NON-EXISTENCE OF PSEUDO-HARMONIC AND PSEUDO-KILLING VECTOR AND TENSOR FIELDS IN COMPACT ORIENTABLE GENERALIZED RIEMANNIAN SPACE (METRIC MANIFOLD WITH TORSION) WITH BOUNDARY

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Yano and Bochner (1953) have defined pseudo-harmonic and pseudo-killing vectors and tensors on metric manifolds with torsion. Non-existence of pseudo-harmonic and pseudo-killing vectors and tensors in this manifold has been studied by Goldberg (1956), by taking an arbitrary symmetric tensor G_{ij} which is a scalar multiple of fundamental tensor g_{ij}. Later on, Yano (1959) and Yano and Takahashi (1958-59) gave some of the conditions under which a harmonic or killing vector (tensor) field could not exist in a manifold with boundary.

In the present paper pseudo-harmonic, pseudo-killing vector (tensor) fields have been studied and conditions have been established under which such fields could not exist in generalized Riemannian manifold with boundary (Rani and Prakash 1965).

1. Consider a compact manifold M which is the closure of an open sub-manifold of an n-dimensional orientable (Yano and Bochner 1953) generalized Riemannian manifold X_n of class $C^r (r \geqslant 2)$ with a positive definite metric

$$ds^2 = a_{\underline{\alpha\beta}} dy^\alpha dy^{\beta *} \text{ [Eisenhart 1951]} \quad \quad (1.1)$$

for $a_{\overset{\vee}{\alpha\beta}} dy^\alpha dy^\beta$ contributes nothing. M is represented in a neighbourhood of each point on the boundary B of class C^r by $y^\alpha \geqslant 0$, therefore it follows that B is an $(n-1)$ dimensional compact orientable submanifold (Chern 1955).

The boundary B is represented locally in parametric form by $y^\alpha = y^\alpha (x^i)$ in $U(P) \cap M$, where $U(P)$ is a coordinate neighbourhood in X_n of a point P and B. Let the metric for the boundary B be

$$ds^2 = g_{\underline{ij}} dx^i dx^j. \quad \quad (1.2)$$

We define

$$\frac{\partial y^\alpha}{\partial x^i} = C_i^\alpha \quad \quad (1.3)$$

* Throughout this paper we adopt the convention that Greek alphabets $\alpha, \beta, \gamma, \delta, \sigma, \ldots$ take the values 1, 2, ... n and Latin alphabets $i, j, k,$ etc. take values 1, 2, ... $n-1$.

and then

$$g_{\underline{ij}} = a_{\underline{\alpha\beta}} C_i^\alpha C_j^\beta. \quad \ldots \quad (1.4)$$

N^α denotes the unit normal to B such that N^α and $C_1^{\cdot\alpha}$, $C_2^{\cdot\alpha} \ldots C_{n-1}^{\cdot\alpha}$ form the positive sense of M.

Thus we have the following relations:

$$a_{\underline{\alpha\beta}} N^\alpha C_j^\beta = 0 \quad \ldots \quad (1.5a)$$

$$a_{\underline{\alpha\beta}} N^\alpha N^\beta = 1 \quad \ldots \quad (1.5b)$$

Symbols ∇_α and $'\nabla_j$ denote the covariant differentiation with respect to $\Delta_{\beta\gamma}^{\alpha}$ and Δ_{ij}^{h}, these being coefficients of connections with regard to manifold and the boundary respectively.

Since the space is generalized Riemannian Δ_{ij}^{h} is not symmetric in indices i and j.

We have instead (Eisenhart 1929)

$$\Gamma_{ij}^{h} = \tfrac{1}{2}(\Delta_{ij}^{h} + \Delta_{ji}^{h}) \quad \ldots \quad (1.6a)$$

$$S_{ij}^{h} = \tfrac{1}{2}(\Delta_{ij}^{h} - \Delta_{ji}^{h}). \quad \ldots \quad (1.6b)$$

Tensor derivative of $C_i^{\cdot\alpha}$ and N^α are given by

$$'\nabla_j C_i^\alpha = \Omega_{ij} N^\alpha + A_{ij}^k C_k^\alpha \quad \ldots \quad (1.7)$$

$$'\nabla_j N^\alpha = -g^{\underline{ih}} \Omega_{ij} C_h^\alpha, \quad \ldots \quad (1.8)$$

where Ω_{ij} is the second fundamental covariant tensor of second order in X_{n-1} and

$$A_{ij}^k = a_{\underline{\alpha\beta}}('\nabla_j C_i^\alpha) C_l^\beta g^{\underline{kl}}. \quad \ldots \quad (1.9)$$

If

$$C_{i\beta} = C_i^\alpha a_{\underline{\alpha\beta}} \quad \ldots \quad (1.10a)$$

$$C_\beta^i = C_j^\alpha g^{\underline{ji}} a_{\underline{\alpha\beta}} \quad \ldots \quad (1.10b)$$

then we have

$$N_\alpha N^\beta + C_{\cdot\alpha}^i C_i^{\cdot\beta} = \delta_\alpha^\beta, \quad \ldots \quad (1.11a)$$

where Kronecker delta δ_α^β takes the value one when $\alpha = \beta$ and zero when $\alpha \neq \beta$.

