = 0$ and $v_x = u$. We divide $\vec{B}$ into a uniform field $\vec{B}_0 = (0, 0, B_0)$ and an induced field $\vec{b} = (b(z, t), 0, 0) : \vec{B} = \vec{B}_0 + \vec{b}$, where $\vec{b}$ is assumed to be small. The induced currents j_y vanish at infinity and (since $z < 0$ is occupied by an insulator) at the surface itself, *i.e.* $\vec{b}(z \to \infty) = \vec{b}(z = 0) = 0$.

3. *Formal Solution*

It is evident from symmetry that all quantities depend only on the coordinate z and the time t. From the equation of continuity $\operatorname{div} \vec{v} = 0$, we therefore have $\frac{\partial v_z}{\partial z} = 0$, whence $v_z = \text{constant} = 0$ from the boundary condition. Since all quantities are independent of the coordinates x and y and since $v_z = 0$, it follows that $(\vec{v} \cdot \nabla)\vec{v} \equiv 0$. Writing $v_x = v$ we observe that v is governed by the scalar equation

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2}{\rho} v. \qquad \ldots \quad (9)$$

We seek a solution of this equation which is periodic in z and t of the form

$$v = u_0 \exp\,[i(kz - \omega t)] \qquad \ldots \quad (10)$$

with the complex amplitude u_0, so that $v = u$ for $z = 0$. Substituting (10) in (9) we find

$$i\omega = k^2 \nu + \frac{\sigma B_0^2}{\rho},$$

whence

$$k = \left[\left(-\frac{\sigma B_0^2}{\rho} + i\omega\right)\Big/\nu\right]^{\frac{1}{2}} = \pm r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i \sin\frac{\theta}{2}\right), \qquad \ldots \quad (11)$$

where

$$r = \left[\left(\frac{\omega}{\nu}\right)^2 + \left(\frac{\sigma B_0^2}{\rho \nu}\right)^2\right]^{\frac{1}{2}} \text{ and } \theta = \pi - \tan^{-1}\left(\frac{\rho\omega}{\sigma B_0^2}\right)$$

so that the velocity v is

$$v = u_0 \exp\left[-\left(r^{\frac{1}{2}} \sin\frac{\theta}{2}\right)z\right] \exp\left\{i\left[\left(r^{\frac{1}{2}} \cos\frac{\theta}{2}\right)z - \omega t\right]\right\}. \qquad \ldots \quad (12)$$

We have taken k to have a positive imaginary part since otherwise the velocity would increase without limit in the interior of the fluid, which is physically impossible. Supposing u_0 to be real and taking the real part of (12) we obtain

$$v = u_0 \cos\left[\left(r^{\frac{1}{2}} \cos\frac{\theta}{2}\right)z - \omega t\right] \exp\left[-\left(r^{\frac{1}{2}} \sin\frac{\theta}{2}\right)z\right]. \qquad \ldots \quad (13)$$

The solution obtained represents a transverse wave: its velocity $v_x = v$ is perpendicular to the direction of propagation. The most important property of this wave is that it is rapidly damped in the interior of the fluid: the amplitude decreases exponentially as the distance z from the solid surface increases. The graphical representation of v is depicted in Figs. 1 and 2.