Multiplying (1.10b) by a $a^{\underline{\beta\alpha_1}} C_i^{\cdot\gamma}$ we find after taking into account (1.11a) and suitable arrangement of indices

$$a^{\underline{\alpha\alpha_1}} - N^\alpha N^{\alpha_1} = g^{\underline{ij}} C_{ij}^{\alpha\alpha_1}. \quad \ldots \quad (1.11b)$$

We also have the following relations:

$$N^{\alpha}C^{i}_{.\alpha}=0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12a)$$

$$C_{j}^{.\alpha}C^{i}_{.\alpha}=\delta^{i}_{j} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12b)$$

Curvature and Pseudo-Ricci tensors are defined by

$$\Delta_{\beta\gamma\delta}^{..\alpha}=\frac{\partial}{\partial y^{\delta}}\Delta_{\beta\gamma}^{\alpha}-\frac{\partial}{\partial y^{\gamma}}\Delta_{\beta\delta}^{\alpha}+\Delta_{\sigma\delta}^{\alpha}\Delta_{\beta\gamma}^{\sigma}-\Delta_{\sigma\gamma}^{\alpha}\Delta_{\beta\delta}^{\sigma} \qquad \ldots \quad \ldots \quad (1.13a)$$

and

$$\Delta_{\gamma\beta\alpha}^{..\alpha}=\Delta_{\gamma\beta}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.13b)$$

Stokes theorem. For an arbitrary vector field U^{α} in M we have

$$\int_{M}\Delta_{\alpha}u^{\alpha}\,d\sigma=\int_{B}u^{\alpha}\cdot N_{\alpha}d\sigma' \qquad \ldots \quad \ldots \quad \ldots \quad (1.14)$$

where $d\sigma$ and $d\sigma'$ are volume elements given by

$$d\sigma=\sqrt{a}\,dy^{1}\Lambda\,dy^{2}\Lambda\ldots\Lambda\,dy^{n},\quad d\sigma'=\sqrt{g}\,dx^{1}\Lambda\,dx^{2}\Lambda\ldots\Lambda\,dx^{n-1}$$

respectively, a and g being the determinants of fundamental tensors $a_{\alpha\beta}$ and g_{ij} (Eisenhart 1951). Components u^{α} in M of a vector field ${}^{/}u^{i}$ of B are given by relation

$$u^{\alpha}=C_{i}^{.\alpha}\,{}^{/}u^{i}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.15)$$

Applying Stokes theorem to $u^{\beta}(\nabla_{\beta}u^{\alpha})-u^{\alpha}(\nabla_{\beta}u^{\beta})$
we have

$$\int_{M}\left\{\Delta_{\alpha\beta}u^{\alpha}u^{\beta}-2S_{\beta\alpha\gamma}(\nabla^{\gamma}u^{\alpha})u^{\beta}+(\nabla^{\alpha}u^{\beta})(\nabla_{\beta}u_{\alpha})-(\nabla_{\alpha}u^{\alpha})(\nabla_{\beta}u^{\beta})\right\}d\sigma$$

$$=\int_{B}\left\{u^{\beta}(\nabla_{\beta}u^{\alpha})-u^{\alpha}(\nabla_{\beta}u^{\beta})\right\}N_{\alpha}d\sigma', \qquad \ldots \quad \ldots \quad (1.16)$$

where

$$S_{\alpha\beta\gamma}=S_{\alpha\beta}^{\ \ \delta}g_{\underline{\gamma\delta}}.$$

Now assume that the vector field u^{α} is tangential to B.
That is

$$u^{\beta}N_{\beta}=0 \text{ on } B. \qquad \ldots \quad \ldots \quad \ldots \quad (1.17)$$

Differentiating (1.17) covariantly along B and using (1.8) and (1.15) we have

$$(\nabla_{\alpha}u_{\beta})u^{\alpha}N^{\beta}=\Omega_{\underline{ij}}\,{}^{/}u^{i}\,{}^{/}u^{j}. \qquad \ldots \quad \ldots \quad \ldots \quad (1.18)$$

Thus in view of (1.17) and (1.18), equation (1.16) can be written in the following form:

$$\int_{M}\left\{\Delta_{\alpha\beta}u^{\alpha}u^{\beta}-2S_{\beta\alpha\gamma}(\nabla^{\gamma}u^{\alpha})u^{\beta}+(\nabla^{\alpha}u^{\beta})(\nabla_{\beta}u_{\alpha})\right.$$

$$\left.-(\nabla_{\alpha}u^{\alpha})(\nabla_{\beta}u^{\beta})\right\}d\sigma=\int_{B}\Omega_{\underline{ij}}\,{}^{/}u^{i}\,{}^{/}u^{j}\,d\sigma'$$

which can be put in the following two forms:

$$\int_M \Big\{ \Delta_{\alpha\beta} u^\alpha u^\beta - 2S_{\beta\alpha\gamma}(\nabla^\gamma u^\alpha)u^\beta + a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} (\nabla^\alpha u^\beta)(\nabla^\gamma u^\delta)$$
$$- \tfrac{1}{2}(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha)(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha) - (\nabla_\alpha u^\alpha)(\nabla_\beta u^\beta) \Big\} d\sigma$$
$$= \int_B \Omega_{\underline{ij}}\, {}^{/}u^i\, {}^{/}u^j\, d\sigma' \quad \ldots \quad (1.19)$$

$$\int_M \Big\{ \Delta_{\alpha\beta} u^\alpha u^\beta - 2S_{\beta\alpha\gamma}(\nabla^\gamma u^\alpha)u^\beta - a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} (\nabla^\alpha u^\beta)(\nabla^\gamma u^\delta) - (\nabla_\alpha u^\alpha)(\nabla_\beta u^\beta)$$
$$+ \tfrac{1}{2}(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha)(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) \Big\} d\sigma = \int_B \Omega_{\underline{ij}}\, {}^{/}u^i\, {}^{/}u^j\, d\sigma'. \quad \ldots \quad (1.20)$$