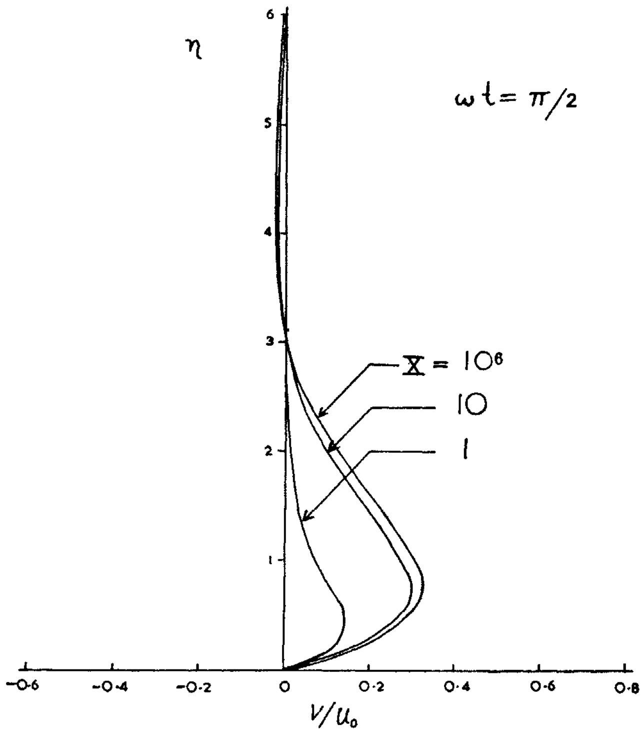

FIG. 1. $X = \dfrac{\rho\omega}{\sigma B_0^2}$; $\eta = zr^{\frac{1}{2}} \sin \dfrac{\theta}{2}$.

The distance δ over which the amplitude falls off by a factor of e is called the depth of penetration. We see from (12) that

$$\delta = 1\Big/\left(r^{\frac{1}{2}} \sin \frac{\theta}{2}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Obviously, for a fixed θ, the depth of penetration diminishes with increasing frequency and the magnetic field, but increases with the kinematic viscosity of the fluid.

The frictional force acting on unit area of the oscillating plane in the viscous fluid is evidently in x-direction and is equal to the component

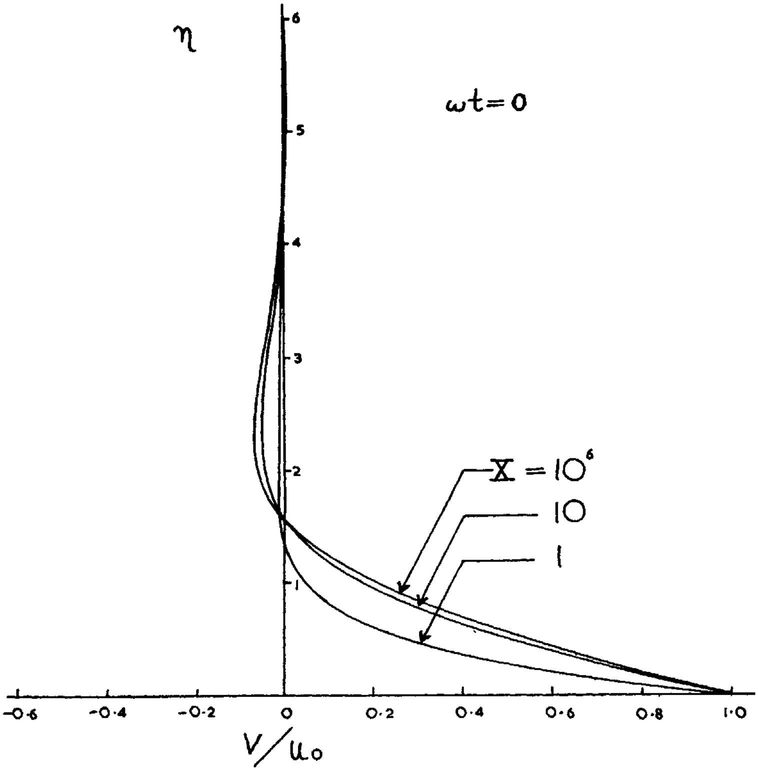

FIG. 2.