If the vector field u^α is pseudo-harmonic and also tangential to B, then (1.19) takes the form

$$\int_M \Big\{ (\Delta_{\alpha\beta} + \Delta_{\beta\alpha}) u^\alpha u^\beta - 2(S_{\beta\alpha\gamma} + S_{\beta\gamma\alpha})(\nabla^\gamma u^\alpha) u^\beta$$
$$+ (a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} + a_{\underline{\alpha\delta}} a_{\underline{\beta\gamma}})(\nabla^\gamma u^\delta)(\nabla^\alpha u^\beta) \Big\} d\sigma = 2 \int_B \Omega_{\underline{ij}}\, {}^{/}u^i\, {}^{/}u^j\, d\sigma'.$$

Thus we have the following theorem:

THEOREM 1. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$N = \left\| \begin{matrix} (\Delta_{\alpha\beta} + \Delta_{\beta\alpha}) & -(S_{\beta\alpha\gamma} + S_{\beta\alpha\gamma}) \\ -(S_{\beta\gamma\alpha} + S_{\beta\gamma\alpha}) & (a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} + a_{\underline{\alpha\delta}} a_{\underline{\beta\gamma}}) \end{matrix} \right\|$$

defines a positive definite form in the variables u^α and $u^{\alpha\beta} = u^{\beta\alpha}$ and the second fundamental form is negative semi-definite, then there does not exist a pseudo-harmonic vector field tangential to B other than the zero vector field.

Now if the vector field is pseudo-killing and also tangential to B, then (1.20) takes the form

$$\int_M \{ (\Delta_{\alpha\beta} + \Delta_{\beta\alpha}) u^\alpha u^\beta - (S_{\beta\alpha\gamma} - S_{\beta\gamma\alpha})(\nabla^\gamma u^\alpha) u^\beta$$
$$- (a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} - a_{\underline{\alpha\delta}} a_{\beta\gamma})(\nabla^\gamma u^\delta)(\nabla^\alpha u^\beta) \}\, d\sigma$$
$$= 2 \int_B \Omega_{\underline{ij}}\, {}^{/}u^i\, {}^{/}u^j\, d\sigma'.$$

Therefore we have the following theorem:

THEOREM 2. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$N^1 = \left\| \begin{matrix} (\Delta_{\alpha\beta} + \Delta_{\beta\alpha}) & -(S_{\beta\alpha\gamma} - S_{\beta\gamma\alpha}) \\ -(S_{\beta\alpha\gamma} - S_{\beta\gamma\alpha}) & -(a_{\underline{\alpha\gamma}} a_{\underline{\beta\delta}} - a_{\underline{\alpha\delta}} a_{\underline{\beta\gamma}}) \end{matrix} \right\|$$

defines a negative definite form the variables u^{α} and $u^{\alpha\beta} = -u^{\beta\alpha}$ and the second fundamental form is positive semi-definite, then there does not exist a pseudo-killing vector field tangential to B other than the zero field.

Let now u^{α} be normal to B.

Then

$$u^{\alpha} = aN^{\alpha}, \qquad \ldots \quad (1.21)$$

where a is a scalar on B.

Differentiating (1.21) covariantly along B and simplifying it with the help of (1.11a), (1.12a) and (1.12b) we get

$$N_{\alpha}\left\{u^{\beta}(\nabla_{\beta}u^{\alpha})-u^{\alpha}(\nabla_{\beta}u^{\beta})\right\} = a^{2}\Omega_{j}^{\cdot j}. \qquad \ldots \quad (1.22)$$

Applying (1.22) to (1.16) Stokes theorem gives the

$$\int_{M}\left\{\Delta_{\alpha\beta}u^{\alpha}u^{\beta}-2S_{\beta\alpha\gamma}(\nabla^{\gamma}u^{\alpha})u^{\beta}+(\nabla^{\alpha}u^{\beta})(\nabla_{\beta}u_{\alpha})-(\nabla_{\alpha}u^{\alpha})(\nabla_{\beta}u^{\beta})\right\}d\sigma = \int_{B}a^{2}\Omega_{j}^{\cdot j}\,d\sigma'$$

which can be written in the following two forms:

$$\int_{M}\left\{\Delta_{\alpha\beta}u^{\alpha}u^{\beta}-2S_{\beta\alpha\gamma}(\nabla^{\gamma}u^{\alpha})u^{\beta}+a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}(\nabla^{\alpha}u^{\beta})(\nabla^{\gamma}u^{\delta}) -\tfrac{1}{2}(\nabla^{\alpha}u^{\beta}-\nabla^{\beta}u^{\alpha})(\nabla_{\alpha}u_{\beta}-\nabla_{\beta}u_{\alpha})-(\nabla_{\alpha}u^{\alpha})(\nabla_{\beta}u^{\beta})\right\}d\sigma = \int_{B}a^{2}\Omega_{j}^{j}\,d\sigma' \quad (1.23)$$

$$\int_{M}\left\{\Delta_{\alpha\beta}u^{\alpha}u^{\beta}-2S_{\beta\alpha\gamma}(\nabla^{\gamma}u^{\alpha})u^{\beta}-a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}(\nabla^{\alpha}u^{\beta})(\nabla^{\gamma}u^{\delta}) +\tfrac{1}{2}(\nabla^{\alpha}u^{\beta}+\nabla^{\beta}u^{\alpha})(\nabla_{\alpha}u_{\beta}+\nabla_{\beta}u_{\alpha})-(\nabla_{\alpha}u^{\alpha})(\nabla_{\beta}u^{\beta})\right\}d\sigma = \int_{B}a^{2}\Omega_{j}^{j}\,d\sigma'. \quad (1.24)$$

If the vector field is pseudo-harmonic and also normal to B, then (1.23) gives

$$\int_{M}\left\{\left(\Delta_{\alpha\beta}+\Delta_{\beta\alpha}\right)u^{\alpha}u^{\beta}-2(S_{\beta\alpha\gamma}+S_{\beta\gamma\alpha})(\nabla^{\gamma}u^{\alpha})u^{\beta} +\left(a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}+a_{\underline{\alpha\delta}}a_{\underline{\beta\gamma}}\right)(\nabla^{\alpha}u^{\beta})(\nabla^{\gamma}u^{\delta})\right\}d\sigma = \int_{B}2a^{2}\Omega_{j}^{j}\,d\sigma'.$$

Hence it follows

THEOREM 3. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$N = \begin{Vmatrix} (\Delta_{\alpha\beta}+\Delta_{\beta\alpha}) & -(S_{\beta\alpha\gamma}+S_{\beta\gamma\alpha}) \\ -(S_{\beta\alpha\gamma}+S_{\beta\gamma\alpha}) & \left(a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}+a_{\underline{\alpha\delta}}a_{\underline{\beta\gamma}}\right) \end{Vmatrix}$$

defines a positive definite form in the variables u^{α} and $u^{\alpha\beta} = u^{\beta\alpha}$ and the mean curvature $\Omega_i^{\cdot i}$ is negative, then there does not exist a pseudo-harmonic vector field normal to B other than the zero vector field.

Now if u^α is pseudo-killing vector field normal to B, then (1.24) reduces to

$$\int_M \Big\{(\Delta_{\alpha\beta}+\Delta_{\beta\alpha})u^\alpha u^\beta - 2(S_{\beta\alpha\gamma}-S_{\beta\gamma\alpha})(\nabla^\gamma u^\alpha)u^\beta$$
$$-\Big(a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}-a_{\underline{\alpha\delta}}a_{\underline{\beta\gamma}}\Big)(\nabla^\alpha u^\beta)(\nabla^\gamma u^\delta)\Big\}\, d\sigma = \int_B 2a^2\Omega_j^j d\sigma'.$$

Hence we have the following theorem:

THEOREM 4. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$N' = \begin{Vmatrix} (\Delta_{\alpha\beta}+\Delta_{\beta\alpha}) & -(S_{\beta\alpha\gamma}-S_{\beta\gamma\alpha}) \\ -(S_{\beta\alpha\gamma}-S_{\beta\gamma\alpha}) & -\Big(a_{\underline{\alpha\gamma}}a_{\underline{\beta\delta}}-a_{\underline{\alpha\delta}}a_{\underline{\beta\gamma}}\Big) \end{Vmatrix}$$

defines a negative definite form in the variables u^α and $u^{\alpha\beta} = -u^{\beta\alpha}$ and the mean curvature $\Omega_i^{\cdot i}$ is positive, then there does not exist a pseudo-killing vector field normal to B other than the zero vector field.

2. *Tensor fields.*—For an arbitrary skew-symmetric tensor field $u^{\alpha_1 \cdots \alpha_p}$ of order p $(1 \leqslant p \leqslant n)$ in M we have

$$\nabla_\beta\Big[u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\beta\alpha_2\ldots\alpha_p}) - u^{\beta}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})\Big]$$
$$=(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p})(\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}) - \Big(\nabla_\beta u^{\beta}_{\alpha_2\ldots\alpha_p}\Big)(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})$$
$$+\Delta_{\gamma\alpha}u^{\alpha}_{\alpha_2\ldots\alpha_p}u^{\gamma\alpha_2\ldots\alpha_p} + \frac{p-1}{2}\Delta_{\gamma\beta\alpha\delta}u^{\gamma\beta}_{\alpha_3\ldots\alpha_p}u^{\alpha\delta\alpha_3\ldots\alpha_p}$$
$$-2S_{\alpha\beta\gamma}u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla^\gamma u^{\beta\alpha_2\ldots\alpha_p}). \qquad \ldots \quad (2.1)$$

Applying Stokes theorem to

$$\Big\{u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\beta\alpha_2\ldots\alpha_p}) - u^{\beta}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})\Big\}$$

we have from (1.14) by virtue of (2.1)

$$\int_M \Big\{\Delta_{\gamma\alpha}a_{\underline{\beta\delta}}u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}_{\alpha_3\ldots\alpha_p} + \frac{p-1}{2}\Delta_{\gamma\beta\alpha\delta}u^{\gamma\beta}_{\alpha_3\ldots\alpha_p}u^{\alpha\delta\alpha_3\ldots\alpha_p}$$
$$+(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p})(\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}) - \Big(\nabla_\beta u^{\beta}_{\alpha_2\ldots\alpha_p}\Big)(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}) \; -$$
$$-2S_{\alpha\beta\gamma}u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla^\gamma u^{\beta\alpha_2\ldots\alpha_p})\Big\}\, d\sigma$$
$$= \int_B N_\beta\Big\{u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\beta\alpha_2\ldots\alpha_p}) - u^{\beta}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})\Big\}\, d\sigma'. \quad (2.2)$$

For a skew-symmetric tensor field $u^{\alpha_1\cdots\alpha_p}$ of order p we have

$$u^{\alpha_1\ldots\alpha_p} = C^{\alpha_1\ldots\alpha_p}_{i_1\ldots i_p}\,'u^{i_1\ldots i_p}, \qquad (2.3)$$

where

$$C^{\alpha_1\ldots\alpha_p}_{i_1\ldots i_p} = C^{\alpha_1}_{i_1}C^{\alpha_2}_{i_2}\ldots C^{\alpha_p}_{i_p}$$

and $'u^{i_1\cdots i_p}$ is skew-symmetric tensor field defined over B.
Now let skew-symmetric tensor field $u^{\alpha_1\cdots\alpha_p}$ be tangential to B,

i.e.

$$u_{\alpha\alpha_2\ldots\alpha_p}N^\alpha = 0 \quad \text{on } B. \qquad (2.4)$$