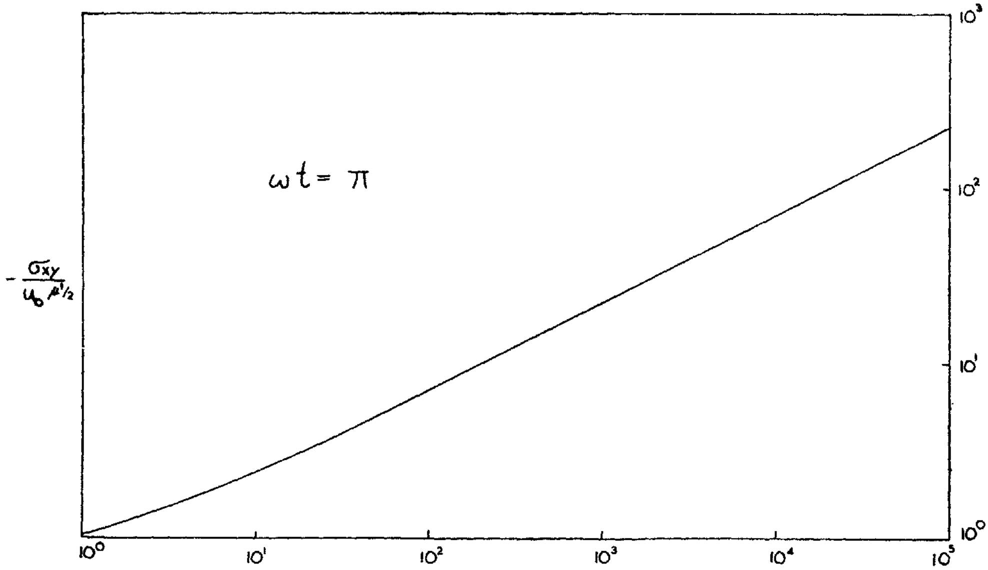

FIG. 3. $\frac{\sigma_{xy}}{u_0\mu^{\frac{1}{2}}} = [X^2+1]^{\frac{1}{4}} \sin\left(\omega t - \frac{\theta}{2}\right)$; $X = \left(\frac{\rho\omega}{\sigma B_0^2}\right)$.

$\sigma_{zx} = \mu \dfrac{\partial v_x}{\partial z}$ of the stress tensor; the value of the derivative must be taken at the surface itself. Substituting (12) and taking the real part, we obtain

$$\sigma_{zx} = u_0\big[(\mu\omega\rho)^2 + \big(\mu\sigma B_0^2\big)^2\big]^{\frac{1}{4}} \sin\left(\omega t - \frac{\theta}{2}\right). \qquad (15)$$

The graphical solution for the skin friction is shown in Fig. 3.

The velocity of the oscillating surface, however, is $u = u_0 \cos \omega t$. There is therefore a phase difference between the velocity and the frictional force.

It is possible to calculate the time average of the energy dissipation in our problem. This is the amount of work done by the frictional forces. The energy dissipated per unit time per unit area of the oscillating plane is equal to the mean value of the real part of the product of the force σ_{zx} and the velocity $u_x = u$:

$$-\overline{\sigma_{zx} u} = \tfrac{1}{2}u_0^2\big[(\mu\omega\rho)^2 + \big(\mu\sigma B_0^2\big)\big]^{\frac{1}{4}} \sin\frac{\theta}{2}. \qquad (16)$$

When $B_0 = 0$, the equations (12), (14), (15) and (16) reduce to the solution of ordinary fluid mechanics problem (Schlichting 1960). Fig. 4 depicts graphically the energy dissipation in our problem.