Differentiating (2.4) covariantly along B and simplifying it with the help of (2.3) we find

$$\nabla_\beta(u_{\alpha\alpha_2\ldots\alpha_p})u^{\beta\alpha_2\cdots\alpha_p}N^\alpha = \Omega_{\underline{ij}}\,'u^{j}_{i_2\ldots i_p}\,'u^{ii_2\cdots i_p}. \qquad (2.5)$$

Applying (2.4) and (2.5) to (2.2) we get

$$2p\int_M\Big\{\Delta_{\gamma\alpha}a_{\underline{\beta\delta}}u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}_{\ \ \alpha_3\ldots\alpha_p}+\frac{p-1}{2}\Delta_{\gamma\beta\alpha\delta}u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}_{\ \ \alpha_3\ldots\alpha_p}$$
$$+\left(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}\right)-\left(\nabla_\beta u^\beta_{\ \alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\right)$$
$$-2S_{\alpha\beta\gamma}a_{\underline{\sigma\delta}}u^{\alpha\sigma\alpha_3\ldots\alpha_p}\left(\nabla^\gamma u^{\beta\delta}_{\ \ \alpha_3\ldots\alpha_p}\right)\Big\}\,d\sigma = 2p\int_B\Omega_{\underline{ij}}\,'u^j_{i_2\ldots i_p}\,'u^{ii_2\ldots i_p}\,d'\sigma$$

which can be written in the following two forms:

$$\int_M\Big\{2p\left(\Delta_{\gamma\alpha}a_{\underline{\beta\delta}}+\frac{p-1}{2}\Delta_{\gamma\beta\alpha\delta}\right)u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}_{\ \ \alpha_3\ldots\alpha_p}+2\left(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p}\right)\left(\nabla_\beta u_{\alpha\alpha_2\ldots\alpha_p}\right)$$
$$-2(p+1)\left(\nabla^{[B}u^{\alpha\alpha_2\ldots\alpha_p]}\right)\left(\nabla_{[B}u_{\alpha\alpha_2\ldots\alpha_p]}\right)-2p\left(\nabla_\beta u^\beta_{\ \alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\right)$$
$$-2p\left(2S_{\alpha\beta\gamma}a_{\underline{\sigma\delta}}\right)u^{\alpha\sigma\alpha_3\ldots\alpha_p}\left(\nabla^\gamma u^{\beta\delta}_{\ \ \alpha_3\ldots\alpha_p}\right)\Big\}\,d\sigma = \int_B 2p\Omega_{\underline{ij}}\,'u^j_{i_2\ldots i_p}\,'u^{ii_2\ldots i_p}\,d'\sigma \qquad (2.6)$$

$$\int_M\Big\{2p\left(\Delta_{\gamma\alpha}a_{\underline{\beta\delta}}+\frac{p-1}{2}\Delta_{\gamma\beta\alpha\delta}\right)u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}_{\ \ \alpha_3\ldots\alpha_p}-2p\left(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p}\right)\left(\nabla_\beta u_{\alpha\alpha_2\ldots\alpha_p}\right)$$
$$+p\left(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p}+\nabla^\alpha u^{\beta\alpha_2\ldots\alpha_p}\right)\left(\nabla_\beta u_{\alpha\alpha_2\ldots\alpha_p}+\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}\right)$$
$$-2p\left(\nabla_\beta u^\beta_{\ \alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\right)-2p\left(2S_{\alpha\beta\gamma}a_{\underline{\sigma\delta}}\right)u^{\alpha\sigma\alpha_3\ldots\alpha_p}\left(\nabla^\gamma u^{\beta\delta}_{\ \ \alpha_3\ldots\alpha_p}\right)\Big\}\,d\sigma$$
$$=\int_B 2p\Omega_{\underline{ij}}\,'u^j_{i_2\ldots i_p}\,'u^{ii_2\ldots i_p}\,d'\sigma. \qquad (2.7)$$

In view of the fact that $u^{\alpha_1\cdots\alpha_p}$ is a skew-symmetric tensor, equations (2.6) and (2.7) can be further put down as

$$\int_M \Big\{ \overset{(p)}{K}_{\gamma\beta\alpha\delta} u^{\alpha\delta\alpha_3\ldots\alpha_p} u^{\gamma\beta}{}_{\alpha_3\ldots\alpha_p} + G_{\alpha\rho\beta\delta\sigma\gamma}\left(\nabla^\beta u^{\alpha\rho\alpha_3\ldots\alpha_p}\right)\left(\nabla^\gamma u^{\delta\sigma}{}_{\alpha_3\ldots\alpha_p}\right)$$

$$-2(p+1)\left(\nabla^{[\beta} u^{\alpha\alpha_2\ldots\alpha_p]}\right)\left(\nabla_{[\beta} u_{\alpha\alpha_2\ldots\alpha_p]}\right) - 2p\left(\nabla_\beta u^\beta{}_{\alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\right)$$

$$-2\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma} u^{\alpha\sigma\alpha_3\ldots\alpha_p}\left(\nabla^\gamma u^{\beta\delta}{}_{\alpha_3\ldots\alpha_p}\right)\Big\} d\sigma = \int_B 2p\Omega_{\underline{ij}}\, 'u^j_{i_2\ldots i_p}\, 'u^{ii_2\ldots i_p}\, d'\sigma \quad (2.8)$$

$$\int_M \Big\{ \overset{(p)}{K}_{\gamma\beta\alpha\delta} u^{\alpha\delta\alpha_3\ldots\alpha_p} u^{\gamma\beta}{}_{\alpha_3\ldots\alpha_p} - pG_{\alpha\rho\beta\delta\sigma\gamma}\left(\nabla^\beta u^{\alpha\rho\alpha_3\ldots\alpha_p}\right)\left(\nabla^\gamma u^{\delta\sigma}{}_{\alpha_3\ldots\alpha_p}\right)$$

$$+p\left(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p} + \nabla^\alpha u^{\beta\alpha_2\ldots\alpha_p}\right)\left(\nabla_\beta u_{\alpha\alpha_2\ldots\alpha_p} + \nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}\right)$$

$$-2p\left(\nabla_\beta u^\beta{}_{\alpha_2\ldots\alpha_p}\right)\left(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\right) - 2\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma} u^{\alpha\sigma\alpha_3\ldots\alpha_p}\left(\nabla^\gamma u^{\beta\delta}{}_{\alpha_3\ldots\alpha_p}\right)\Big\} d\sigma$$

$$= \int_B 2p\Omega_{\underline{ij}}\, 'u^j_{i_2\ldots i_p}\, 'u^{ii_2\ldots i_p}\, d'\sigma, \quad \ldots \quad (2.9)$$

where (Yano and Bochner 1953)

$$\overset{(p)}{K}_{\gamma\beta\delta} = \frac{p}{2}\left(\Delta_{\gamma\alpha} a_{\underline{\delta\beta}} - \Delta_{\beta\alpha} a_{\underline{\delta\gamma}} - \Delta_{\gamma\delta} a_{\underline{\alpha\beta}} + \Delta_{\beta\delta} a_{\underline{\alpha\gamma}}\right)$$

$$-\frac{p(p-1)}{2}\left(\Delta_{\gamma\alpha\delta\beta} - \Delta_{\beta\alpha\delta\gamma} - \Delta_{\gamma\delta\alpha\beta} + \Delta_{\beta\delta\alpha\gamma}\right)$$

$$\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma} = \frac{p}{2}\left(S_{\alpha\beta\gamma} a_{\underline{\sigma\delta}} - S_{\sigma\beta\gamma} a_{\underline{\alpha\delta}} - S_{\alpha\delta\gamma} a_{\underline{\sigma\beta}} - S_{\sigma\delta\gamma} a_{\underline{\alpha\beta}}\right)$$

$$G_{\alpha\rho\beta\delta\sigma\gamma} = \left(a_{\underline{\alpha\delta}} a_{\underline{\rho\sigma}} - a_{\underline{\alpha\sigma}} a_{\underline{\rho\delta}}\right) a_{\underline{\beta\gamma}}.$$

Thus from equations (2.8) and (2.9) we can state the following theorems:

THEOREM 5. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\ \ \gamma}$ the matrix

$$P_1 = \left\| \begin{matrix} \overset{(p)}{K}_{\gamma\beta\alpha\delta} & -\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma} \\ -\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma} & G_{\alpha\rho\beta\delta\sigma\gamma} \end{matrix} \right\|$$

defines a positive definite form in the variables $u^{\alpha\beta} = -u^{\beta\alpha}$ and $u^{\alpha\beta\gamma} = -u^{\beta\alpha\gamma}$ and the second fundamental form is negative semi-definite, then there does not exist a pseudo-harmonic tensor field of order p tangential to B other than the zero tensor field.

THEOREM 6. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\ \ \gamma}$ the matrix

$$P_2 = \begin{Vmatrix} K^{(p)}_{\gamma\beta\alpha\delta} & -S^{(p)}_{\alpha\sigma\beta\delta\gamma} \\ -S^{(p)}_{\alpha\sigma\beta\delta\gamma} & -pG_{\alpha\rho\beta\delta\sigma\gamma} \end{Vmatrix}$$

defines a negative definite form in the variables $u^{\alpha\beta} = -u^{\beta\alpha}$ and $u^{\alpha\beta\gamma} = -u^{\beta\alpha\gamma}$ and the second fundamental form is positive semi-definite, then there does not exist a pseudo-killing tensor field of order p tangential to B other than the zero tensor field.

Let now skew-symmetric tensor field $u^{\alpha_1\cdots\alpha_p}$ be normal to B.

Then

$$u_{\alpha_1\ldots\alpha_p} C^{\alpha_1\ldots\alpha_p}_{i_1\ldots i_p} = 0. \qquad \ldots \quad (2.10)$$

We also have the relations

$$(u_{\alpha_1\ldots\alpha_p}) N^{\alpha_1} N^{\alpha_k} = 0 \quad \text{for } (2 \leqslant k \leqslant p) \qquad \ldots \quad (2.11a)$$

$$(\nabla_\alpha u_{\alpha_1\ldots\alpha_p}) N^{\alpha_1} N^{\alpha_k} = 0. \qquad \ldots \quad (2.11b)$$

Multiplying (2.10) by $C^{i_1\cdots i_p}_{\beta_1\ldots\beta_p} a^{\underline{\beta_1\alpha}} a^{\underline{\beta_2\alpha_2}} \ldots a^{\underline{\beta_p\alpha_p}}$ and simplifying with the help of (1.11a) and (2.11a) we get

$$N^{\alpha} u^{\beta\alpha_2\ldots\alpha_p} N_\beta = u^{\alpha\alpha_2\ldots\alpha_p} - N^{\alpha_2}(u^{\alpha\beta\alpha_3\ldots\alpha_p} N_\beta) - \ldots$$
$$\ldots - N^{\alpha_p}(u^{\alpha\alpha_2\ldots\alpha_{p-1}\beta} N_\beta). \qquad \ldots \quad (2.12)$$