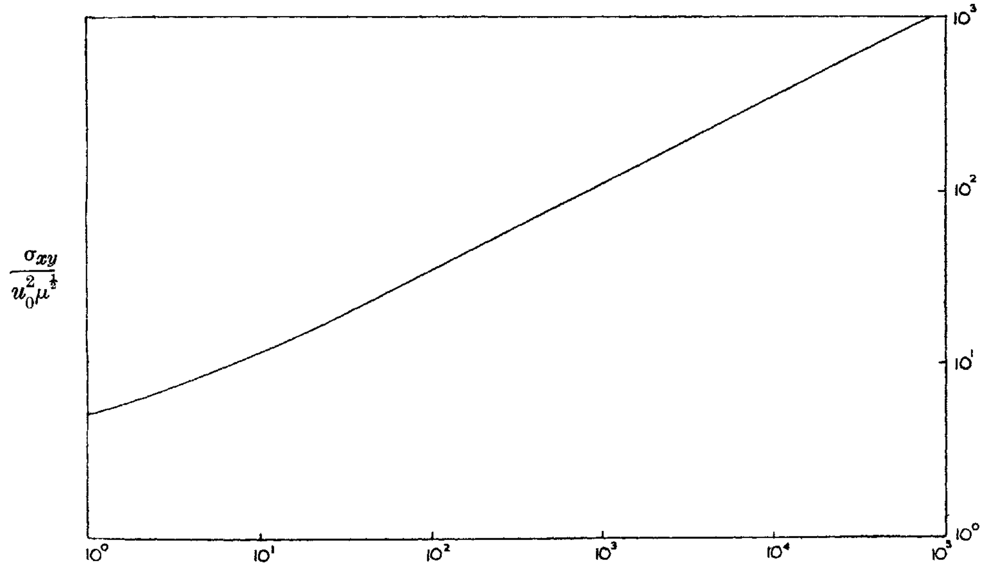

FIG. 4. $\dfrac{\sigma_{xy}}{u_0^2\mu^{\frac{1}{2}}} = \frac{1}{2}[X^2+1]^{\frac{1}{4}} \sin\dfrac{\theta}{2}$; $X = \left(\dfrac{\rho\omega}{\sigma B_0^2}\right)$.

III. SOLUTION OF THE GENERAL CASE

Let us now consider the general case of an oscillating body of arbitrary shape. In this case, the term $(\vec{v}\,.\,\text{grad})\vec{v}$ does not vanish identically. We shall assume, however, that this term is small in comparison with others so that it may be neglected. The conditions under which $(\vec{v}\,.\,\text{grad})\vec{v}$ may be neglected will be examined below.

Taking the curl on both sides of (4) and using the assumption that $\vec{b} \ll \vec{B}_0$, we obtain

$$\frac{\partial}{\partial t}\left(\text{curl}\,\vec{v}\right) = \nu\nabla^2\,\text{curl}\,\vec{v} - \frac{\sigma}{\rho}B_0^2\,\text{curl}\,\vec{v}. \qquad \ldots \quad \ldots \quad (17)$$

Thus curl $\vec{v}$ satisfies (9). We have seen above, however, that such an equation gives rise to a solution which is damped in the interior of the fluid. Consequently, we can say that the vorticity decreases towards the interior of the fluid. This means that the motion of the fluid caused by the oscillations of the body is rotational in a certain layer close to the body, whereas at large distances it rapidly changes to an electrically conducting ideal flow. It is easily verified that the depth of penetration δ is

$$\delta \sim 1/r^{\frac{1}{2}}\sin\frac{\theta}{2}.$$

Two important limiting cases are possible here: δ may be large or small compared with the dimension of the body. Suppose l is the order of magnitude of this dimension. Let us first consider the case where $\delta \gg l$. Besides this condition, let us also assume that the Reynolds number is small. Mathematically this means that

$$\left(l\sin\frac{\theta}{2}\right)^2 r \ll 1,\ \frac{\omega a l}{\nu} \ll 1, \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

where a is the amplitude of the oscillations. This corresponds to the case of low frequency of oscillations which in turn means that the velocity changes slowly with time. Therefore, we can neglect $\dfrac{\partial\vec{v}}{\partial t}$ in the equation of motion. The term $(\vec{v}\,.\,\text{grad})\vec{v}$, on the other hand, can be neglected because the Reynolds number is small.

Thus for $\delta \gg l$, the flow may be regarded as steady at any given instant. This means that the flow at any given instant is what it would be if the body were moving uniformly with its instantaneous velocity.

Let us examine the case when $l \gg \delta$. In order that the term $(\vec{v}\,.\,\text{grad})\vec{v}$ should again be negligible, it is necessary that the amplitude of the oscillations be small in comparison with the dimensions of the body:

$$\left(l\sin\frac{\theta}{2}\right)^2 r \gg 1,\ a \ll l. \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

It should be noticed that the Reynolds number need not be small in this case. The operator $(\vec{v}\,.\,\text{grad})$ denotes differentiation in the direction of velocity. Near the surface of the body, however, the velocity is nearly tangential. In the tangential direction the velocity changes appreciably only over distances of the order of dimensions of the body. Hence

$$(\vec{v}\,.\,\text{grad})\vec{v} \sim \frac{a^2\omega^2}{l}.$$

The derivative $\frac{\partial \vec{v}}{\partial t}$, on the other hand, is of the order of $a\omega^2$. Comparing these we see that

$$(\vec{v}\,.\,\text{grad})\vec{v} \ll \frac{\partial \vec{v}}{\partial t} \text{ if } a \ll l.$$

Let us now discuss the flow round an oscillating body when the conditions (19) hold. The flow is rotational in a thin layer near the surface of the body whereas in the rest of the fluid we have potential flow. Hence the flow everywhere, except in a thin layer close to the body, is given by

$$\text{curl}\,\vec{v} = 0, \quad \text{div}\,\vec{v} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (20)$$

Hence it follows that $\nabla^2\vec{v} = 0$ and the equation of motion reduces to the one for electrically conducting ideal fluid. The boundary condition therefore for solving (20) in order to determine the flow everywhere except in a thin surface layer should be that on the surface of the body, the fluid velocity must equal that of the body.

Although equations (20) are not applicable in the surface layer, the velocity distribution obtained by solving them satisfies the necessary boundary condition for the normal velocity component. The actual variation of the normal component near the surface is not significant. The tangential component would be found, by solving the equations (20) to have some value different from the corresponding component of the body, whereas these velocity components should be equal also. Hence the tangential velocity component must change rapidly in the surface layer. To study the nature of this variation, consider any portion of the body, of dimension large compared with δ, but small compared with that of the body. Such a portion may be regarded approximately as a plane and therefore the results obtained for the plane surface may be used. Suppose the z-axis is directed along the normal to the portion considered and the x-axis parallel to the tangential velocity component of the surface there. Let v_x denote the tangential component of the fluid velocity relative to the body; v_x must vanish on the body. Let $v_0e^{-i\omega t}$ be the value of v_x found by solving the equations (20). Then, by using (12), we can say that in the surface layer v_x will fall off towards the surface according to the law:

$$v_x = v_0e^{-i\omega t}\left\{1-\exp\left[-r^{\frac{1}{2}}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)z\right]\right\}. \qquad \ldots \quad \ldots \quad (21)$$

The total amount of energy dissipated per unit time is given by

$$\overline{\dot{E}} = -\tfrac{1}{2}\mu r^{\frac{1}{2}} \sin \frac{\theta}{2} \oint |v_0|^2\, ds \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

taken over the surface of the oscillating body.

We conclude the discussion by making the following general remark about the drag on the oscillating body. Writing the velocity of the body in the complex form $u = u_0 e^{-i\omega t}$, we obtain a drag F proportional to the velocity and also complex: $F = \beta u$, where $\beta = \beta(r, \theta)$ is complex. Alternately,

$$F = (\beta_1 + i\beta_2)u = \beta_1 u - \beta_2 \dot{u}/\omega. \qquad \ldots \quad \ldots \quad \ldots \quad (23)$$

The time average of the energy dissipation is given by the mean product of the real part of the drag and the velocity:

$$\overline{Fu} = \tfrac{1}{4}(\beta + \beta^*)|u_0|^2 = \tfrac{1}{2}\beta_1 |u_0|^2. \qquad \ldots \quad \ldots \quad \ldots \quad (24)$$

Thus the energy dissipation arises only from the real part of β; the corresponding part of drag in (23), proportional to velocity, may be called the dissipative part. The other part of the drag proportional to the acceleration and determined by the imaginary part of β does not involve any dissipation of energy and may be called the inertial part.

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