Let

$$u_{\alpha\alpha_2\ldots\alpha_p} N^{\alpha} C^{\alpha_2\ldots\alpha_p}_{i_2\ldots i_p} = {'u}^{i_2\ldots i_p}, \qquad \ldots \quad (2.13a)$$

where ${'u}^{i_2\cdots i_p}$ is skew-symmetric in all indices. (2.13a) can be alternatively put as

$$u^{\beta\alpha_2\ldots\alpha_p} N_\beta = {'u}^{i_2\ldots i_p} C^{\alpha_2\ldots\alpha_p}_{i_2\ldots i_p}. \qquad \ldots \quad (2.13b)$$

If we now differentiate (2.10) covariantly along B and make use of (1.7), (1.11b), (2.3), (2.13a) we obtain

$$\left(\nabla_\alpha u^{\alpha}_{\ \alpha_2\ldots\alpha_p}\right) u^{\beta\alpha_2\ldots\alpha_p} N_\beta - N^{\alpha} N^{\alpha_1} (\nabla_\alpha u_{\alpha_1\ldots\alpha_p}) u^{\beta\alpha_2\ldots\alpha_p} N_\beta =$$

$$-\Omega^{i}_{\cdot i}\, {'u}_{i_2\ldots i_p}\, {'u}^{i_2\ldots i_p} + \Omega^{i}_{\cdot i_2}\, {'u}_{i i_3\ldots i_p}\, {'u}^{i_2\ldots i_p} + \ldots \Omega^{i}_{\cdot i_p}\, {'u}_{i_2\ldots i_{p-1} i}\, {'u}_{i_2\ldots i_p}$$

$$-\Big\{\left({'u}^{i_2\ldots i_p} a_{\underline{\alpha_1\alpha_1'}} \ldots a_{\underline{\alpha_p\alpha_p'}}\right) \times \left(C^{\cdot\,\alpha_1'}_{i_1'} \ldots C^{\cdot\,\alpha_p'}_{i_p'}\, {'u}^{i_1'\ldots i_p}\right)\left(C^{\alpha_1\alpha_2\ldots\alpha_p}_{k i_2\ldots i_p} A^{k\ \ i_1}_{\ \ i_1}\right.$$

$$\left.+ C^{\alpha_1\ldots\alpha_p}_{i_1 k i_3\ldots i_p} A^{k\ \ i_1}_{\ \ i_2} + \ldots + \ldots C^{\alpha_1\ldots\alpha_p}_{i_1\ldots i_{p-1}k} A^{k\ \ i_1}_{\ \ i_p}\right)\Big\} = 0.$$

A substitution of the value of $N^\alpha u^{\beta\alpha_2\ldots\alpha_p}N_\beta$ from (2.12) and its simplification in view of (2.11b) yields

$$N_\beta\Big\{u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\beta\alpha_2\ldots\alpha_p})-u^{\beta}_{\alpha_2\ldots\alpha_p}(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})\Big\}$$
$$=\Omega^{i}_{i}\,{}'u_{i_2\ldots i_p}\,{}'u^{i_2\ldots i_p}-(p-1)\Omega_{\underline{ij}}\,{}'u^{j}_{i_3\ldots i_p}\,{}'u^{ii_3\ldots i_p}$$
$$+{}'u^{i_2\ldots i_p}\Big\{(p+1)A^{k}{}_{[i_1}{}^{i_1}\,{}'u_{ki_2\ldots i_p]}+A^{k}{}_{k}{}^{i_1}\,{}'u_{i_1\ldots i_p}\Big\}. \qquad (2.14)$$

Applying (2.14) to (2.2) we get

$$\int_M\Big\{\Big(\Delta_{\alpha\gamma}a_{\underline{\beta\delta}}+\frac{p-1}{2}\Delta_{\gamma\beta\dot\alpha\delta}\Big)u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}{}_{\alpha_3\ldots\alpha_p}+(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p})(\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p})$$
$$-\Big(\nabla_\beta u^{\beta}_{\alpha_2\ldots\alpha_p}\Big)(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p})-2S_{\alpha\beta\gamma}u^{\alpha}_{\alpha_2\ldots\alpha_p}(\nabla^\gamma u^{\beta\alpha_2\ldots\alpha_p})\,d\sigma$$
$$=\int_B\Big[\Omega^{i}_{i}\,{}'u_{i_2\ldots i_p}\,{}'u^{i_2\ldots i_p}-(p-1)\Omega_{\underline{ij}}\,{}'u^{j}_{i_3\ldots i_p}\,{}'u^{ii_3\ldots i_p}$$
$$+{}'u^{i_2\ldots i_p}\Big\{(p+1)A^{k}{}_{[i_1}{}^{i_1}\,{}'u_{ki_2\ldots i_p]}+A^{k}{}_{k}{}^{i_1}\,{}'u_{i_1\ldots i_p}\Big\}\Big]d\sigma'. \qquad (2.15)$$

For a skew-symmetric tensor $u^{\alpha_1\ldots\alpha_p}$ of order p normal to B (2.15) can be written in the following two forms:

$$\int_M\Big\{\overset{(p)}{K}_{\gamma\beta\alpha\delta}u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\gamma\beta}{}_{\alpha_3\ldots\alpha_p}+G_{\alpha\rho\beta\delta\sigma\gamma}(\nabla^\beta u^{\alpha\rho\alpha_3\ldots\alpha_p})\Big(\nabla^\gamma u^{\delta\sigma}{}_{\alpha_3\ldots\alpha_p}\Big)$$
$$-2(p+1)(\nabla^{[\beta}u^{\alpha\alpha_2\ldots\alpha_p]})(\nabla_{[\beta}u_{\alpha\alpha_2\ldots\alpha_p]})-2p\Big(\nabla_\beta u^{\beta}_{\alpha_2\ldots\alpha_p}\Big)\Big(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\Big)$$
$$-2\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma}u^{\alpha\sigma\alpha_3\ldots\alpha_p}\Big(\nabla^\gamma u^{\beta\delta}{}_{\alpha_3\ldots\alpha_p}\Big)\Big\}\,d\sigma$$
$$=2p\int_B\Big[\Omega^{i}_{i}\,{}'u_{i_2\ldots i_p}\,{}'u^{i_2\ldots i_p}-(p-1)\Omega_{\underline{ij}}\,{}'u^{j}_{i_3\ldots i_p}\,{}'u^{ii_3\ldots i_p}$$
$$+{}'u^{i_2\ldots i_p}\Big\{(p+1)A^{k}{}_{[i_1}{}^{i_1}\,{}'u_{ki_2\ldots i_p]}+A^{k}{}_{k}{}^{i_1}\,{}'u_{i_1\ldots i_p}\Big\}\Big]d\sigma' \qquad (2.16)$$

and

$$\int_M\Big\{\overset{(p)}{K}_{\gamma\beta\alpha\delta}u^{\alpha\delta\alpha_3\ldots\alpha_p}u^{\alpha\beta}{}_{\alpha_3\ldots\alpha_p}-pG_{\alpha\rho\beta\delta\sigma\gamma}\Big(\nabla^\beta u^{\alpha\rho\alpha_3\ldots\alpha_p}\Big)\Big(\nabla^\gamma u^{\delta\sigma}{}_{\alpha_3\ldots\alpha_p}\Big)$$
$$+p\Big(\nabla^\beta u^{\alpha\alpha_2\ldots\alpha_p}+\nabla^\alpha u^{\beta\alpha_2\ldots\alpha_p}\Big)\Big(\nabla_\beta u_{\alpha\alpha_2\ldots\alpha_p}+\nabla_\alpha u_{\beta\alpha_2\ldots\alpha_p}\Big)$$
$$-2p\Big(\nabla_\beta u^{\beta}_{\alpha_2\ldots\alpha_p}\Big)\Big(\nabla_\alpha u^{\alpha\alpha_2\ldots\alpha_p}\Big)-2\overset{(p)}{S}_{\alpha\sigma\beta\delta\gamma}u^{\alpha\sigma\alpha_3\ldots\alpha_p}\Big(\nabla^\gamma u^{\beta\delta}{}_{\alpha_3\ldots\alpha_p}\Big)\Big\}\,d\sigma$$
$$=2p\int_B\Big[\Omega_{i}^{\cdot\,i}\,{}'u_{i_2\ldots i_p}\,{}'u^{i_2\ldots i_p}-(p-1)\Omega_{ij}\,{}'u^{j}_{i_3\ldots i_p}\,{}'u^{ii_3\ldots i_p}$$
$$+{}'u^{i_2\ldots i_p}\Big\{(p+1)A^{k}{}_{[i_1}{}^{i_1}\,{}'u_{ki_2\ldots i_p]}+A^{k}{}_{k}{}^{i_1}\,{}'u_{i_1\ldots i_p}\Big\}\Big]\,d\sigma'. \qquad (2.17)$$

Hence the following two theorems follow.

THEOREM 7. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$P = \left\| \begin{matrix} K^{(p)}_{\gamma\beta\alpha\delta\cdot} & -S^{(p)}_{\alpha\sigma\beta\delta\gamma} \\ -S^{(p)}_{\alpha\sigma\beta\delta\gamma} & G_{\alpha\rho\beta\delta\sigma\gamma} \end{matrix} \right\|$$

defines a positive definite form in the variables $u^{\alpha\beta} = -u^{\beta\alpha}$ and $u^{\alpha\beta\gamma} = -u^{\beta\alpha\gamma}$ and the quadratic form

$$G(u, u) = \Omega^{i}_{i}\, 'u_{i_2 \ldots i_p}\, 'u^{i_2 \ldots i_p} - (p-1)\Omega_{ij}\, 'u^{ii_3 \ldots i_p}\, 'u^{j}_{i_3 \ldots i_p}$$
$$+ 'u^{i_2 \ldots i_p} \left\{ (p+1) A^{k}{}_{[i_1}{}^{i_1}\, 'u_{ki_2 \ldots i_p]} + A^{k}{}_{k}{}^{i_1}\, 'u_{i_1 \ldots i_p} \right\}$$

is negative semi-definite, then there does not exist a pseudo-harmonic tensor field of order p normal to B other than the zero tensor.

THEOREM 8. If in a compact orientable manifold M with boundary B and torsion $S_{\alpha\beta}^{\gamma}$ the matrix

$$P_2 = \left\| \begin{matrix} K^{(p)}_{\gamma\beta\alpha\delta} & -S^{(p)}_{\alpha\sigma\beta\delta\gamma} \\ -S^{(p)}_{\alpha\sigma\beta\delta\gamma} & -pG_{\alpha\rho\beta\delta\sigma\gamma} \end{matrix} \right\|$$

defines a negative definite form in the variables $u^{\alpha\beta} = -u^{\beta\alpha}$ and $u^{\alpha\beta\gamma} = -u^{\beta\alpha\gamma}$ and the quadratic form

$$G(u, u) = \Omega^{\cdot\, i}_{i}\, 'u_{i_2 \ldots i_p}\, 'u^{i_2 \ldots i_p} - (p-1)\Omega_{ij}\, 'u^{ii_3 \ldots i_p}\, 'u^{j}_{i_3 \ldots i_p}$$
$$+ 'u^{i_2 \ldots i_p} \left\{ (p+1) A^{k}{}_{[i_1}{}^{i_1}\, 'u_{ki_2 \ldots i_p]} + A^{k}{}_{k}{}^{i}\, 'u_{i_1 \ldots i_p} \right\}$$

is positive semi-definite, then there does not exist a pseudo-killing tensor field of order p normal to B other than the zero tensor field.

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